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Solving Differential Equations by Symmetry Groups

John Starrett

1. WHICH ODEs ARE EASY TO SOLVE? Consider the first order ODEs

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = g(y).$$

It is easy to see that the solutions are found by computing

$$\int dy = \int f(x) dx, \quad \int \frac{dy}{g(y)} = \int dx,$$

respectively.

In an elementary course in differential equations, we learn that equations of the form $dy/dx = f(x)g(y)$ are *separable* and are simple to solve because we can separate terms involving only x from those involving only $y = y(x)$. In fact, as we will see, the deeper property that lets us solve these is the presence of a Lie group¹ symmetry: a continuous transformation that takes each solution curve $y = \phi(x, \alpha)$ into another. The constant of integration α can be thought of as an adjustable parameter of the continuous group action that maps one solution curve into another.

In these two cases, the symmetries are particularly simple. In the first case, because $dy/dx = f(x)$, the *slopes* dy/dx of the solution curves $y = \phi(x, \alpha)$ are independent of y . Therefore we can slide any solution curve in the y -direction into any of the other solution curves by means of the correspondence $(x, y) \mapsto (x, y + \alpha)$, as in the left-hand graph of Figure 1. Similarly, for $dy/dx = g(y)$ the slopes dy/dx of the solution curves $y = \psi(x, \beta)$ are independent of x , so we can slide any of these curves in the x -direction into any other via $(x, y) \mapsto (x + \beta, y)$, as in the right-hand graph of Figure 1. These two continuous transformations exemplify the group idea in this context: any sequence of two translations is also a translation, there is an identity translation (no translation), and for any translation there is an inverse translation that undoes it (resulting in the identity translation).

Now consider the differential equation

$$\frac{dy}{dx} = \frac{y^3 + x^2y - x - y}{x^3 + xy^2 - x + y}, \quad (1)$$

which, at first glance, may seem quite difficult to solve. However, a change of variables to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ results in an equation in r and θ ,

$$\frac{dr}{d\theta} = r(1 - r^2),$$

that is separable in the new variables. We see that the slope $dr/d\theta$ of this differential equation is independent of θ , so translation in the θ -direction will take one solution

¹A *Lie group* is a topological group (i.e., a group endowed with a topology with respect to which the group operations are continuous) that is also a manifold.

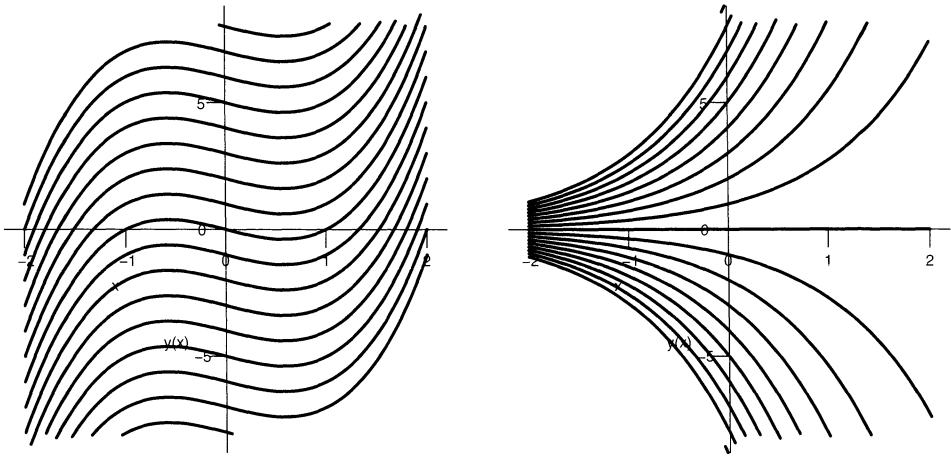


Figure 1. On the left, a family of curves that is invariant under a shift in the y -direction; on the right, a family of curves that is invariant under a shift in the x -direction.

curve $r = h(\theta, \gamma)$ into another. The solutions to (1) are invariant under the continuous group of transformations $(r, \theta) \mapsto (r, \theta + \gamma)$ that represent rotations about the origin. A graph of a number of solution curves to this equation is shown in Figure 2.

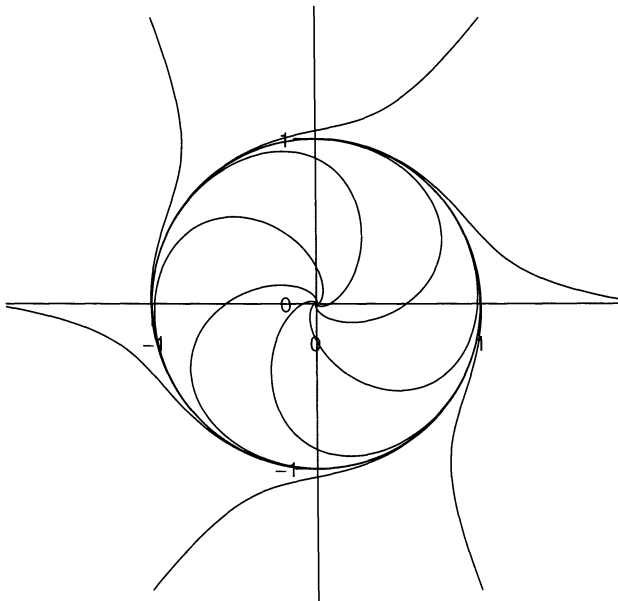


Figure 2. A family of curves that is invariant under a shift in the θ -direction.

These simple examples suggest that a first-order ODE can be transformed into a separable equation if its set of solution curves is invariant under translation in *some* coordinate system. Given a first-order ODE, our goal is to find a general method to determine this coordinate system so that the simplified equation can be integrated: this reduces the order of the first-order equation by one. In fact, this is the general procedure for higher order ODEs, where we have symmetry groups involving more than one parameter. We aim to reduce the order step-by-step, one step per parameter,

until we can integrate the final first order ODE. Unfortunately, this goal is not always achievable. We do not consider second-order (or higher order) ODEs in this article, nor Lie group methods for PDEs. The subject is experiencing a renewal in interest, and there are several good modern texts listed in the bibliography for those interested in pursuing these ideas further.

Finding differential equations for families of curves. Suppose we have a family of curves that can map into each other along some direction. Can we find a differential equation for which this family is a set of solution curves? For instance, consider the family of concentric circles centered at the origin described by the equations $x^2 + y^2 = c^2$ ($c > 0$). The easiest way to arrive at an associated differential equation is to differentiate this equation implicitly until all the arbitrary constants are gone: $D_x(x^2 + y^2 = c^2)$ becomes $x + yy' = 0$ or

$$y' = -x/y. \tag{2}$$

Here, D_x signifies the *total derivative* operator defined by

$$D_x = \frac{dx}{dx} \frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} + \frac{dy'}{dx} \frac{\partial}{\partial y'} + \dots = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots,$$

which accomplishes implicit differentiation when it is applied to an equation in $x, y(x), y'(x), \dots$. We can easily solve the separable equation (2):

$$\int y \, dy = - \int x \, dx,$$

which leads to the implicit solution $y^2 + x^2 = k$ for arbitrary nonnegative k , and thus to our original family of curves (except for the degenerate case $k = 0$). Now, even though equation (2) is separable, there is a change of variables $(r, s) = (x, y^2/2)$ that results in a differential equation $ds/dr = -r$ the slopes of whose solution curves are independent of the dependent variable s . This means that solution curves can be translated into one another in the s -direction. The change of variables is easily computed as follows:

$$\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{D_x \frac{1}{2} y^2}{D_x x} = yy'.$$

Then, substitution of $y' = -x/y$ leads to $ds/dr = -x = -r$. Even simpler, we can see by inspection that the circles are described by the differential equation $dr/d\theta = 0$ in polar coordinates.

Now consider the family of curves $y = (x - c)^2 - c^2$ pictured in Figure 3. By application of the total derivative operator we obtain the differential equation

$$y' = y/x + x.$$

Unlike the case of the family of circles, it is not obvious from inspection of the given family of parabolas what a simplifying coordinate system might be.

More generally, a family of curves with n free parameters is a solution to an n th-order differential equation. The family of ellipses $x^2/a^2 + y^2/b^2 = c^2$ ($c > 0$), for instance, has a second-order differential equation

$$y'' = \frac{y'(y - xy')}{xy},$$

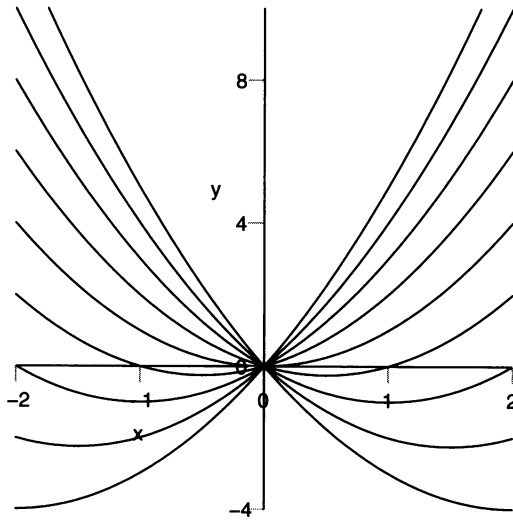


Figure 3. The family of curves $y = (x - c)^2 - c^2$.

as can be seen by applying the total derivative operator twice, substituting and simplifying along the way. The reader can discover some sophisticated tricks for solving differential equations by starting with a family of curves, repeatedly taking the total derivative, and then figuring out how to work backwards from the final differential equation.

At this point we need to look more carefully at Lie group symmetries so that we can assemble a systematic approach to finding these simplifying coordinate transformations. This we do in the next section.

2. SYMMETRIES AND ODEs. A *symmetry* is a mapping of one mathematical object into itself or into another mathematical object that preserves some property of the object. The easiest symmetries to see are the discrete symmetries of geometrical objects, such as the rotational symmetries of the objects in Figure 4. Note that the sphere in the middle is invariant under a *continuous* group of rotational symmetries, not just a discrete group.

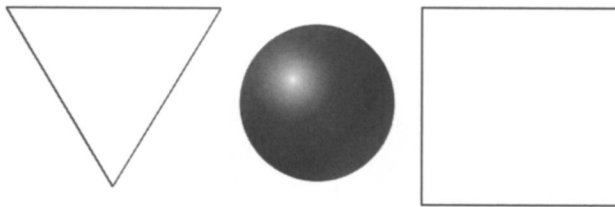


Figure 4. Geometric objects that have symmetries.

In the previous section, we gave three examples of differential equations whose solutions map into each other in some very simple way: the solutions just slid into each other in some coordinate direction. These solution curves are *isometric*, that is, they have exactly the same shape, so that they can be mapped into each other by rigid motions.

Suppose that the solutions of a differential equation can be mapped into each other nonisometrically, as in the examples of the family of circles or ellipses. Can we still solve the differential equation by an appropriate change of coordinates? For first-order differential equations the answer is a qualified yes. In effect, we find a change of coordinates that “straightens out” the direction in which we “slide” the curves. For example, if we write $dy/dx = -x/y$ in polar coordinates, we obtain the equation $dr/d\theta = 0$ whose solution is $r = C$. In polar coordinates, solution curves slide into one another along the r -direction.

Lie group symmetries. We have been speaking implicitly about groups of transformations of families of curves $\phi(x, y) = c$ in the plane. Let $\mathbf{x} = (x, y)$ and $\mathbf{X} = (X, Y)$ be points in the Euclidean plane, and for λ in \mathbb{R} , let $P_\lambda : \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \lambda) = \mathbf{X}$ be a transformation, depending on the parameter λ , that takes points \mathbf{x} to points \mathbf{X} . There will be a composition rule $\psi(\mu, \nu)$ defining the composition of two transformations with parameters μ and ν , but we can always reparameterize the group so that the composition is additive, that is, $\psi(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$. (With this parameterization, the identity transformation corresponds to $\lambda = 0$.) We say the set of transformations P_λ is a (additive) *transformation group* G if the following conditions are satisfied:

1. P_λ is one-to-one and onto;
2. $P_{\lambda_2} \circ P_{\lambda_1} = P_{\lambda_2 + \lambda_1}$, that is, $\mathbf{f}(\mathbf{f}(\mathbf{x}, \lambda_1), \lambda_2) = \mathbf{f}(\mathbf{x}, \lambda_2 + \lambda_1)$;
3. $P_0 = I$ (i.e., $\mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$);
4. For each λ_1 there exists a unique λ_2 (in fact, $\lambda_2 = -\lambda_1$) such that $P_{\lambda_2} \circ P_{\lambda_1} = P_0 = I$, that is, $\mathbf{f}(\mathbf{f}(\mathbf{x}, \lambda_1), \lambda_2) = \mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$.

If, in addition to these four group properties, \mathbf{f} is infinitely differentiable with respect to \mathbf{x} and analytic with respect to λ , we say G is a *one-parameter Lie group* (or a *Lie point transformation*). A point transformation maps points in the Euclidean plane into other points in the plane.² If $\lambda = 0$, then

$$P_\lambda : (x, y) \mapsto (X, Y) \Big|_{\lambda=0} = (f(x, y, 0), g(x, y, 0)) = (x, y).$$

For example, if $P_\lambda : (x, y) \mapsto (x + \lambda, y - \lambda)$ or $P_\lambda : (x, y) \mapsto (e^\lambda x, y)$, then when $\lambda = 0$ we have $(X, Y) = (x, y)$.

The Lie group symmetries we are considering are symmetries under a *local* group, that is, the group action may not be defined over the whole plane. For instance, the group action

$$P_\lambda : (x, y) \mapsto (X, Y) = \left(\frac{x}{1 - \lambda x}, \frac{y}{1 - \lambda x} \right)$$

is defined only if $\lambda < 1/x$ when $x > 0$ and $\lambda > 1/x$ when $x < 0$.

The ODE $dy/dx = 0$ has many (in fact, like all first-order differential equations, an infinite number of) symmetries, among them $P_\lambda : (x, y) \mapsto (x, e^\lambda y)$, $P_\lambda : (x, y) \mapsto (x + \lambda, e^\lambda y)$, $P_\lambda : (x, y) \mapsto (x, \lambda + y)$, $P_\lambda : (x, y) \mapsto (e^\lambda x, y)$, etc. It is easy to see why: the solutions to this ODE, shown in Figure 5, take the form $y = c$, which has as its family of solution curves the family of horizontal lines in the plane. Stretch them, shift them, stretch and shift them in either the x - or the y -direction or both, and you *still* have a family of horizontal lines in the plane. The symmetries $P_\lambda : (x, y) \mapsto (e^\lambda x, y)$

²There are more general transformations, such as contact transformations and Lie-Backlund transformations, but we ignore them in this introduction.

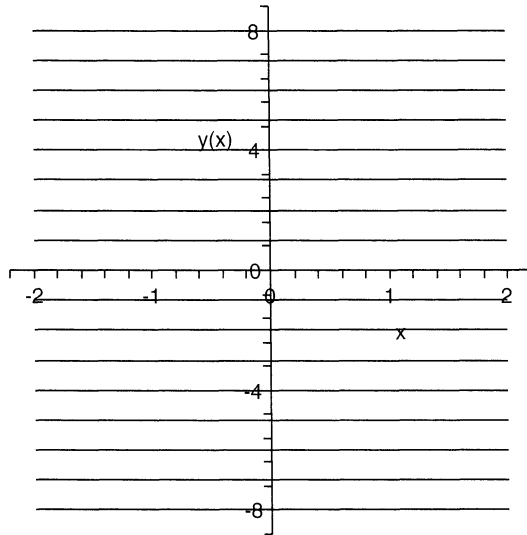


Figure 5. Solution curves to $dy/dx = 0$.

and $P_\lambda : (x, y) \rightarrow (x + \lambda, y)$ are called *trivial*, because for this particular equation they take each solution curve into itself. Trivial symmetries, as well as discrete symmetries, are not useful for our purposes.

Orbits of solutions. Consider a particular point (x_0, y_0) and the action of an additive Lie group

$$P_\lambda : (x_0, y_0) \mapsto (X_0, Y_0) = (f(x_0, y_0, \lambda), g(x_0, y_0, \lambda)).$$

As λ varies, the point (X_0, Y_0) moves about the plane tracing out a continuous curve, as in Figure 6. This curve is called the *orbit* of (x_0, y_0) under the group, or just an orbit of the group. If the Lie group is a (nontrivial) symmetry group of a differential equation $dy/dx = h(x, y)$, then an orbit of the group takes a continuous path transverse to

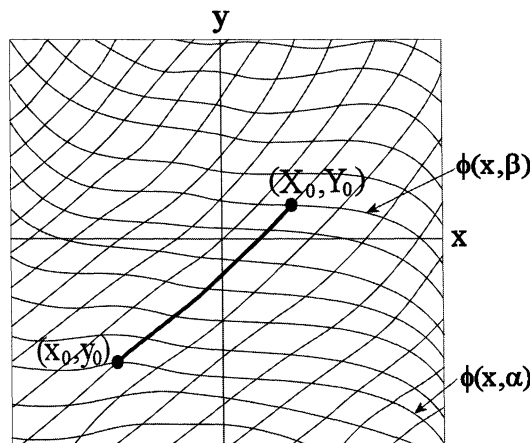


Figure 6. One portion of an orbit of a symmetry is shown darkened. The labels $\phi(\mathbf{x}, \alpha)$ and $\phi(\mathbf{x}, \beta)$ refer to solutions $\phi(\mathbf{x}) = c_1, \phi(\mathbf{x}) = c_2$ of the differential equation for two different parameter values $\lambda = \alpha$ and $\lambda = \beta$.

solution curves of the differential equation. An orbit of solution curves is a continuous family $\phi(x, y) = c(\lambda)$. Along this orbit, changes in λ map solution curves to other solution curves.

An orbit of a particular point (x_0, y_0) in a solution curve is a candidate for one of the coordinates in a coordinate system in which the differential equation becomes easy to integrate, because in this “direction” solution curves slide into one another. As a specific example, consider the set of solutions to the Bernoulli equation $y' = y(x - y)/x^2$, whose graphs are shown in Figure 7. By a method we will discuss a little later, we are able to find a coordinate system $(r, s) = (x, -x/y)$ in which solutions translate into each other in one of the coordinate directions. When the Bernoulli equation is written in these coordinates, it transforms, using $ds/dr = D_x s / D_x r$, to $ds/dr = 1/r$, the right-hand side of which is independent of s . Thus, solution curves can be translated in the s -direction into each other, and the equation is separable in these variables. Solving the new equation and substituting the original variables in the result gives the solution we seek, $y = x/(\ln x + C)$.

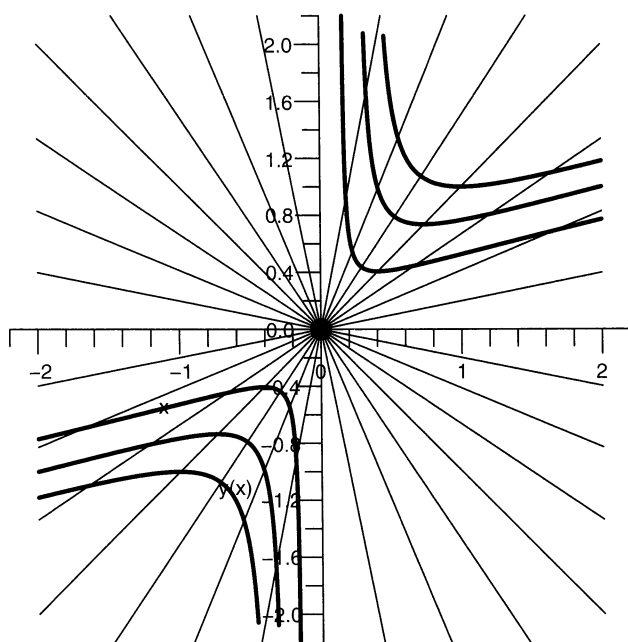


Figure 7. A family of solutions (heavy lines) to the Bernoulli equation, along with a canonical coordinate system $r = x, s = -x/y$.

One of the standard techniques for solving an equation like this is to notice that

$$y' = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2.$$

Then we make a change of variable $u = x/y$ or $v = y/x$, differentiate, substitute, and solve the differential equation. This technique, like most standard solution methods, is really a special case of solution by Lie group methods.

When we slide solution curves into each other along a canonical coordinate direction, we are using implicitly the relation between the original curves and the curves in

the new variables. Suppose we have a differential equation

$$\frac{dy}{dx} = h(x, y) \tag{3}$$

and know a symmetry $P_\lambda : (x, y) \mapsto (X, Y)$. Then, because the symmetry group takes solutions into other solutions, upon substitution of (X, Y) into (3) we should have

$$\frac{dY}{dX} = h(X, Y)$$

when $dy/dx = h(x, y)$, that is, the differential equation should have exactly the same form.

Because we know an explicit form $(X, Y) = (f(x, y, \lambda), g(x, y, \lambda))$ for the symmetry, we can easily calculate $h(X, Y)$. In order to write the transformed differential equation explicitly, we need to calculate dY/dX . Using the total derivative, we find that

$$\frac{dY}{dX} = \frac{D_x Y}{D_x X} = \frac{Y_x + y'Y_y}{X_x + y'X_y} = h(X, Y).$$

Now, because $y' = h(x, y)$, we can write

$$\frac{Y_x + h(x, y)Y_y}{X_x + h(x, y)X_y} = h(X, Y). \tag{4}$$

This nonlinear partial differential equation gives us the symmetry $(x, y) \mapsto (X, Y)$ we seek implicitly in terms of (x, y) . If we could solve (4) for X and Y , we could find our simplifying coordinate system, but in practice it is usually impossible to do so. However, if we linearize this equation, we get something that frequently *can* be solved, and from that solution we can find the simplifying coordinate system. In the next section, we see how this can be done.

Vector fields of group orbits. We now develop the machinery to find the vector field (linearization) of the group orbit. This vector field is everywhere tangent to the coordinate curves of our new coordinate system. Suppose that we knew a symmetry group $P_\lambda : (x, y) \mapsto (X, Y) = (f(x, y, \lambda), g(x, y, \lambda))$ for our differential equation. Then, along the orbits, we would have

$$\frac{dX}{d\lambda} = \frac{df}{d\lambda} = \xi(X, Y), \quad \frac{dY}{d\lambda} = \frac{dg}{d\lambda} = \eta(X, Y),$$

with

$$\left(\frac{dX}{d\lambda} \Big|_{\lambda=0}, \frac{dY}{d\lambda} \Big|_{\lambda=0} \right) = (\xi(x, y), \eta(x, y)).$$

The functions ξ and η , sometimes called the *symbols* of the infinitesimal transformation, are central to the process of finding new coordinates in which an equation gets simplified. They are the tangents to the coordinate curves we seek.

In order to compute the vector field of the group of orbits, we expand X , Y , and $h(X, Y)$ in Taylor series around $\lambda = 0$:

$$\begin{aligned} X &= x + \lambda \xi(x, y) + \mathcal{O}(\lambda^2), & Y &= y + \lambda \eta(x, y) + \mathcal{O}(\lambda^2), \\ h(X, Y) &= h(x, y) + \lambda(h_x(x, y)\xi(x, y) + h_y(x, y)\eta(x, y)) + \mathcal{O}(\lambda^2). \end{aligned} \tag{5}$$

By ignoring terms of order λ^2 and higher, we obtain linearizations of X , Y , and $h(X, Y)$. Because linearization gives the slope of a curve, it defines a vector field, and therefore the orbit curves can be retrieved by integration.

In order to simplify the symmetry condition (4), we linearize by substituting (5) into equation (4), ignoring terms of order λ^2 and higher:

$$\frac{h + \lambda(\eta_x + h\eta_y)}{1 + \lambda(\xi_x + h\xi_y)} = h + \lambda(h_x\xi + h_y\eta).$$

Multiplying and again disregarding terms of order λ^2 and higher, we find that

$$h + \lambda(\eta_x + (\eta_y - \xi_x)h - \xi_y h^2) = h + \lambda(\xi h_x + \eta h_y),$$

which simplifies to

$$\eta_x - \xi_y h^2 + (\eta_y - \xi_x)h - (\xi h_x + \eta h_y) = 0, \tag{6}$$

the *linearized* symmetry condition for first-order differential equations. This equation is, in general, impossible to solve unless we make an assumption about the form of the solution.

Example 1. Consider the differential equation $dy/dx = h = y^2/x$. Substitute $h = y^2/x$ into the linearized symmetry condition $\eta_x - \xi_y h^2 + (\eta_y - \xi_x)h - (\xi h_x + \eta h_y) = 0$ to obtain

$$\eta_x - \xi_y \frac{y^4}{x^2} + (\eta_y - \xi_x) \frac{y^2}{x} + \xi \frac{y^2}{x^2} - \eta \frac{2y}{x} = 0.$$

In this form it is still not easy to solve, so we make a simplifying assumption: namely, that ξ depends only on x , η only on y . If $(\xi, \eta) = (\xi(x), \eta(y))$, then

$$\eta_y - 2\frac{\eta}{y} = \xi_x - \frac{\xi}{x}.$$

This is a partial differential equation in which the variables separate. We find solutions by setting each side equal to a constant c (for the sake of simplicity we choose $c = 0$). The symbols are therefore $\xi = c_1 x$ and $\eta = c_2 y^2$. In the next section we will see how to find a simplifying coordinate system from the symbols.

Canonical coordinates. In canonical coordinates $(r(x, y), s(x, y))$, a given differential equation becomes separable: for the simplest cases we obtain either $ds/dr = f(r)$ or $ds/dr = f(s)$. For the sake of definiteness, we treat only the case $ds/dr = f(r)$. Then $(R, S) = (r, s + \lambda)$, that is, in the new coordinates there is a point symmetry $P_\lambda : (r, s) \mapsto (R, S) = (r, s + \lambda)$ that amounts to translation in the s -direction. Then the tangent vector at (r, s) will be

$$\left. \frac{dR}{d\lambda} \right|_{\lambda=0} = 0, \quad \left. \frac{dS}{d\lambda} \right|_{\lambda=0} = 1.$$

Taking derivatives with respect to λ at $\lambda = 0$ leads to

$$\begin{aligned} \left. \frac{dR}{d\lambda} \right|_{\lambda=0} &= \left. \frac{dR}{dx} \frac{dx}{d\lambda} \right|_{\lambda=0} + \left. \frac{dR}{dy} \frac{dy}{d\lambda} \right|_{\lambda=0} = \frac{dr}{dx} \xi + \frac{dr}{dy} \eta = 0, \\ \left. \frac{dS}{d\lambda} \right|_{\lambda=0} &= \left. \frac{dS}{dx} \frac{dx}{d\lambda} \right|_{\lambda=0} + \left. \frac{dS}{dy} \frac{dy}{d\lambda} \right|_{\lambda=0} = \frac{ds}{dx} \xi + \frac{ds}{dy} \eta = 1 \end{aligned}$$

or

$$r_x \xi + r_y \eta = 0, \quad s_x \xi + s_y \eta = 1. \quad (7)$$

Now equations (7) are first-order linear partial differential equations for $r = r(x, y)$ and $s = s(x, y)$, and solutions $r = c$ and $s = k$ are *first integrals* (i.e., functions $\phi(x, y)$ constant along solution curves, but generally with different constants along different solution curves). Geometrically, solutions to equations (7) may be represented by surfaces, and the projections of the level curves $r(x, y) = c, s(x, y) = k$ on the (x, y) -plane form the family of curves that will become our simplifying coordinate system.

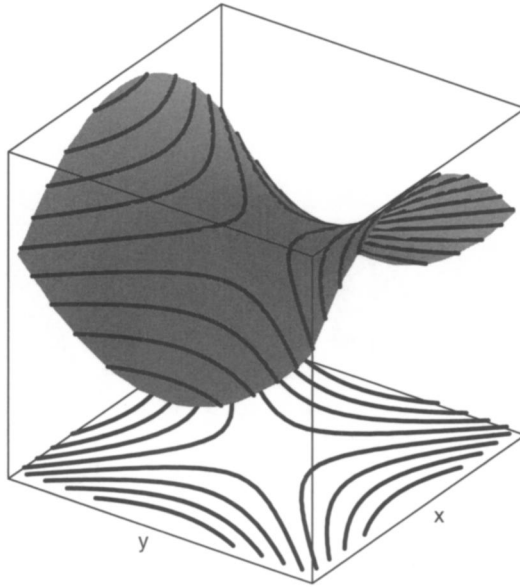


Figure 8. Solution surface for a first-order linear PDE along with the projection of some of its level curves.

If $\xi(x, y) = dX/d\lambda|_{\lambda=0} = 0$ or $\eta(x, y) = dY/d\lambda|_{\lambda=0} = 0$, then we can solve equations (7) directly to obtain $(r, s) = (x, \int dy/\eta)$ or $(r, s) = (y, \int ds/\xi)$, respectively. Otherwise, we solve (7) by the *method of characteristics*, whereby we can find the solution to a linear PDE by solving a system of ODEs.

Consider the linear partial differential equation in $u(x, y)$

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y), \quad (8)$$

whose solutions $u(x, y)$ each define a surface $S = S(x, y, u(x, y)) = S(x, y, z)$. We have from (8) that a, b, c , and u must satisfy $\langle a, b, c \rangle \cdot \langle u_x, u_y, -1 \rangle = 0$, and

because any solution surface S has a normal vector $\mathbf{n} = \langle u_x, u_y, -1 \rangle$, the vector $\langle a(x, y), b(x, y), c(x, y) \rangle$ must lie in the tangent plane to S for each (x, y) . Now consider a curve C parameterized by $t \mapsto (x(t), y(t), z(t))$ and lying in the surface S whose tangent vector is in the direction $\langle a(x(t), y(t)), b(x(t), y(t)), c(x(t), y(t)) \rangle$. Then we must have

$$\frac{dx}{dt} = Ca, \quad \frac{dy}{dt} = Cb, \quad \frac{dz}{dt} = Cc$$

for some positive constant C , or

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}.$$

These are called the *characteristic equations* of (8).

For the particular case of (7), characteristic equations are, respectively,

$$\frac{dx}{\xi} = \frac{dy}{\eta}, \quad \frac{dx}{\xi} = \frac{dy}{\eta} = ds,$$

and we may find r and s by solving these systems. The solution to the first takes the form

$$r = k = g(x, y),$$

and we can solve $dx/\xi = ds$ by integrating

$$s = \int ds = \int \frac{dx}{\xi}.$$

Example 2. As an example, suppose that some differential equation has the symmetry $\xi = 1, \eta = x$. In this case we must solve $dy/x = dx/1$ to get $y = x^2/2 + c$, so $r = c = y - x^2/2$ gives us the r -coordinate. Then we integrate $\int ds = \int dx/1$ to obtain $s = x$, so that $(r, s) = (y - x^2/2, x)$. If the symmetries were instead of the form $(\xi, \eta) = (0, x^2)$, then $r = x = c$ and we would find s by solving $dy/x^2 = ds$: $\int dy/x^2 = \int ds \rightarrow s = y/x^2$. Therefore, our simplifying coordinates are $(r, s) = (x, y/x^2)$.

The equation in canonical coordinates. The antepenultimate step is to transform the given equation to canonical coordinates. This is a simple process for a first-order ODE. The nonlinear symmetry condition is

$$\frac{ds}{dr} = \frac{s_x + h(x, y)s_y}{r_x + h(x, y)r_y}, \quad (9)$$

and we have every element we need: we know the canonical coordinates $r(x, y)$ and $s(x, y)$, and we have $h(x, y)$ from the original equation. After this step, all that remains is to solve the simplified ODE and translate the solution into the original coordinates. Of course, even for a separable ODE, we may not be able to find a solution in terms of simple functions. We consider the ODE solved when it has been reduced to quadrature.

3. SOLVING FIRST-ORDER ODES. We are now ready to solve a full-fledged example from start to finish. The steps are:

1. Find Lie symmetries of the unknown solutions of the differential equation, say $dy/dx = h(x, y)$, by appealing to the linearized symmetry condition (6)

$$\eta_x - \xi_y h^2 + (\eta_y - \xi_x)h - (\xi h_x + \eta h_y) = 0.$$

To do this, we must make a guess at the form of a symmetry (remember, there are an infinite number, so a few guesses usually suffice).

2. Use the solutions to the linearized symmetry condition (the symbols ξ and η) to find a coordinate system (r, s) in which the solutions depend on only one of the variables. To do this, we integrate the characteristic equations $dx/\xi = dy/\eta$ and $dx/\xi = dy/\eta = ds$ of the orbit.
3. Substitute the canonical coordinates into

$$\frac{ds}{dr} = \frac{s_x + h(x, y)s_y}{r_x + h(x, y)r_y}$$

and solve the differential equation in the canonical coordinate system.

4. Express the solution in the original coordinates.

Example 3. As a first easy example, and one that we know how to solve with the aid of an integrating factor, we solve the differential equation $y' = y/x + x$ for the family of curves described by $y = (x - c)^2 - c^2$ (c real) that we considered earlier (Figure 3). Our goal is to make this a separable equation in its canonical coordinates. The linearized symmetry condition (6) reduces for our particular equation to

$$\eta_x - \xi_y(y/x + x)^2 + (\eta_y - \xi_x)(y/x + x) - (\xi(1 - y/x^2) + \eta(1/x)) = 0.$$

As usual with first-order ODEs, the symmetry condition is too difficult to deal with as it stands. We look for symbols ξ and η of the type $\xi = 0$, $\eta = \eta(x)$. Substituting this into the symmetry condition, we obtain $\eta_x = \eta/x$, which we can easily solve to get $\eta = cx$. Why did we choose this form? There are many possibilities, so we picked a simple one, hoping that it would do the trick. If our choice had failed, we would try another. Usually, it pays to try the simplest polynomials of a single variable $\xi = P(x)$, $\eta = Q(y)$ or $\xi = P(y)$, $\eta = Q(x)$ first, then sums $\xi = P_1(x) + Q_1(y)$, $\eta = P_2(x) + Q_2(y)$, products $\xi = P_1(x)Q_1(y)$, $\eta = P_2(x)Q_2(y)$, quotients, etc.

Next we find the canonical coordinates. As before, because $\xi = 0$ we have $r = c = x$, and we integrate $ds = dy/\eta = dy/cx$ to get $s = y/cx$. In canonical coordinates the equation becomes

$$\frac{ds}{dr} = \frac{s_x + h(x, y)s_y}{r_x + h(x, y)r_y}$$

or

$$\frac{ds}{dr} = \frac{-y/x^2 + (y/x + x)1/x}{1 + 0} = 1.$$

Accordingly, $s = r + k$. Finally, substitution yields $y = x^2 + xk$, which we can readily check to be a solution of the given ODE.

Example 4. The Bernoulli equation $dy/dx = y + y^{-1}e^x$, when substituted into condition (6), leads to

$$\eta_x - \xi_y(y + y^{-1}e^x)^2 + (\eta_y - \xi_x)(y + y^{-1}e^x) - (\xi(y^{-1}e^x) + \eta(1 - e^x/y^2)) = 0.$$

This, again, is too difficult as it sits, so we try a few simplifying assumptions before we discover that $\xi = 1$, $\eta = \eta(y)$ yields

$$\eta_y(y + y^{-1}e^x) - (y^{-1}e^x) - \eta(1 - y^{-2}e^x) = 0.$$

Because some terms depend only on y , we solve $y\eta_y - \eta = 0$ to obtain $\eta = cy$. Inserting this form of η into the remaining equation $\eta_y + y^{-1}\eta - 1 = 0$, we arrive at $\eta = y/2$.

Now that we have settled on the symbols $(\xi, \eta) = (1, y/2)$, we find canonical coordinates by solving $dy/dx = \eta/\xi = y/2$ to get r and s . Remember that we seek families of functions that remain constant for r , so $r = c = ye^{-x/2}$. The second coordinate s is found by integrating $ds = dx/1$ to get $s = x$.

The next step is to find the differential equation in the canonical coordinates by computing

$$\frac{ds}{dr} = \frac{s_x + s_y h}{r_x + r_y h}.$$

We learn that

$$\frac{ds}{dr} = \frac{1}{-\frac{1}{2}ye^{-x/2} + e^{-x/2}(y + y^{-1}e^x)} = \frac{1}{\frac{1}{2}ye^{-x/2} + y^{-1}e^{x/2}}.$$

Expressing $\frac{1}{2}ye^{-x/2} + y^{-1}e^{x/2}$ in terms of r and s leads to $r/2 + 1/r$, whence

$$\frac{ds}{dr} = \frac{r}{r^2/2 + 1}.$$

This integrates to

$$s = \ln(r^2/2 + 1) + c.$$

Returning to the original coordinates, we obtain

$$y = \pm\sqrt{ce^{2x} - 2e^x}.$$

Example 5. Consider the ODE $y' = y/(x - y)$. We can solve this by exchanging x and y and computing

$$\frac{dx}{dy} = \frac{D_x x}{D_x y} = \frac{x - y}{y}.$$

The ODE is now easily solvable using an integrating factor, but we want to go back to the original equation and solve it using Lie group methods from the start. The symmetry condition (6) takes the form

$$(y^2 - 2yx + x^2)\eta_x + (yx - y^2)\eta_y + (y^2 - yx)\xi_x - y^2\xi_y + y\xi - x\eta = 0,$$

but we can see that it simplifies to

$$(y^2 - 2yx + x^2)\eta_x + (xy - y^2)\eta_y - x\eta - y^2\xi_y + y\xi = 0$$

if we choose ξ to be a function of y alone. Because of the dependence of ξ upon y , $y\xi - y^2\xi_y = 0$ separately, and we can solve this to obtain $\xi = y$. Now, the remainder in η is difficult to solve but is identically satisfied by $\eta = 0$. Therefore, we try $(\xi, \eta) = (y, 0)$. Integrating the characteristic equations, we get $(r, s) = (y, x/y)$. Computing

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'},$$

we see that $ds/dr = -1/r$, which upon substitution of the original variables becomes $x/y = -\ln y + c$. We can easily verify by implicit differentiation that this satisfies the original ODE.

Example 6. Consider the ODE $y' = 1 + (1 - y^2)/(xy)$ (taken from [3, p. 32]). This nonlinear ODE does not appear to be solvable by any of the standard methods. The key, as usual, is to find the point symmetries of this equation. Our assumption for the tangent vector field is that

$$\xi = \alpha(x), \quad \eta = \beta(x)y + \gamma(x).$$

Taking derivatives and substituting into condition (6), we get

$$\beta'y + \gamma' + (\beta - \alpha') \left(1 + \frac{1 - y^2}{xy}\right) = \alpha \left(\frac{y^2 - 1}{x^2 y}\right) - (\beta y + \gamma) \left(\frac{1 + y^2}{xy^2}\right).$$

We can compare coefficients of powers of y , which reveals the following information:

1. y^{-2} : $\gamma = 0$;
2. y^{-1} : $(\beta - \alpha')/x = -\alpha/x^2 - \beta/x$;
3. y^0 : $\beta = \alpha'$.

From the second and third statements we infer that $\alpha' + \alpha/x = 0$, with solution $\alpha = cx^{-1}$, so $\beta = -cx^{-2}$. The tangent vector field of the Lie symmetry group thus has the form

$$(\xi, \eta) = (cx^{-1}, -cx^{-2}).$$

Since $\xi \neq 0$, we can solve $dy/dx = \eta/\xi = -y/x$ to get $r = xy$ and $s = \int dx/\xi = c \int x dx$. We conclude that

$$r = xy, \quad s = \frac{1}{2}x^2.$$

Because of the symmetry condition, we know that

$$\frac{ds}{dr} = \frac{s_x + hs_y}{r_x + hr_y} = \frac{x + 0}{y + x \left(\frac{1 - y^2 + xy}{xy}\right)}.$$

Simplifying and substituting r and s for x and y , we arrive at

$$\frac{ds}{dr} = \frac{r}{1 + r},$$

which has the solution

$$s = r - \ln(1 - r)$$

or

$$\frac{1}{2}x^2 = xy - \ln(1 + xy)$$

in the original coordinates.

4. SUMMARY. In this paper we have attempted to give a simple, self-contained introduction to the use of Lie group methods for the solution of first-order ODEs. The method applied to such equations is particularly nice in that a geometric interpretation can be given. The Lie group method of solving higher order ODEs, PDEs, and systems of differential equations is more involved, but the basic idea is the same: we find a coordinate system in which the equations are simpler and exploit this simplification.

Although the best known applications of Sophus Lie's theory of continuous groups are in differential geometry, relativity, classical and quantum mechanics, continuum mechanics, and control theory, there is now a renewed interest in his original application to solutions of differential equations, and a number of fine texts have appeared. Some of these are listed in the bibliography. The introduction of at least some of these ideas into an elementary course on differential equations seems reasonable, practical, and desirable.

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