Nonlinear Dynamics **15:** 115–136, 1998.

Lie–Bäcklund and Noether Symmetries with Applications

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(Received: 25 March 1997; accepted: 14 August 1997)

Abstract. New identities relating the Euler–Lagrange, Lie–Bäcklund and Noether operators are obtained. Some important results are shown to be consequences of these fundamental identities. Furthermore, we generalise an interesting example presented by Noether in her celebrated paper and prove that any Noether symmetry is equivalent to a strict Noether symmetry, i.e. a Noether symmetry with zero divergence. We then use the symmetry based results deduced from the new identities to construct Lagrangians for partial differential equations. In particular, we show how the knowledge of a symmetry and its corresponding conservation law of a given partial differential equation can be utilised to construct a Lagrangian for the equation. Several examples are given.

Key words: Lie–Bäcklund, Euler–Lagrange, Noether, symmetries, Lagrangians, conservation laws.

1. Introduction

The Euler–Lagrange, Lie–Bäcklund and Noether operators play a central role in the study of invariances in the calculus of variations and differential equations. Lie–Bäcklund symmetries are an important generalisation of Lie point symmetries. For example, they have application in accounting for the hidden symmetry associated with the Laplace vector of the well-known Kepler problem. Noether symmetries form the basis of a simple systematic way of determining conservation laws for systems of Euler–Lagrange equations. The *Noether identity* is a fundamental identity that connects the Euler–Lagrange, Lie–Bäcklund and Noether operators.

In this paper, we, *inter alia*, present new identities relating these basic operators. Moreover, in the following sections we discuss the fundamental identities and point out their applications. In particular, we determine necessary conditions, using symmetries and associated conservation laws, for the construction of Lagrangians, both for ordinary and partial differential equations. Furthermore, it is well known that Noether's fundamental theorem gives a constructive way of determining conservation laws for Euler–Lagrange equations once their symmetry properties are known. We generalise an interesting example presented by Noether in her celebrated paper. Namely, we prove that any Noether symmetry is equivalent to a strict Noether symmetry, i.e., a Noether symmetry with zero divergence.

In more detail, the outline of the paper is as follows. In Section 2, we outline the nomenclature used in the paper. Section 3 reviews the Noether identity. In Section 4 we present a new representation of the Lie–Bäcklund operator. Section 5 deals with two new operator identities which are given in terms of commutator relations. These identities have applications in classical mechanics and variational calculus and are investigated in the latter sections. In Section 6, Noether symmetries are considered. Here it is proved that any Noether symmetry is equivalent to a strict Noether symmetry. In the following Section 7, examples of this result are given. The application of the identities of Section 5 to inverse problems in mechanics and variational calculus are deduced in Section 8. In particular, the relationship between Noether symmetries and the Noether conserved vector is presented. In Section 9, we pursue aspects in inverse problems. The results of Section 8 which, *inter alia*, are applied to various examples are given in Section 10. Finally, in Section 11, a utilization of a conservation law in the construction of an alternative Lagrangian for a system is given.

2. Main Operators

We first remind the reader of the universal space $\mathcal{A}$ of differential functions introduced by Ibragimov [1] (see also [2, p. 56]). The summation convention is adopted throughout.

Let

$$x = (x^1, \ldots, x^n)$$

be the independent variable with coordinates x^i, and

$$u = (u^1, \ldots, u^m)$$

the dependent variable with coordinates u^α. The derivatives of u with respect to x are

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u^\alpha), \quad \ldots, \tag{1}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n \tag{2}$$

is the *operator of total differentiation*. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$. Similarly, the collections of all higher-order derivatives are denoted by $u_{(2)}, u_{(3)}, \ldots$.

Following Lie, in group analysis it is expedient to consider all variables $x, u, u_{(1)}, u_{(2)}, \ldots$ as functionally independent connected only by the differential relations (1). Consequently, the u^αs are referred to as *differential variables*.

Intrinsic to modern group analysis of differential equations is the universal space $\mathcal{A}$ defined as follows:

We denote by z the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \ldots) \tag{3}$$

with elements z^ν, $\nu \geq 1$, where, e.g.,

$$z^i = x^i, \quad 1 \leq i \leq n, \quad z^{n+\alpha} = u^\alpha, \quad 1 \leq \alpha \leq m,$$

with the remaining elements representing the derivatives of u. However, in applications one invariably utilizes only finite subsequences of z which are denoted by $[z]$.

A locally analytic function $f(x, u, u_{(1)}, \ldots, u_{(k)})$ of a finite number of variables is called a *differential function of order k* and for brevity is written as $f([z])$. The space $\mathcal{A}$ is the vector space of all differential functions of all finite orders. A total derivative (2) converts any differential function of order k to a differential function of order $k+1$. Hence, the space $\mathcal{A}$ is closed under total derivations D_i.

The main operators introduced below are correctly defined in the space $\mathcal{A}$. Precisely, this means that the operators defined as formal sums truncate when they act on differential functions.

DEFINITION 1. The *Euler–Lagrange* operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,, \quad \alpha = 1,\ldots,m. \tag{4}$$

The operator (4) is sometimes referred to as the *Euler operator*, named after Euler (1744) who first introduced it in a geometrical manner for the one-dimensional case. Also, it is called the *Lagrange operator*, bearing the name of Lagrange (1762) who considered the multi-dimensional case and established its use in a *variational* sense (see, e.g., [3] for a history of the calculus of variations). Following Lagrange, Equation (4) is frequently referred to as a *variational derivative*. In the modern literature, the terminology Euler–Lagrange and variational derivative are used interchangeably as (4) usually arises in considering a variational problem.

DEFINITION 2. The Lie–Bäcklund operator is given by

$$X = \xi^i\,\frac{\partial}{\partial x^i} + \eta^\alpha\,\frac{\partial}{\partial u^\alpha}\,, \quad \xi^i,\ \eta^\alpha \in \mathcal{A}. \tag{5}$$

This operator is in fact an abbreviated form of the following infinite formal sum:

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i\frac{\partial}{\partial u^\alpha_i} + \zeta^\alpha_{i_1i_2}\frac{\partial}{\partial u^\alpha_{i_1i_2}} + \cdots, \tag{6}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ \zeta^\alpha_{i_1i_2} &= D_{i_1}D_{i_2}(W^\alpha) + \xi^j u^\alpha_{ji_1i_2}, \\ &\ldots \end{aligned} \tag{7}$$

In (7), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{8}$$

One can write the Lie–Bäcklund operator (6) in the form

$$X = \xi^i D_i + W^\alpha\frac{\partial}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial}{\partial u^\alpha_i} + D_{i_1}D_{i_2}(W^\alpha)\frac{\partial}{\partial u^\alpha_{i_1i_2}} + \cdots. \tag{9}$$

It should be remarked that in classical Lie theory dealing with point and first-order contact transformations only, it was natural to indicate the prolongations of the operator X to finite order derivatives of u with respect to x by another symbol, e.g., $X_{(1)}$ for the first prolongation.

Now with the introduction of the universal space $\mathcal{A}$, all Lie point, Lie contact and Lie–Bäcklund operators naturally act in the space $\mathcal{A}$ after prolongation. Accordingly, the same symbol X is used both for the operator and any of its prolongations.

In modern group analysis, there exists a variety of so-called *generalised symmetries* which generalise Lie's point and contact infinitesimal group generators. However, the problem still remains whether these generalised symmetries generate, via the Lie equations, a group. The problem thus far is solved for Lie–Bäcklund operators (6). That is, the Lie equation is uniquely solvable, in the space $[[\mathcal{A}]]$ of formal power series with coefficients from $\mathcal{A}$ (the proof can be found in [4, 5]), for any Lie–Bäcklund operator (6).

In the above sense, Lie–Bäcklund symmetries are distinguished from all other generalised symmetries. Furthermore, the corresponding formal transformation group leaves invariant the contact conditions of any order. The possible existence of higher-order contact transformations were extensively discussed by Lie and Bäcklund during the period 1874–1876. In recognition of their fundamental contribution, the above generalisation of Lie point and first-order contact transformations was given the name *Lie–Bäcklund transformations* by Ibragimov and Anderson [6]. The corresponding infinitesimal generator (6) is naturally called the *Lie–Bäcklund operator*. It should be noted that the prolongation formulae (7) are obtained as a direct consequence of the invariance of the infinite-order contact conditions.

The set of Lie–Bäcklund operators constitute an infinite dimensional Lie algebra L and is called the *Lie–Bäcklund algebra*. It contains the total derivations (2) as well as $X_* = \lambda^i D_i$ for any $\lambda^i \in \mathcal{A}$. Now let L_* denote the set of operators $\{X_*\}$. Then L_* is an ideal of L, i.e. the Lie bracket $[X, X_*] \in L_*$ for any $X \in L$ (see, e.g. [4]).

DEFINITION 3. The Lie–Bäcklund operators $\tilde{X}$ and X are said to be *equivalent* if $X - \tilde{X} \in L_*$. That is

$$X - \tilde{X} = \lambda^i D_i, \quad \lambda^i \in \mathcal{A}.$$

In particular, a Lie–Bäcklund operator of the form $\tilde{X} = \eta^\alpha \partial/\partial u^\alpha + \cdots$ is called a *canonical* representation of X.

DEFINITION 4. The Noether operator associated with a Lie–Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}, \quad i = 1, \ldots, n, \tag{10}$$

where the Euler–Lagrange operators with respect to derivatives of u^α are obtained from (4) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \tag{11}$$

The operator (10) was introduced by Ibragimov [7] and the name *Noether operator* was given in recognition of Noether's contribution [8]. As a consequence of the operator (10), the proof of Noether's theorem becomes purely algebraic and independent of variational calculus. The algebraic proof is based on the identity presented in the next section.

3. Noether Identity

THEOREM 1. *The Euler–Lagrange, Lie–Bäcklund and Noether operators are connected by the operator identity*

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{12}$$

Here, $D_i(\xi^i)$ is a differential function which is a sum of functions obtained by total derivations D_i of differential functions ξ^i. That is, $D_i(\xi^i)$ is a divergence of the vector $\xi = (\xi^1, \ldots, \xi^n)$, viz., $\operatorname{div}\xi$ whereas, $D_i N^i$ is an operator obtained as a sum of products of operators D_i on N^i, i.e., it is a scalar product of vector operators $D = (D_1, \ldots, D_n)$ and $N = (N^1, \ldots, N^n)$.

The identity (12) is due to Ibragimov [7] (see also [5]) and is called the *Noether identity* because of its close relation to the Noether theorem.

4. A Representation of the Lie–Bäcklund Operator

For one independent variable x, the Lie–Bäcklund operator (9) is

$$X = \xi D + W^\alpha \frac{\partial}{\partial u^\alpha} + D(W^\alpha)\frac{\partial}{\partial u_x^\alpha} + D^2(W^\alpha)\frac{\partial}{\partial u_{xx}^\alpha} + \cdots, \tag{13}$$

where $W^\alpha = \eta^\alpha - \xi u_x^\alpha$ and from (10), N takes the form

$$N = \xi + W^\alpha \frac{\delta}{\delta u_x^\alpha} + D(W^\alpha)\frac{\delta}{\delta u_{xx}^\alpha} + D^2(W^\alpha)\ \frac{\delta}{\delta u_{xxx}^\alpha} + \cdots. \tag{14}$$

LEMMA.

$$\frac{\delta}{\delta u^\alpha}\ D = 0, \tag{15}$$

$$\frac{\delta}{\delta u_{k+1}^\alpha}\ D = \frac{\partial}{\partial u_k^\alpha}\ , \quad k = 0, 1, 2, \ldots, \tag{16}$$

where u_k^α denotes the k-th total derivative of u^α with respect to x.

Proof. For Equation (15):

$$\begin{aligned}
\frac{\delta}{\delta u^\alpha}\ D \;&=\; \left(\frac{\partial}{\partial u^\alpha} - D\frac{\partial}{\partial u_1^\alpha} + D^2\frac{\partial}{\partial u_2^\alpha} - D^3\frac{\partial}{\partial u_3^\alpha} + \cdots\right)\\
&\quad\times\left(\frac{\partial}{\partial x} + u_1^\beta\frac{\partial}{\partial u^\beta} + u_2^\beta\frac{\partial}{\partial u_1^\beta} + \cdots\right)\\
&= \frac{\partial}{\partial u^\alpha}\ D - D\frac{\partial}{\partial u^\alpha} - D^2\frac{\partial}{\partial u_1^\alpha} + D^2\frac{\partial}{\partial u_1^\alpha} + D^3\frac{\partial}{\partial u_2^\alpha} - D^3\frac{\partial}{\partial u_2^\alpha}\cdots\\
&= \frac{\partial}{\partial u^\alpha}\ D - D\frac{\partial}{\partial u^\alpha}\ .
\end{aligned}$$

However,

$$\frac{\partial}{\partial u^\alpha}\ D = D\frac{\partial}{\partial u^\alpha}\ .$$

Hence,

$$\frac{\delta}{\delta u^\alpha}\, D = 0.$$

For Equation (16):

$$\begin{aligned}
\frac{\delta}{\delta u_1^\alpha}\, D &= \left(\frac{\partial}{\partial u_1^\alpha} - D\frac{\partial}{\partial u_2^\alpha} + D^2\frac{\partial}{\partial u_3^\alpha} - D^3\frac{\partial}{\partial u_4^\alpha} + \cdots\right) \\
&\times \left(\frac{\partial}{\partial x} + u_1^\beta\frac{\partial}{\partial u^\beta} + u_2^\beta\frac{\partial}{\partial u_1^\beta} + u_3^\beta\frac{\partial}{\partial u_2^\beta} + u_4^\beta\frac{\partial}{\partial u_3^\beta} + \cdots\right) \\
&= \frac{\partial}{\partial u^\alpha} + D\frac{\partial}{\partial u_1^\alpha} - D\frac{\partial}{\partial u_1^\alpha} - D^2\frac{\partial}{\partial u_2^\alpha} + D^2\frac{\partial}{\partial u_2^\alpha} + D^3\frac{\partial}{\partial u_3^\alpha} - D^3\frac{\partial}{\partial u_3^\alpha} - \cdots \\
&= \frac{\partial}{\partial u^\alpha}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta}{\delta u_2^\alpha}\, D &= \left(\frac{\partial}{\partial u_2^\alpha} - D\frac{\partial}{\partial u_3^\alpha} + D^2\frac{\partial}{\partial u_4^\alpha} - D^3\frac{\partial}{\partial u_5^\alpha} + \cdots\right) \\
&\times \left(\frac{\partial}{\partial x} + u_1^\beta\frac{\partial}{\partial u^\beta} + u_2^\beta\frac{\partial}{\partial u_1^\beta} + u_3^\beta\frac{\partial}{\partial u_2^\beta} + u_4^\beta\frac{\partial}{\partial u_3^\beta} + \cdots\right) \\
&= \frac{\partial}{\partial u_1^\alpha} + D\frac{\partial}{\partial u_2^\alpha} - D\frac{\partial}{\partial u_2^\alpha} - D^2\frac{\partial}{\partial u_3^\alpha} + D^2\frac{\partial}{\partial u_3^\alpha} + D^3\frac{\partial}{\partial u_4^\alpha} - D^3\frac{\partial}{\partial u_4^\alpha} - \cdots \\
&= \frac{\partial}{\partial u_1^\alpha}\,.
\end{aligned}$$

In a similar manner, one can easily verify (16) for $k \geq 2$. □

THEOREM 2. *The following operator equality holds:*

$$X = ND, \tag{17}$$

where X is an arbitrary Lie–Bäcklund operator and N is the Noether operator (14).

Proof. We have

$$ND = \xi D + W^\alpha\frac{\delta D}{\delta u_1^\alpha} + D(W^\alpha)\frac{\delta D}{\delta u_2^\alpha} + D^2(W^\alpha)\frac{\delta D}{\delta u_3^\alpha} + \cdots .$$

Invoking (16), we obtain

$$\begin{aligned}
ND &= \xi D + W^\alpha\frac{\partial}{\partial u^\alpha} + D(W^\alpha)\frac{\partial}{\partial u_1^\alpha} + D^2(W^\alpha)\frac{\partial}{\partial u_2^\alpha} + \cdots \\
&= X.
\end{aligned}$$

□

5. New Identities

In this section, we present another new identity relating two of the main operators, viz., the Lie–Bäcklund and the Noether operators. This operator identity is expressed in terms of a commutator relation.

We first consider, in Section 5.1, the one-dimensional case (one independent variable x) because of its importance in classical mechanics. We then generalise the result to the multi-dimensional case in Section 5.2.

5.1. One-Dimensional Case

THEOREM 3. *The following operator commutator relation holds:*

$$[X, N] = ND(\xi), \tag{18}$$

where $[X, N] = XN - NX$.

Note that in (18), $ND(\xi)$ is the product of the multiplication operator $D(\xi)$ and the operator N. That is, it acts on any differential function $f \in \mathcal{A}$ as follows:

$$ND(\xi)(f) = N(D(\xi)f).$$

The relation (18) is proved by straightforward, albeit tedious, computation. It would certainly be a useful exercise for the interested reader to verify identity (18), e.g., in the case of differential functions of first-order. One could, e.g., take ξ and η in X to depend on x, u, u' ($u' = du/dx$) and act by the left and right-hand sides of the operator identity (18) on a function $f(x, u, u')$. To encourage the reader, note that it is useful to write X in the form

$$X = \xi D + W\frac{\partial}{\partial u} + D(W)\frac{\partial}{\partial u'} + \cdots,$$

where $W = \eta - \xi u'$ and from (10) N in the form

$$N = \xi + W\frac{\partial}{\partial u'} + [D(W) - WD]\frac{\partial}{\partial u''} + \cdots.$$

5.2. Multi-Dimensional Case

In the case of $n > 1$ independent variables x^i, the equivalent of identity (18) display features that differ from the point of view of applications. This will be pointed out later in Section 8.

THEOREM 4. *The Lie–Bäcklund and Noether operators are related by the operator identity*

$$[X + D_k(\xi^k), N^i] = D_k(\xi^i)N^k. \tag{19}$$

The proof is similar to the one of Theorem 3 and is by direct computation.

6. Noether Symmetries

Consider a k-th order differential equation

$$E^\alpha(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, \tilde{m}. \tag{20}$$

DEFINITION 5. A *conserved vector* of (20) is a tuple $T = (T^1, \ldots, T^n)$, $T^j = T^j(x, u, u_{(1)}, \ldots, u_{(k-1)}) \in \mathcal{A}$, $j = 1, \ldots, n$, such that

$$D_i(T^i) = 0 \qquad (21)$$

is satisfied for all solutions of (20).

REMARK. When Definition 5 is satisfied, (21) is called a *conservation law* for (20).

We now discuss conservation laws of *Euler–Lagrange* equations. That is, differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \qquad (22)$$

where $L = L(x, u, u_{(1)}, \ldots, u_{(l)}) \in \mathcal{A}$, $l \leq k$, k being the order of (22), is a Lagrangian and $\delta/\delta u^\alpha$ is the *Euler–Lagrange* operator defined by (4).

DEFINITION 6. A Lie–Bäcklund operator X of the form (6) is called a *Noether symmetry* corresponding to a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, \ldots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \qquad (23)$$

If in Equation (23) $B^i = 0$, $i = 1, \ldots, n$, then X is referred to as a *strict Noether symmetry* corresponding to a Lagrangian $L \in \mathcal{A}$.

THEOREM 5. *For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, defined by*

$$T^i = N^i(L) - B^i, \quad i = 1, \ldots, n, \qquad (24)$$

which is a conserved vector of Equation (22), i.e., $D_i(T^i) = 0$ on the solutions of (22).

The above Theorem 5 is due to Noether [8] and is called the *Noether Theorem*.

In her famous fundamental paper on symmetries and conservation laws, Noether [8, section III] presented an example of a symmetry

$$X = -2\frac{u}{u'^2}\frac{\partial}{\partial x} + (x - 2\frac{u}{u'})\frac{\partial}{\partial u} + \cdots$$

of the Lagrangian $L = u'^2/2$ corresponding to the second-order ordinary differential equation $u'' = 0$. The symmetry X is a strict Noether symmetry, i.e. a Noether symmetry with zero divergence, of L since $X(L) = 0$.

It should be noted that X is not a Lie point nor a Lie contact symmetry. It is what is today termed a Lie–Bäcklund symmetry (see Equation (6)). However, X is equivalent to a Lie point symmetry $\tilde{X} = x\partial/\partial u$ in the sense that $\tilde{X} = X + \xi(x, u, u')D_x$ (see Editor's footnote in [9]), where $D_x = \partial/\partial x + u'\partial/\partial u + \cdots$ is the operator of total differentiation and $\xi(x, u, u')D_x$ is admitted by any ordinary differential equation (see also Editor's footnote in [9]). The point symmetry $\tilde{X}$ is a Noether symmetry, $\tilde{X}(L) = D_x(u)$, albeit not a strict one. Thus, it is seen that for the example considered that on the one hand X is a strict Noether symmetry (zero divergence term) and on the other hand $\tilde{X}$ is a Noether symmetry (non-zero divergence u) even though X and $\tilde{X}$ are equivalent symmetries in the sense mentioned above. The question then arises whether this applies in general. That is, is any Noether symmetry equivalent to a strict Noether symmetry? The answer is in the affirmative and this constitutes the following theorem. Also, examples are presented.

THEOREM 6. *Let two Lie–Bäcklund operators X and $\tilde{X}$ be equivalent, i.e. $\tilde{X} = X + \lambda^i D_i$, $\lambda^i \in \mathcal{A}$. Then X is a Noether symmetry if and only if $\tilde{X}$ is.*

Proof. Let X be a Noether symmetry, i.e.

$$X(L) + LD_i(\xi^i) = D_i(B^i)$$

for a Lagrangian $L \in \mathcal{A}$ and some vector $B = (B^1, \ldots, B^n)$, where $B^i \in \mathcal{A}$. Let

$$\begin{aligned}\tilde{X} &= X + \lambda^i D_i, \quad \lambda^i \in \mathcal{A}\\ &= (\xi^i + \lambda^i)\frac{\partial}{\partial x^i} + (\eta^\alpha + \lambda^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots.\end{aligned}$$

We substitute $X = \tilde{X} - \lambda^i D_i$ into (23). This yields

$$\begin{aligned}\tilde{X}L - \lambda^i D_i L + LD_i(\xi^i) &= D_i(B^i),\\ \tilde{X}L + LD_i(\xi^i) + LD_i(\lambda^i) &= \lambda^i D_i L + LD_i(\lambda^i) + D_i(B^i),\\ \tilde{X}L + LD_i(\xi^i + \lambda^i) &= D_i(\lambda^i L + B^i). \qquad (25)\end{aligned}$$

Hence, $\tilde{X}$ is a Noether symmetry of $L \in \mathcal{A}$, namely $\tilde{X}$ satisfies (23) with $\tilde{B}^i = \lambda^i L + B^i$. The steps of the above calculation are reversible and thus proves the theorem.

COROLLARY 1. *Any Noether symmetry is equivalent to a strict Noether symmetry.*

Indeed, it follows from (25) that $\tilde{X} = X + \lambda^i D_i$ with $\lambda^i = -(1/L)B^i$ being a strict Noether symmetry of $L \in \mathcal{A}$, i.e.,

$$\tilde{X}L + LD_i(\xi^i - \frac{1}{L}B^i) = 0 \qquad (26)$$

holds with no divergence. Thus the operator

$$\tilde{X} = \left(\xi^i - \frac{1}{L}\,B^i\right)\frac{\partial}{\partial x^i} + \left(\eta^\alpha - \frac{1}{L}\,B^i u_i^\alpha\right)\frac{\partial}{\partial u^\alpha} + \cdots$$

equivalent to X is a strict Noether symmetry.

COROLLARY 2. *X is a Noether symmetry, if and only if the canonical operator $\bar{X}$ associated with X is also Noether.*

Indeed, if one puts $\lambda^i = -\xi^i$ in (25), one gets

$$\bar{X}L = D_i(B^i - L\xi^i).$$

Hence, $\bar{X} = (\eta^\alpha - \xi^i u_i^\alpha)(\partial/\partial u^\alpha) + \cdots$ is a Noether symmetry with divergence $\bar{B}^i = B^i - L\xi^i$. Thus, X is a Noether symmetry, if and only if the canonical operator $\bar{X}$ associated with X is also Noether. This corresponds to a result in [10].

REMARK. If $\xi^i = B^i/L$ in Equation (26), then $\tilde{X}$ is a symmetry of the Lagrangian (i.e., $\tilde{X}(L) = 0$) given by the canonical operator

$$\tilde{X} = \left(\eta^\alpha - \frac{1}{L} B^i u_i^\alpha\right) \frac{\partial}{\partial u^\alpha} + \cdots.$$

7. Examples on Theorem 6

7.1. Scalar Differential Equations

7.1.1. *One-Dimensional Free Particle Equation*

The one-dimensional free particle equation

$$m\ddot{u} = 0,$$

where m is the mass of the particle and $\dot{} = \mathrm{d}/\mathrm{d}t$, has the usual Lagrangian

$$L = \frac{1}{2}\, m\dot{u}^2.$$

It is well known that there are five Noether point symmetries associated with L. They are (see, e.g., [11])

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial u}, \quad X_3 = 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \cdots,$$
$$X_4 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u} + \cdots, \quad X_5 = t\frac{\partial}{\partial u} + \cdots.$$

The symmetries X_4 and X_5 are not strict Noether symmetries whereas the others are Noether. It is simple to check that $X_4(L) + LD(t^2) = D(mu^2/2)$ and $X_5(L) = D(mu)$, where D is the total differentiation operator with respect to t.

We now obtain the operators equivalent to X_4 and X_5 which are strict Noether symmetries of L. Corollary 1 is invoked.

Firstly, we consider X_4. We set $\lambda = -(B/L) = -(u^2/\dot{u}^2)$, where B is the divergence term. The operator $\tilde{X}_4$ is then

$$\tilde{X}_4 = (t^2 - u^2/\dot{u}^2)\frac{\partial}{\partial t} + (tu - u^2/\dot{u})\frac{\partial}{\partial u} + \cdots$$

which is a strict Noether symmetry of L equivalent to X_4. Note that

$$\bar{X}_4 = (tu - t^2\dot{u})\frac{\partial}{\partial u} + \cdots$$

is the canonical operator associated with X_4 and by the above Corollary 2 it is a Noether symmetry with divergence $\bar{B} = mu^2/2 - mt^2\dot{u}^2/2$.

For X_5 we have that $\lambda = -2u/\dot{u}^2$ and the strict Noether symmetry equivalent to X_5 is

$$\tilde{X}_5 = -2\frac{u}{\dot{u}^2}\frac{\partial}{\partial t} + (t - 2\frac{u}{\dot{u}})\frac{\partial}{\partial u} + \cdots.$$

Here it should be pointed out that X_5 which is a canonical operator is not a strict Noether symmetry whereas its equivalent $\tilde{X}_5$ is a strict Noether symmetry and is clearly not canonical.

7.1.2. *Stationary Transonic Gas Flow Equation*

The second-order partial differential equation

$$u_x u_{xx} + u_{yy} = 0$$

describing stationary transonic gas flow has a Lagrangian

$$L = \frac{1}{6}\, u_x^3 + \frac{1}{2}\, u_y^2.$$

The point symmetries admitted by this equation are known and given in, e.g., [5]. The Lie algebra L_6 is six-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u},$$

$$X_4 = y\frac{\partial}{\partial u} + \cdots, \quad X_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u} + \cdots, \quad X_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} + \cdots.$$

The Lie algebra of the Noether point symmetries is a subalgebra L_5 of L_6 and is generated by the three translations, X_4 and $5X_5 + 7X_6$. The only non-strict Noether symmetry is X_4 since $X_4(L) = D_y(u)$, where D_y is the operator of differentiation with respect to y. We utilise Corollary 1 to construct the strict Noether symmetry equivalent to X_4. This is manifest once we know λ^1 and λ^2. We find that $\lambda^1 = 0$ ($B^1 = 0$) and $\lambda^2 = -B^2/L = -6u/(3u_y^2 + u_x^3)$. Hence

$$\tilde{X}_4 = -\frac{6u}{3u_y^2 + u_x^3}\,\frac{\partial}{\partial y} + \left(y - \frac{6uu_y}{3u_y^2 + u_x^3}\right)\frac{\partial}{\partial u} + \cdots.$$

It is seen that X_4 is canonical. Its equivalent $\tilde{X}_4$ which we have shown to be a strict Noether symmetry is evidently not canonical. The only symmetry that is both canonical and strict is $X_3 = \partial/\partial u$. As further examples, one can verify by Corollary 2 that the canonical operators associated with X_5, X_6 and indeed the translations in x and y are non-strict Noether symmetries.

7.2. Systems of Differential Equations

7.2.1. *Three-Dimensional Free Particle Equation*

The free particle motion of a point in three-dimensional space is (cf. Section 7.1.1)

$$m\ddot{u} = 0,$$

where m is the mass of the particle, $u = (u^1, u^2, u^3)$ and $\dot{} = \mathrm{d}/\mathrm{d}t$. This equation has the Lagrangian

$$L = \frac{1}{2}\, m\dot{u}^2$$

which admits Noether symmetries that generate the Galilean group [5]. Among these symmetries there are only the canonical operators

$$X_\alpha = t\frac{\partial}{\partial u^\alpha} + \cdots, \quad \alpha = 1, 2, 3$$

that are non-strict Noether symmetries since $X_\alpha(L) = D(mu^\alpha)$, $\alpha = 1, 2, 3$ (cf. Section 7.1.1). If we set $\lambda_\alpha = (-B_\alpha/L) = (-2u^\alpha/\dot{u}^2)$, where B_α is the divergence for $\alpha = 1, 2, 3$, then the strict Noether symmetry equivalent to X_α is

$$\tilde{X}_\alpha = -2\frac{u^\alpha}{\dot{u}^2}\,\frac{\partial}{\partial t} + \left(t - 2\frac{u^\alpha}{\dot{u}^2}\dot{u}^\beta\right)\frac{\partial}{\partial u^\beta} + \cdots, \quad \alpha = 1, 2, 3.$$

7.2.2. *Kepler Problem*

The classical two-body or Kepler problem which models the gravitational interaction of two bodies is described by the equation of motion (see, e.g. [12] for a review)

$$m\ddot{u} = \mu\frac{u}{r^3}\,,$$

where m is the reduced mass of the two bodies, μ is a physical constant, $r = |u|$ and $u = (u^1, u^2, u^3)$. A Lagrangian for this equation is

$$L = \frac{1}{2}\,m\dot{u}^2 - \frac{\mu}{r}\,.$$

The canonical Lie–Bäcklund symmetries of the equation of motion which is at most linear in the velocity components $\dot{u}^\alpha$ are given in [5]. They are

$$X_\alpha = (2u^\alpha\dot{u}^\beta - u^\beta\dot{u}^\alpha - (u\cdot\dot{u})\delta^\beta_\alpha)\frac{\partial}{\partial u^\beta} + \cdots, \quad \alpha = 1, 2, 3.$$

These are Noether symmetries with respect to the Lagrangian L and via the Noether theorem give rise to the three components of the Laplace vector [5]. Indeed, it is not difficult to verify that $X_\alpha(L) = D(-2\mu(u^\alpha/r))$. Thus the canonical operators X_α are not strict Noether symmetries. However, there are corresponding equivalent operators which are strict Noether symmetries. To see this we invoke Corollary 1. We obtain $\lambda_\alpha = -B_\alpha/L = 4\mu u^\alpha/(mr\dot{u}^2 - 2\mu)$. Hence, the strict Noether symmetries equivalent to X_α are

$$\tilde{X}_\alpha = \lambda_\alpha\frac{\partial}{\partial t} + (\eta^\beta + \lambda_\alpha\dot{u}^\beta)\frac{\partial}{\partial u^\beta} + \cdots, \quad \alpha = 1, 2, 3,$$

where λ_α is as given above and $\eta^\beta = 2u^\alpha\dot{u}^\beta - u^\beta\dot{u}^\alpha - (u\cdot\dot{u})\delta^\beta_\alpha$ for $\alpha = 1, 2, 3$.

7.2.3. *Newton–Cotes Potential*

Finally, we consider the two body problem (m, μ, u and r are as described in Section 7.2.2 above) given by the equation

$$m\ddot{u} = 2\mu\frac{u}{r^4}$$

which describes the motion of a particle in the Newton–Cotes potential. This equation is invariant under the projective transformations (see, e.g., [5]) with operator

$$X = t^2\frac{\partial}{\partial t} + tu^\alpha \frac{\partial}{\partial u^\alpha} + \cdots$$

which is a Noether symmetry [5] corresponding to the Lagrangian

$$L = \frac{1}{2}\, m\dot{u}^2 - \frac{\mu}{r^2}\,.$$

It is rather straightforward to verify that $X(L) + LD(t^2) = D(mr^2/2)$. By Corollary 1 we have that the strict Noether symmetry equivalent to X is

$$\tilde{X} = \left(t^2 + \frac{mr^4}{2\mu - mr^2\dot{u}^2}\right)\frac{\partial}{\partial t} + \left(tu^\alpha + \frac{mr^4}{2\mu - mr^2\dot{u}^2}u^\alpha\right)\frac{\partial}{\partial u^\alpha} + \cdots.$$

By Corollary 2, the canonical operator (cf. Section 7.1.1)

$$\bar{X} = (tu^\alpha - t^2\dot{u}^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots$$

is a Noether symmetry with divergence $\bar{B} = mr^2/2 - mt^2\dot{u}^2/2 + t^2\mu/r^2$.

This concludes our examples.

8. Application of the Identities

The identities stated in Sections 3–5 have important applications, e.g., in the inverse problem in the Calculus of Variations and in the reduction of order in differential equations.

In particular, for the inverse problem, we show below how necessary conditions, using symmetries and associated conservation laws, can be deduced for the construction of Lagrangians.

We first discuss the one-dimensional case (one independent variable x). By Theorem 3, invoking Equation (18) and operating the left-hand and right-hand sides of (18) on a Lagrangian $L \in \mathcal{A}$, we have

$$\begin{aligned} X(NL) &= N(XL) + N(D(\xi)L) \\ &= N(XL + D(\xi)L). \end{aligned}$$

If X is a Noether symmetry, the above relation becomes

$$X(NL) = ND(B), \tag{27}$$

where $B \in \mathcal{A}$. Moreover, the Noether conserved vector associated with the Noether symmetry X is

$$T = NL - B. \tag{28}$$

Substituting Equation (28) into (27) implies

$$X(T) = ND(B) - X(B),$$

which by Theorem 2, operator identity (17), yields

$$X(T) = 0.$$

We have in fact generalised a result of Sarlet and Cartrijn [13] to any Lie–Bäcklund operator X.

THEOREM 7. *The Noether conserved quantity T given by (28) is a differential invariant of the Lie–Bäcklund operator X which is a generator of a Noether symmetry, i.e.,*

$$X(T) = 0. \tag{29}$$

Thus, a necessary condition for a Lagrangian for a system of ordinary differential equations to exist is that relation (29) holds, where X is a generator of a symmetry and T its corresponding invariant.

We now consider the multi-dimensional case, i.e., the situation for independent variables x^i. Invoking the identity of Theorem 4, viz. (19), and applying it on a Lagrangian $L \in \mathcal{A}$, we obtain

$$X(N^i L) = N^i(X + D_k(\xi^k))L + D_k(\xi^i)N^k L - D_k(\xi^k)N^i L.$$

For X a Noether symmetry, this relation becomes

$$X(N^i L) = N^i(D_k(B^k)) + D_k(\xi^i)N^k L - D_k(\xi^k)N^i L, \tag{30}$$

where each $B^i \in \mathcal{A}$. The components of the Noether conserved vector associated with the Noether symmetry X are given by (see Equation (24))

$$T^i = N^i L - B^i. \tag{31}$$

Inserting Equation (31) into (30) yields

$$\begin{aligned} &X(T^i) + D_k(\xi^k)(T^i) - D_k(\xi^i)(T^k) \\ &\qquad = N^i(D_k(B^k)) + D_k(\xi^i)(B^k) - D_k(\xi^k)(B^i) - X(B^i). \end{aligned}$$

Hence, we have the following theorem:

THEOREM 8. *The components of the Noether conserved vector T^i, given by (31), associated with the Lie–Bäcklund operator X which is a generator of a Noether symmetry, satisfy*

$$\begin{aligned} &X(T^i) + D_k(\xi^k)(T^i) - D_k(\xi^i)(T^k) \\ &\qquad = N^i(D_k(B^k)) + D_k(\xi^i)(B^k) - D_k(\xi^k)(B^i) - X(B^i). \end{aligned} \tag{32}$$

The relation (32) is by far more complicated in form than (29). However, a particular case of (32) was considered in [14] and [15] to construct conservation laws for some physical systems.

Finally, we mention that a necessary condition for a Lagrangian for a system of partial differential equations to exist is that relation (32) be satisfied, where the Lie–Bäcklund operator X is a generator of a symmetry and T^i its associated conserved vector components.

9. Aspects in Inverse Problems

The *Inverse problem* in the calculus of variations involves the determination of a Lagrangian, $L \in \mathcal{A}$ if it exists, which corresponds to a given differential equation. That is, given Equation (20), find a Lagrangian $L \in \mathcal{A}$ such that the Euler–Lagrange equation (22) is equivalent to the given Equation (20).

We provide the following definition of alternative Lagrangians [16].

DEFINITION 7. Two Lagrangians, $L \in \mathcal{A}$ and $\bar{L} \in \mathcal{A}$, are said to be *alternative Lagrangians* for a given differential equation if their respective Euler–Lagrange equations imply each other, i.e., $\delta L/\delta u^\alpha = 0$ if and only if $\delta \bar{L}/\delta u^\alpha = 0$.

The *strong requirement* of the inverse problem amounts to the determination of an $L \in \mathcal{A}$ such that $\delta L/\delta u^\alpha = E^\alpha$, $\alpha = 1, \dots, m$. In general, a given Equation (20) is equivalent to the Euler–Lagrange equation (22) for some $L \in \mathcal{A}$ if there exists an invertible matrix f^α_β such that

$$\frac{\delta L}{\delta u^\alpha} = f^\alpha_\beta E^\beta, \quad \alpha = 1, \dots, m, \tag{33}$$

where $f^\alpha_\beta = f^\alpha_\beta(x, u, u_{(1)}, \dots, u_{(k-1)})$, $\alpha, \beta = 1, \dots, m$. This is the *weak requirement*.

Investigations in the area of inverse problems in mechanics and variational calculus are extensive. For example, we refer the reader to [17] and the references therein. This paper contains results on the Lagrangian formulation of quasi-linear second-order partial differential equations. A complete classification of Lagrangians for scalar second-order differential equations according to Noether point symmetries was solved in [11].

It is well known that many equations do not admit a Lagrangian. This is even true in the case of the weak requirement.

In the scalar case, (33) is

$$\frac{\delta L}{\delta u} = fE. \tag{34}$$

Let us consider the following example:

$$u_{tx} + u_x + u^2 = 0 \tag{35}$$

which arises in the description of Maxwellian tails (see, e.g., [18]). Equation (34) becomes

$$\frac{\delta L}{\delta u} = fu_{tx} + fu_x + fu^2. \tag{36}$$

Expanding $\delta L/\delta u$ and equating the coefficients of u_{tt} and u_{xx} in (36) give rise to

$$L = c(t, x, u)u_t u_x + d(t, x, u)u_t + e(t, x, u)u_x + g(t, x, u).$$

The coefficient of u_{tx} in (36) then implies

$$f = -2c.$$

The remaining terms of (36) are

$$-c_u u_t u_x - c_t u_x - c_x u_t - d_t - e_x + g_u = fu_x + fu^2. \tag{37}$$

The strong requirement is equivalent to the construction of L for (35) with $f = 1$. It is easy to verify that this requirement imposed on (37) gives rise to $c = -1/2$ and $c_t = -1$ which is not possible. Therefore, Equation (35) does not admit a Lagrangian in the absence of a multiplier.

For the same problem (35) with f as yet arbitrary in (37), we obtain $c = \alpha \exp 2t$ (α constant) and $d_t + e_x - g_u = 2\alpha e^{2t} u^2$. We can, without loss of generality, set $e = g = 0$ so that $d = \alpha u^2 \exp(2t)$. Hence (35), with multiplier $f = -2\alpha e^{2t}$, has a Lagrangian

$$L = \alpha e^{2t}(u_t u_x + u^2 u_t).$$

Another Lagrangian, for example, can be obtained by setting $d = e = 0$ and $g = -2/3\alpha e^{2t}u^3$. This gives

$$L = \alpha e^{2t}(u_t u_x - \frac{2}{3}u^3).$$

In a similar manner, one can obtain more Lagrangians.

Next we consider two examples in which one can utilise the the invariance condition (23) with $B^i = 0$ in conjunction with (33).

The following equation

$$\nabla^2\Phi = 0, \quad \nabla = (\partial/\partial x^1, \ldots, \partial/\partial x^n), \tag{38}$$

where x^ν $(\nu = 1, \ldots, n)$ are space variables and n takes on values 1,2 and 3 for one-dimensional, plane and three-dimensional motions, respectively, arises in the study of perfect compressible fluids, where $\Phi(t, x)$ is the potential. A Lagrangian (see [5]) for (38) is

$$L = \Phi_t + \frac{1}{2}\,|\nabla\Phi|^2. \tag{39}$$

A well-known symmetry of (38) is the scaling symmetry

$$X = x^\nu\frac{\partial}{\partial x^\nu} - \frac{n-2}{2}\,\Phi\frac{\partial}{\partial\Phi}\ . \tag{40}$$

In fact, X is a Noether symmetry of (39), viz.,

$$X(L) + D_i(\xi^i)L = D_t\left(\frac{8-n}{2}\Phi\right) + D_\nu\left(\frac{3-n}{2}\Phi\nabla\Phi\right)$$

on the solution of the Laplace equation (38). One can also construct a Lagrangian for which the operator (40) is a strict Noether symmetry ($B^i = 0$ in Equation (23)). In this manner one obtains an alternative Lagrangian [5]

$$L = |x|^{-(n+2)/2}\Phi_t + |\nabla\Phi|^2. \tag{41}$$

We further consider the following example which is the potential form of the perfect polytropic gas equation,

$$\Phi_{tt} + 2\nabla\Phi\cdot\nabla\Phi_t + \nabla\Phi\cdot(\nabla\Phi\cdot\nabla)\nabla\Phi + (\gamma-1)\left(\Phi_t + \frac{1}{2}\,|\nabla\Phi|^2\right)\nabla^2\Phi = 0.$$

In [4], the Lagrangian

$$L = \left(\Phi_t + \frac{1}{2}\,|\nabla\Phi|^2\right)^{\gamma/(\gamma-1)}$$

was constructed using the strict form of (23) (i.e., with zero divergence) with known symmetries X and (33). This yields the multiplier

$$f = -\frac{\gamma}{(\gamma-1)^2}\left(\Phi_t + \frac{1}{2}\,|\nabla\Phi|^2\right)^{(2-\gamma)/(\gamma-1)}.$$

For two space variables, $f = 1$ and hence L satisfies the strong requirement.

We use another symmetry based approach in the following section. This is based on Theorems 5 and 8.

10. Applications to Inverse Problems

In this section, we apply Theorems 5 and 8 to the algorithmic construction of Lagrangians for some physically important problems.

The examples presented here are restricted to scalar second-order partial differential equations and serve as prototype examples. The reader can try other examples. Also, we take $(x_1, x_2, x_3, x_4) = (t, x, y, z)$ and the derivatives are denoted by the appropriate subscripts u_t, u_x, etc. The total derivative operator will be denoted by D_t, D_x, etc.

EXAMPLE 1. It is easy to see that

$$u_{tx} + u_x u_{xx} - u_{yy} = 0, \tag{42}$$

which arises in the study of two-dimensional non-steady state transonic gas flow, can be written in the conserved form

$$D_t T^1 + D_x T^2 + D_y T^3 = 0, \tag{43}$$

where $T^1 = u_x$, $T^2 = (1/2)u_x^2$ and $T^3 = -u_y$. The necessary condition (32) is satisfied with $X = \partial/\partial u$ provided that $N^i(D_k(B^k)) = X(B^i)$, $i = 1, 2, 3$ for some B^is. Then Equation (31) becomes

$$u_x = \frac{\partial L}{\partial u_t} - B^1, \tag{44a}$$

$$\frac{1}{2}\, u_x^2 = \frac{\partial L}{\partial u_x} - B^2, \tag{44b}$$

$$-u_y = \frac{\partial L}{\partial u_y} - B^3. \tag{44c}$$

Assume that B^1 is independent of u_t and solve (44a). This yields

$$L = u_t u_x + B^1 u_t + A,$$

where A is independent of u_t. Then (44b) and (44c) are

$$u_t + \frac{\partial B^1}{\partial u_x}\, u_t + \frac{\partial A}{\partial u_x} = B^2 + \frac{1}{2}\, u_x^2,$$

$$-u_y = \frac{\partial B^1}{\partial u_y}\, u_t + \frac{\partial A}{\partial u_y} - B^3. \tag{45}$$

We take $B^1 = -(1/2)u_x$ and $B^2 = (1/2)u_t$ from which other differential consequences imply

$$L = \frac{1}{2}\, u_t u_x + \frac{1}{6}\, u_x^3 - \frac{1}{2}\, u_y^2 + \alpha(t, x, y, u) + B^3(t, x, y, u) u_y. \tag{46}$$

Note that the constraints

$$N^1(D_tB^1 + D_xB^2) = X(B^1),$$
$$N^2(D_tB^1 + D_xB^2) = X(B^2)$$

are satisfied for the above choices of B^1 and B^2. It can be shown, by determining the Euler–Lagrange equation associated with L, that Equation (46) is a Lagrangian for (42) if $\alpha_u = B^3_y$.

EXAMPLE 2. The nonlinear equation

$$2u_{tx} + u_xu_{xx} - u_{yy} - u_{zz} = 0 \tag{47}$$

is the three-dimensional version of non-steady state transonic gas flow (cf. Example 1) and as such is not different to the first example. However, in this example we choose the B_is differently. It is easily verifiable that (47) can be written in the conserved form

$$D_tT^1 + D_xT^2 + D_yT^3 + D_zT^4 = 0, \tag{48}$$

where

$$\begin{aligned} T^1 &= -\frac{1}{2}\,u_y^2 + \frac{1}{6}\,u_x^3 - \frac{1}{2}\,u_z^2, \\ T^2 &= -u_t^2 - \frac{1}{2}\,u_x^2u_t, \\ T^3 &= u_tu_y, \\ T^4 &= u_tu_z. \end{aligned} \tag{49}$$

One can easily check that the necessary condition (32) is satisfied for (49) with $X = \partial/\partial t$. Once again, by substitution in (31), for $i = 1, 2, 3$, we get

$$T^1 = -B^1 - u_t\frac{\partial L}{\partial u_t} + L, \tag{50a}$$

$$T^2 = -B^2 - u_t\frac{\partial L}{\partial u_x}\,, \tag{50b}$$

$$T^3 = -B^3 - u_t\frac{\partial L}{\partial u_y}\,, \tag{50c}$$

$$T^4 = -B^4 - u_t\frac{\partial L}{\partial u_z}\,. \tag{50d}$$

For this example, we assume that the B^is are independent of derivatives. Then, differentiating the four equations of (50) with respect to u_t, u_x, u_y and u_z, respectively, and solving we obtain L to be

$$\begin{aligned} L = {} & Ru_tu_xu_yu_z + Su_tu_xu_y + Tu_tu_xu_z + Uu_xu_t + Vu_tu_yu_z + Wu_tu_y \\ & + Xu_tu_z + Yu_t + \frac{1}{6}u_x^3 + Au_xu_yu_z + Bu_xu_y + Cu_xu_z \\ & + Du_x - \frac{1}{2}u_y^2 + Eu_zu_y + Fu_y + -\frac{1}{2}u_z^2 + Ou_z + P, \end{aligned} \tag{51}$$

where the unknown coefficients are all constants. Then, the differentiation of (50a) with respect to u_x and the substitution of (51), give $U = 1$ and $V = 0$ when (50c) is differentiated with respect to u_z. Proceeding in this manner, one can show that all the remaining constants are zero. Hence, a Lagrangian for (47) is

$$L = \frac{1}{6}u_x^3 + u_t u_x - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2.$$

EXAMPLE 3. We now consider the form of the BBM (Benjamin–Bona–Mahony) equation (see, e.g. [19, ch. 11] and references therein)

$$u_{xt} + u_{xx} + u_x u_{xx} - u_{xxxt} = 0. \tag{52}$$

It is easy to check that Equation (52) can be written in the conserved form

$$D_t T^1 + D_x T^2 = 0,$$

where $T^1 = u_x - u_{xxx}$ and $T^2 = u_x + \frac{1}{2}u_x^2$. The operator $X = \partial/\partial u$ is a symmetry of the components (T^1, T^2) if the necessary condition (32) is satisfied. We have $XT^1 = 0 = XT^2$. Now, Equation (31) becomes

$$T^1 = \frac{\partial L}{\partial u_t} - D_t \frac{\partial L}{\partial u_{tt}} - D_x \frac{\partial L}{\partial u_{tx}} - B^1, \tag{53a}$$

$$T^2 = \frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_t \frac{\partial L}{\partial u_{xt}} + D_x^2 \frac{\partial L}{\partial u_{xxx}} - B^2. \tag{53b}$$

Suppose L is independent of u_{tt} and u_{tx} and take $B^1 = -(1/2)u_x$. Then (53a) simplifies to

$$\frac{1}{2}\, u_x - u_{xxx} = \frac{\partial L}{\partial u_t}\ ,$$

so that

$$L = \frac{1}{2}\, u_x u_t - u_{xxx} u_t + f(t, x, u_x, u_{xx}, u_{xt}). \tag{54}$$

The insertion of Equation (54) into (53b), choosing $B^2 = (1/2)u_t$, and the comparison of the coefficients of the derivatives of u (the calculations are routine and are left out) give rise to

$$f = -\frac{1}{2}\, u_{xt} u_{xx} + \frac{1}{2}\, u_x^2 + \frac{1}{6}\, u_x^3. \tag{55}$$

We get the third-order Lagrangian

$$L = \frac{1}{2}\, u_x u_t - u_t u_{xxx} - \frac{1}{2}\, u_{xt} u_{xx} + \frac{1}{2}\, u_x^2 + \frac{1}{6}\, u_x^3 \tag{56}$$

which is a Lagrangian of Equation (52).

REMARK. By making L independent of u_{tt} and u_{tx} and choosing $B^1 = u_{xxx} - (1/2)u_x$ and $B^2 = (1/2)u_t - u_{xxt}$, one can show that the following second-order Lagrangian arises for Equation (52), viz.,

$$L = \frac{1}{2}\, u_t u_x + \frac{1}{2}\, u_{xx} u_{xt} + \frac{1}{2}\, u_x^2 + \frac{1}{6}\, u_x^3.$$

Here too, the necessary condition (32) is satisfied.

In the next section we will consider an example in which the Lagrangian and the equation is of second-order. Also, the B^is are set to be zero.

EXAMPLE 4. To conclude this section, we obtain a Lagrangian for

$$u_{tx} - (uu_x)_x - u_{yy} + \frac{1}{2}\, u_x^2 = 0. \tag{57}$$

The components of a conserved vector for (57) are

$$\begin{aligned} T^1 &= -\frac{1}{2}\, u_x u_y, \\ T^2 &= uu_x u_y - \frac{1}{2}\, u_t u_y \\ T^3 &= \frac{1}{2}\, u_t u_x + \frac{1}{2}\, u_y^2 - \frac{1}{2}\, uu_x^2. \end{aligned} \tag{58}$$

The necessary condition (32) is satisfied for $X = \partial/\partial y$ and $B^i = 0$. Then Equation (31) is

$$\begin{aligned} &-\frac{1}{2}\, u_x u_y = -u_y \frac{\partial L}{\partial u_t}\,, \\ &-\frac{1}{2}\, u_t u_y + uu_x u_y = -u_y \frac{\partial L}{\partial u_x}\,, \\ &\frac{1}{2}\, u_t u_x + \frac{1}{2}\, u_y^2 - \frac{1}{2}\, uu_x^2 = L - u_y \frac{\partial L}{\partial u_y}\,. \end{aligned} \tag{59}$$

A solution of (59) is straightforward and gives

$$L = \frac{1}{2}\, u_t u_x - \frac{1}{2}\, uu_x^2 - \frac{1}{2}\, u_y^2.$$

11. Conservation Laws and Alternative Lagrangians

Here we consider an example that highlights the utility of conservation laws for alternative Lagrangians. Here again, we use the method of Section 10.

We discuss the following interesting equation

$$u_{xx}u_{yy} - u_{xy}^2 + kx^{-4}\phi^2\left(\frac{y}{x}\right) = 0. \tag{60}$$

This equation is considered in [21] (see also [20, ch. 9]) and is a special case of the Monge–Ampére equation. A Lagrangian for (60) is

$$L = u_y^2 u_{xx} + u_x^2 u_{yy} - 2b(x, y)u_{xy}, \tag{61}$$

where $b_{xy} = 2kx^{-4}\phi^2(y/x)$. Noether's theorem [3] implies the conserved vector $T = (T^1, T^2)$ defined by

$$\begin{aligned} T^1 &= u_x u_{yy}, \\ T^2 &= -u_x u_{xy} + kx^{-3}\Phi\left(\frac{y}{x}\right), \quad \Phi'\left(\frac{y}{x}\right) = \phi^2\left(\frac{y}{x}\right). \end{aligned} \tag{62}$$

This corresponds to the operator $X = \partial/\partial u$. In fact, the application of Noether's theorem gives rise to

$$T^1_* = 4u_x u_{yy} + 2D_y(b - u_x u_y),$$
$$T^2_* = -4u_x u_{xy} + 4kx^{-3}\Phi\left(\frac{y}{x}\right) - 2D_x(b - u_x u_y). \tag{63}$$

The conservation laws corresponding to (62) and (63) are equivalent.

We now construct an alternative Lagrangian for (60) corresponding to (62) with $X = \partial/\partial u$. The method of the previous section is used. It is easy to verify that condition (32) is satisfied for (62) with $X = \partial/\partial u$. Equation (31), with $B^i = 0$, $i = 1, 2$, yields

$$T^1 = \frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}},$$
$$T^2 = \frac{\partial L}{\partial u_y} - D_y \frac{\partial L}{\partial u_{yy}} - D_x \frac{\partial L}{\partial u_{yx}}. \tag{64}$$

The subtraction of T^2 from T^1 yields a first-order equation in L if we suppose that the algebraic sum of the higher derivative terms in L vanish. This gives rise to

$$L = \frac{1}{2}\, u_x^2 u_{yy} + \frac{1}{2}\, u_x^2 u_{xy} - kx^{-3} u_x \Phi + \gamma(u_x + u_y), \tag{65}$$

where γ is obtained from the solution of

$$D_y \frac{\partial L}{\partial u_{yy}} + D_x \frac{\partial L}{\partial u_{yx}} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}} = 0. \tag{66}$$

This results in L having the form

$$L = \frac{1}{2}\, u_x^2 u_{yy} + \frac{1}{2}\, u_x^2 u_{xy} + kx^{-3} u_y \Phi\left(\frac{y}{x}\right). \tag{67}$$

It should be noted that L is independent of u_{yx} and, thus, the term $D_x \partial L/\partial u_{yx}$ in (64) vanishes. This comment is also applicable to the Euler–Lagrange equation (22).

Acknowledgement

This work was supported in part by a grant from the F.R.D. of South Africa.

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