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# SIMILARITY METHODS
# FOR
# DIFFERENTIAL EQUATIONS

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by

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## Preface

These notes are a slightly expanded and revised version of the notes handed out in the short course number CE 1708 given by David Logan, June 17 to June 24, 1980 at Lawrence Livermore National Laboratory. Videotapes of the lecture are available. Make arrangements for viewing them by contacting the LLNL multimedia center.

The author wishes to thank Professor W. D. Curtis of Kansas State University for his discussions and critical remarks on these topics and Dr. David Margolies of Lawrence Livermore National Laboratory for his direction of this project. Finally, thanks go to Dr. Carl Rosenkilde for providing the impetus to these lectures.

## Introduction

We begin with a few remarks on the scope of these lecture notes. The purpose is to give an elementary introduction to similarity methods for partial differential equations for those who have had little or no experience with these techniques. The emphasis will be on motivation and practical calculations involving several simple examples. From time-to-time we will allude to the differential-geometric structure which underlies these concepts; for, what we are really talking about is the invariance of a partial differential equation under the action of a local, Lie group of transformations. However, a deep understanding of these general concepts is not a prerequisite for being able to apply similarity methods to a given system.

The first lecture is mostly an introductory lecture on the nature of self-similar solutions. The second lecture discusses dimensional analysis and the Buckingham Pi Theorem and how dimensional arguments lead in a natural way to similarity solutions. The third and fourth lectures develop the concept of invariance under a group of transformations and, given the group, show how solutions can be constructed. In the fifth and sixth lectures we show how the invariance group can be calculated. The final lecture deals with a detailed physical example taken from the area of detonation physics.

LECTURE 1

SIMILARITY METHODS

## 1. The Problem

The problem that we address in this set of lecture notes is one of finding so-called self-similar (or similarity) solutions to one or more partial differential equations with boundary or initial conditions given. To fix the idea, let us consider a single partial differential equation (PDE)

$$H(x,t,u,u_t,u_x,\ldots) = 0.$$

where x and t are the independent variables (we think of them as representing space and time) and $u = u(x,t)$ is the unknown function. Subscripts denote partial derivatives, i.e.,

$$u_t = \frac{\partial u}{\partial t},\ u_x = \frac{\partial u}{\partial x},\ u_{tx} = \frac{\partial^2 u}{\partial x \partial t},\ \text{and so on.}$$

In general, x and t range over some domain D in space and time where the solution is desired. In addition, there may be auxiliary conditions which must be satisfied of the form

$$B(x,t,u,u_x,u_t,\ldots) = 0,$$

holding on some set of points $w(t,x) = 0$ consisting of the boundary of D. (See Figure 1). By a solution we mean a function $u = u(t,x)$ which satisfies the PDE and auxiliary conditions identically. An auxiliary condition given at time $t = 0$ is called an initial condition; conditions given on the other boundaries of D are called boundary conditions. The problem of determining a function u(x,t), (x,t) ε D, which satisfies the differential equation and initial conditions is called an initial-value problem. If the auxiliary conditions are boundary conditions, then it is a boundary value problem. If both initial and boundary conditions are specified, then we have an initial-boundary-value problem, or a mixed problem.

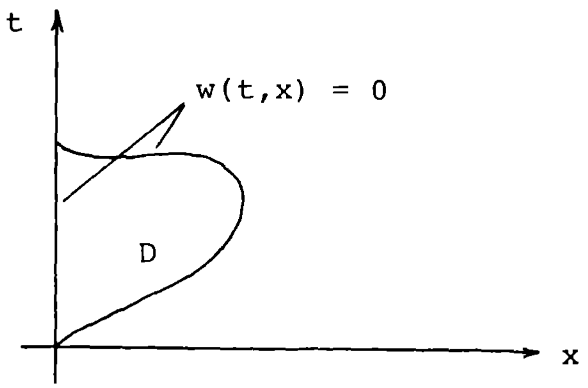



Figure 1. Schematic of the spacetime region D and the boundary w(t,x) = 0.

There are several techniques for solving problems of these types. We list some of the most familiar:

1. Separation of variables or the eigenfunction expansion method.
2. Transform Methods (Laplace transforms, Fourier transforms, ...)
3. Numerical or computer Methods.
4. Perturbation and Asymptotic Methods.
5. Similarity Methods.

The point of these lectures is to discuss the last of these methods, the similarity methods. These methods, in theory, are exact in that they yield analytic solutions. In practice, however, real problems often require approximate methods after the similarity method is used to simplify the problem.

## 2. The Nature of Similary Solutions

Similarity solutions are special kinds of solutions which possess a type of invariance under certain transformations (e.g. translations, rotations, or stretchings). This means that the self-similar motion of a system is one in which the dependent variables or parameters which characterize the system vary in such a way that, as time evolves, the spatial variation of these parameters remains geometrically similar. The scale which characterizes the spatial variation may change with time according to specified rules. The accompanying Figure 2 illustrates typical snapshots of the spatial distribution of a solution $u(x,t_0)$, $u(x,t_1)$ and $u(x,t_2)$. All of these graphs are similar and are related by a transformation called a similarity transformation.

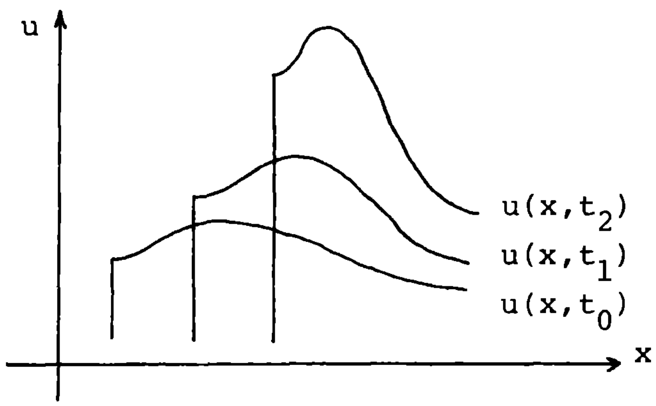



Figure 2. Time snapshots of a similarity solution.

## 3. Physical Meaning

Nature exhibits certain symmetries and the mathematical equations which model nature often exhibit these same symmetries. The symmetries in a system give rise to transformations which leave the system invariant. For example, a system with rotational symmetry may be expected to be invariant under rotations. It is the invariance of the equations under these symmetry transformations that cause similarity solutions to exist.

Although the differential equations which govern the motion of a system may possess these invariant properties, arbitrary initial or boundary conditions may not. This is why self-similar solutions do not exist for some problems with general auxiliary conditions; too much has been specified and, as a result, the symmetry has been lost. One is often satisfied, therefore, with a self-similar solution to a problem in which the initial conditions are ignored. Hence, similarity solutions are solutions which hold after the system has evolved for such a time that the initial conditions no longer affect the motion. In this way, similarity solutions are not unlike intermediate asymptotic solutions or long-time solutions to differential equations.

## 4. The Value of Similarity Solutions

Similarity methods have been developed in great detail in many branches of physics and engineering. Their development in hydrodynamics has been particularly intense and rewarding. In some of the most important problems in gas dynamics the study of the self-similar solutions has enabled researchers to reach important conclusions concerning more general types of motion and establish the governing laws in many cases of practical interest. These methods have been used in shock wave and detonation physics, in problems involving explosions, blast waves and the escape of explosion products, just to mention a few. Of equal importance is the sobering fact that the similarity method is the only analytic tool available for some problems. Even though we may not be able to solve the real problem that we wish, a similarity solution of a simpler problem with some conditions relaxed not only can indicate facts relevant to the original problem but can also be useful in itself in providing, for example, an analytic check for computer solutions or other approximate methods.

## 5. The Determination of Similarity Solutions

The analytic conditions that a self-similar solution exist for a given system lead to the definition of a new independent variable, called the similarity variable, which is a function of the independent variables t and x (refer to the notation in Paragraph 1). With the introduction of this new variable a significant simplification occurs in that the partial differential equation reduces to an ordinary differential equation. The solution of the ordinary differential equation then provides the self-similar solution or self-similar motions of the system. In general, for equations with several independent variables the number of independent variables is reduced (usually by one) in the case of self-similar motion. This is the fact that makes similarity methods useful to those seeking analytic solutions to partial differential equations. To illustrate how this simplification takes place, let us consider two examples.

Example 1. (Burger's Equation) $u_t + uu_x = \nu u_{xx}$.

Let us attempt to find for this nonlinear equation a traveling wave solution of the form $u(x,t) = f(x-ct)$.

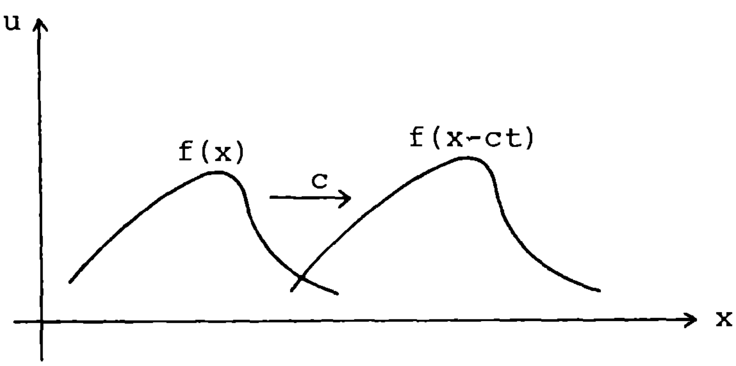



Figure 2. Traveling wave profile snapshots at $t = 0$ and $t = t_0$.

Then

$$u_t = -cf'(x-ct),\ u_x = f'(x-ct),\ u_{xx} = f''(x-ct).$$

Substituting into the PDE gives

$$-cf'(s) + f(s)f'(s) = \nu f''(s),\ s = x-ct.$$

This represents a second order ordinary differential equation (ODE). Thus, introducing a new independent variable s into the problem leads to an ODE in this new variable for the wave profile f(s).

Example 2. Let us consider two equations in the two unknowns $u = u(x,y)$, $v = v(x,y)$, namely

$$u_y - v_x = 0,\ v_y + uu_x = 0.$$

Let $s = x/y$ and assume that $u = f(s)$ and $v = g(s)$.

Then $u_x = f'(s)(1/y)$, $u_y = f'(s)(-x/y^2)$

$v_x = g'(s)(1/y)$, $v_y = g'(s)(-x/y^2)$

Substituting into the system of PDEs gives a system of ODEs for $f(s)$ and $g(s)$, namely

$$-\frac{x}{y^2} f' - \frac{1}{y} g' = 0, \quad -\frac{x}{y^2} g' + \frac{1}{y} ff' = 0$$

or

$$s^2 f' + sg' = 0 \quad -s^2 g' + sfg = 0.$$

One can solve this system as follows. Dividing the second equation by s and adding yields

$$s^2 f'(s) + f(s)f'(s) = 0,$$

or

$$f'(s)\ (s^2 + f(s)) = 0.$$

Hence $f(s)$ = constant (which is not an interesting solution), or $f(s) = -s^2$. In this latter case, $g(s) = 2s^3/3$, which is found from either one of the preceeding equations involving f and g. Therefore, we have a solution

$$u(x,y) = -x^2/y^2,\ v(x,y) = 2x^3/3y^3$$

to the original problem. Again, introduction of the new independent variable s, called a similarity variable, allowed a substantial simplification of our original problem, and we were able to obtain a solution

LECTURE 2

SIMILARITY SOLUTIONS VIA

DIMENSIONAL ANALYSIS

## 1. Dimensional Analysis

The classical method of determining similarity solutions was developed in the 1940's by several investigators using arguments based on dimensional analysis; much of the work was done in the areas of gas dynamics and fluid mechanics.

Dimensional analysis can be described briefly as follows. Physical laws should be independent of the units used to express the variables. Dimensional analysis is the study of the restrictions and consequences that arise as a result of this independence to both the equations and their solutions. It leads to self-similar solutions because this "invariance under units" assumption.

The Buckingham Pi Theorem, which is the cornerstone of the theory, gives an explicit statement of some of these restrictions. Suppose there is a relationship between the variables $a_1, a_2, \ldots a_m$ of the form

$$f(a_1, a_2, \ldots a_m) = 0 \tag{1}$$

This equation may, for example, represent a physical law. Now, let $\ell_1, \ldots, \ell_n$ be fundamental dimensions ($n < m$), e.g. length time, mass, and so on, and suppose

$$[a_i] = \ell_1^{\alpha_{1i}} \ell_2^{\alpha_{2i}} \ldots \ell_n^{\alpha_{ni}},$$

where $[a_i]$ denotes the dimensions of $a_i$. Then Buckingham's Pi Theorem asserts that there are m-r dimensionless quantities $\pi_1,\ldots,\pi_{m-r}$ which can be formed using $a_1,\ldots,a_m$ and, moreover, the physical law (1) is equivalent to an equation of the form

$$F(\pi_1,\pi_2,\ldots,\pi_{m-r}) = 0,$$

only in terms of the dimensionless variables. Here r is the rank of the matrix $(\alpha_{ij})$, which is called the dimension matrix.

The following example illustrates how this theorem can be used to discover interesting relations among the variables in a heat conduction problem.

Example

At time t = 0 an amount of enery e, concentrated at a point in space, is allowed to diffuse outward. If r denotes the radial distance from the source and t is time, the problem is to determine the temperature u = u(r,t) as a function of r and t.

We might conjecture that a relation of the form

$$g(t,r,u,e,k,c) = 0$$

exists, where k is the diffussivity and c is the heat capacity. Letting the fundamental units be

T(time) , $\ell$ (length) , $\tau$ (temperature) , E (energy), we have:

$$[t] = T \qquad [e] = E$$

$$[r] = \ell \qquad [k] = \ell^2 T^{-1}$$

$$[u] = \tau \qquad [c] = E\tau^{-1}\ell^{-3}.$$

Recall that k = thermal conductivity/(density × specific heat) and c = density × specific heat. Here, m = 6 (the number of variables), and n = 4 (number of independent fundamental units.

So, we form the <u>dimension</u> <u>matrix</u>

| | t | r | u | e | k | c |
|---|---|---|---|---|---|---|
| T | 1 | 0 | 0 | 0 | -1 | 0 |
| $\ell$ | 0 | 1 | 0 | 0 | 2 | -3 |
| $\tau$ | 0 | 0 | 1 | 0 | 0 | -1 |
| E | 0 | 0 | 0 | 1 | 0 | 1 |

Clearly the rank of this matrix is r = 4. Hence there are 6-4, or two dimensionless quantities which can be formed among t, r, u, e, k, and c. To find them we proceed as follows:

If $\pi$ is dimensionless, then

$$\begin{aligned} 1 = [\pi] &= [t^{\alpha_1} r^{\alpha_2} u^{\alpha_3} e^{\alpha_4} k^{\alpha_5} c^{\alpha_6}] \\ &= T^{\alpha_1} \ell^{\alpha_2} \tau^{\alpha_3} E^{\alpha_4} (\ell^2 T^{-1})^{\alpha_5} (E\tau^{-1}\ell^{-3})^{\alpha_6} \\ &= T^{\alpha_1 - \alpha_5} \ell^{\alpha_2 + 2\alpha_5 - 3\alpha_6} \tau^{\alpha_3 - \alpha_6} E^{\alpha_4 + \alpha_6} \end{aligned}$$

Hence

$$\alpha_1 - \alpha_5 = 0$$

$$\alpha_2 + 2\alpha_5 - 3\alpha_6 = 0$$

$$\alpha_3 - \alpha_6 = 0$$

$$\alpha_4 + \alpha_6 = 0 .$$

There are two linearly independent solutions since the rank of this homogeneous system in six independent variables is four. One solution is

$$\alpha_1 = -1/2, \ \alpha_2 = 1, \ \alpha_3 = \alpha_4 = 0, \ \alpha_5 = -1/2, \ \alpha_6 = 0.$$

This gives a dimensionless quantity

$$\pi_1 = \frac{r}{\sqrt{kt}} .$$

Another solution is

$$\alpha_1 = 3/2, \ \alpha_2 = 0, \ \alpha_3 = 1, \ \alpha_4 = -1, \ \alpha_5 = 3/2, \ \alpha_6 = 1,$$

which gives

$$\pi_2 = \frac{uc}{e} (kt)^{3/2}.$$

So, by the Pi theorem, the physical law $g(t,r,u,e,k,c) = 0$ is equivalent to

$$F(\pi_1,\pi_2) = 0,$$

or

$$\pi_2 = f(\pi_1),$$

or $$u = \frac{e}{c}(kt)^{-3/2} f\left(\frac{r}{\sqrt{kt}}\right). \tag{2}$$

This gives a variable s, the similarity variable, which we define as

$$s = r/\sqrt{kt}.$$

To determine the unknown function f we substitute eqn. (2) into the spherically symmetric heat equation

$$u_t - \frac{k}{r^2} \frac{\partial}{\partial r} (r^2 u_r) = 0,$$

or

$$u_t - ku_{rr} - \frac{2k}{r} u_r = 0. \tag{3}$$

Now, the similarity law (2) gives

$$u_t = -\frac{1}{2} \frac{e}{c} k^{-2} t^{-3} rf'(s) - \frac{3}{2} \frac{e}{c} k^{-3/2} t^{-5/2} f(s),$$

$$u_r = \frac{e}{c} k^{-2} t^{-2} f'(s),$$

and

$$u_{rr} = \frac{e}{c} k^{-5/2} t^{-5/2} f''(s).$$

Substitution into Equation (3) yields an ordinary differential equation for f(s), namely

$$f''(s) + \frac{2}{s} f'(s) + \frac{s}{2} f'(s) + \frac{3}{2} f(s) = 0.$$

A solution of this equation is

$$f(s) = A \exp(-s^2/4).$$

Hence, a self-similar solution is given by

$$u(r,t) = \frac{Ae}{c}(kt)^{-3/2}e^{-r^2/4kt}$$

## 2. A Boundary Value Problem

Consider now the following problem which consists of the spherically symmetric heat equation subject to initial and boundary conditions:

$$u_t - \frac{k}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0,\ t > 0,\ r > r^* > 0$$

$$u(r,0) = 0,\ r > r^*$$

$$u(\infty,t) = 0,\ t > 0$$

$$4\pi c \int_0^{r^*} r^2 u(r,0)dr = e.$$

The last condition expresses the fact that at t = 0 the energy e is concentrated inside a spherical ball of radius r*, which is small. Our solution above clearly satisfies

$$\lim_{r\to\infty} u(r,t) = 0,\ t > 0$$

$$\lim_{t\to 0} u(r,t) = 0,\ r > r^*.$$

The constant A can be evaluated as follows. Since the amount of heat energy is always the same,

$$4\pi c \int_0^{\infty} r^2 u(r,t)dr = e,$$

or

$$4\pi c \int_0^{\infty} r^2 \frac{Ae}{c}(kt)^{-3/2}e^{-r^2/4kt}\, dr = e.$$

Letting $s = r/\sqrt{kt}$, we obtain

$$4\pi A \int_0^\infty s^2 e^{-s^2/4}\, ds = 1.$$

The value of the integral is easily seen to be $2\sqrt{\pi}$, and so $A = 1/8\pi^{3/2}$ and

$$u(r,t) = \frac{e/c}{(4\pi kt)^{3/2}} \exp(-r^2/4kt).$$

Hence, we have succeeded in solving an initial-boundary-value problem using dimensional analysis. The Buckingham Pi Theorem allowed us to determine the proper form of the similarity variable s and the self-similar form of the solution.

## 3. Interpretation of the Solution

In Fig. 1 we have sketched the similarity curves $s = r/\sqrt{kt}$ = constant, which represent a family of parabolas in rt-space. Figure 2 shows snapshots of the temperature profiles at times $t_o$ and $t_1$. We observe, consistent with our earlier remark, that as time evolves the spatial variation of the dependent variable is geometrically similar. Figure 3 shows the basic form of the self-similar solutions, $f(s) = \exp(-s^2/4)$. We notice

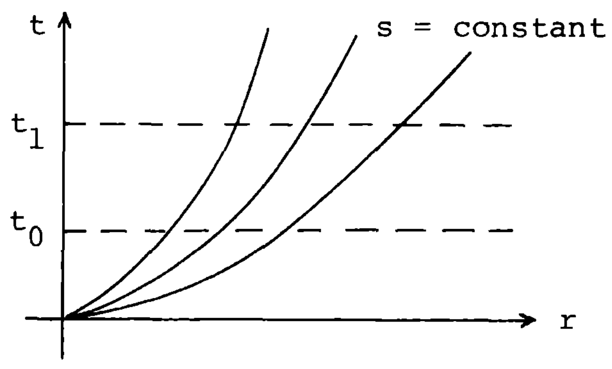



Figure 1. Similarity curves s = constants.

that the solution u(r,t) is of the form

$$u(r,t) = (\text{time dependent scaling})\cdot f(s).$$

Hence, the time snapshots are formed by taking f(s) and stretching it according to the scaling factor.

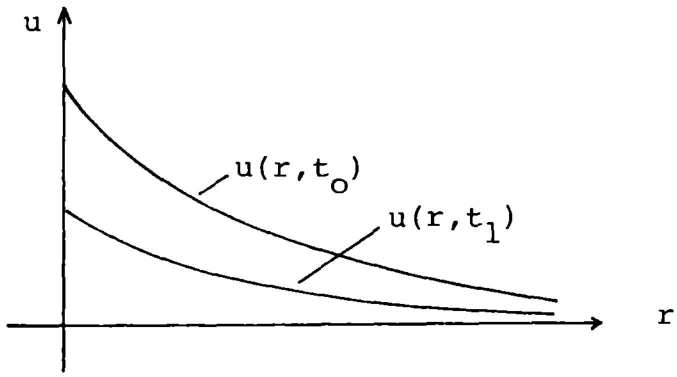



Figure 2. Snapshots at times $t_0$ and $t_1$ with $t_0 < t_1$

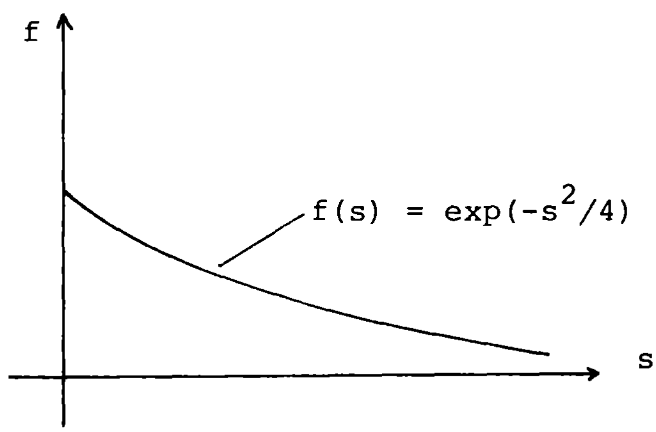
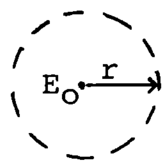



Figure 3. Basic similarity profile.

## 4. Exercises

A. Consider a point explosion where at t = 0 there is an amount of energy E released into surrounding air of pressure $P_0$, density $\rho_0$, and ratio of specific heats $\gamma$. $r = r(t)$ is the radius of the wave front at time t. Using time, length, and force as fundamental units, derive, using dimensional arguments, the equation



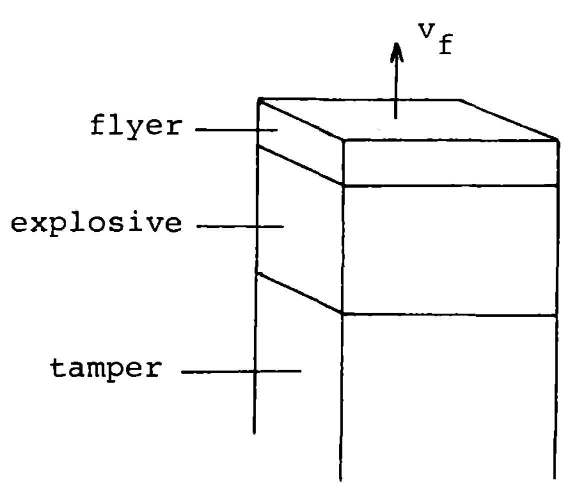

$$\rho_0 r^5/Et^2 = G\left(\frac{P_0 r^3}{E}, \gamma\right).$$

Assuming that $P_0 \simeq 0$, obtain the classic formula

$$r = \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} f(\gamma)$$

for the radius of the blast wave, where f is some function of $\gamma$.

B. A flyer of mass per unit area $M_f$ sits upon an explosive of mass per unit area $M_e$ which is backed by a tamper of essentially infinite mass. When the explosive detonates, the flyer is driven off at velocity $v_f$. If $E_g$ is the Gurney energy (joules/kg) of the explosive, that is, the energy available in the explosive to do work on the flyer, show that dimensional analysis methods give the relation



$$v_f = \sqrt{E_g}\, f\left(\frac{M_f}{M_e}\right),$$

if the fundamental units are time, length, and mass. (Using conservation of energy and momentum it can be shown that the true relationship is

$$v_f = \sqrt{2E_g}\left(\frac{M_f}{M_e} + \frac{1}{3}\right)^{-1/2}.$$

## 5. Bibliographic Remarks

An extremely readable account of dimensional analysis and additional exercises can be found in C. C. Lin and L. A. Segel (see references). G. Birkhoff's book Hydrodynamics provides an in-depth study of dimensional methods and a proof of the Pi Theorem. In a recent work, Curtis, Logan, and Parker have given a purely linear algebra version of the theorem; additional references can be found there.

LECTURE 3

GROUP THEORETIC METHODS

## 1. Introduction

The central idea behind group theoretic methods for finding self-similar solutions is this:

1. Determine a local, Lie group of transformations under which the problem (PDEs, BCs, ICs) is invariant.
2. From the group determine a new coordinate system (canonical coordinates) which simplifies the problem in that one fewer independent variable is required.
3. Solve the problem in this new coordinate system.

We begin with an example which illustrates the type of transformations we consider as well as what we mean by invariance. We use the so-called method of stretchings, where we assume particular transformations.

Example (Heat equation) $u_t - u_{xx} = 0$

We attempt to find a transformation (stretching group)

$$\bar{t} = \varepsilon^{\alpha} t, \ \bar{x} = \varepsilon^{\beta} x, \ \bar{u} = u$$

under which the equation is invariant. Here, $\varepsilon$ is a parameter and $\alpha$ and $\beta$ are numbers which are to be calculated. Let us compute the transformed equation:

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial u}{\partial \bar{t}} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{t}} = \varepsilon^{-\alpha} u_t$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial \bar{x}} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \bar{x}} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{x}} = \varepsilon^{-\beta} u_x$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial}{\partial \bar{x}} (\varepsilon^{-\beta} u_x) = \varepsilon^{-\beta} \frac{\partial u_x}{\partial \bar{x}} = \varepsilon^{-\beta} \left[ \frac{\partial u_x}{\partial t} \frac{\partial t}{\partial \bar{x}} + \frac{\partial u_x}{\partial x} \frac{\partial x}{\partial \bar{x}} \right] = \varepsilon^{-2\beta} u_{xx}$$

Hence, in the transformed coordinates,

$$\bar{u}_{\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = \varepsilon^{-\alpha} u_t - \varepsilon^{-2\beta} u_{xx} .$$

If we pick $\beta = 1$ and $\alpha = 2$, then

$$\bar{u}_{\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = \varepsilon^{-2}(u_t - u_{xx}).$$

Consequently, under the transformation

$$\bar{t} = \varepsilon^2 t, \ \bar{x} = \varepsilon x, \ \bar{u} = u,$$

whenever $u = u(x,t)$ is a solution to $u_t - u_{xx} = 0$, then $\bar{u}(\bar{x},\bar{t})$ is a solution to $\bar{u}_{\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = 0$. We say in this case that the PDE is invariant under the given transformation.

## 2. The General Concept

Consider a PDE

$$H(t,x,u,u_x,u_t) = 0.$$

The type of transformations we consider is a family of transformations from txu - space to $\bar{t}\bar{x}\bar{u}$ - space indexed by a small parameter $\varepsilon$. We assume that the set of these transformations form a group under composition. That is, we assume closure (locally, for small $\varepsilon$), associativity, identity, and inverse. Suppose the transformations are given by the formulas:

$$\bar{t} = \phi(t,x,u,\varepsilon)$$
$$\bar{x} = \psi(t,x,u,\varepsilon) \tag{4}$$
$$\bar{u} = \Omega(t,x,u,\varepsilon)$$

where $\phi$ $\psi$, and $\Omega$ are sufficiently differentiable for our needs. We assume for $\varepsilon = \varepsilon_0 = 0$ we get the identity transformation $\bar{t} = t$, $\bar{x} = x$, $\bar{u} = u$. Such a one-parameter group of transformations is called a local, Lie group of transformations.

<u>Example</u>

$$\bar{t} = t\cos\varepsilon - x\sin\varepsilon$$
$$\bar{x} = t\ \sin\varepsilon + x\cos\varepsilon$$
$$\bar{u} = u + \varepsilon.$$

Geometrically this is a rotation of angle $\varepsilon$ in tx-space and a translation by $\varepsilon$ in u space. We can view this in two ways; both are helpful. First, we can regard it as a coordinate transformation, or a change of coordinates txu to $\bar{t}\bar{x}\bar{u}$. Second, we can view it as a point transformation which moves points $(t,x,u)$ to $(\bar{t},\bar{x},\bar{u})$ in the coordinate system txu. (See figures 1 and 2)

Associated with the local, Lie group (4) is the so-called <u>infinitesimal transformation</u>, which is the principle linear part of the transformation. If we denote $T = \frac{\partial}{\partial\varepsilon}\phi(t,x,u,0)$, $X = \frac{\partial}{\partial\varepsilon}\psi(t,x,u,0)$, and $U = \frac{\partial}{\partial\varepsilon}\Omega(t,x,u,0)$, and expand the right hand sides of (4) in a Taylor series about $\varepsilon = 0$, we obtain

$$\bar{t} = t + \varepsilon\ T\ (t,x,u) + 0(\varepsilon^2)$$
$$\bar{x} = x + \varepsilon\ X\ (t,x,u) + 0(\varepsilon^2)$$
$$\bar{u} = u + \varepsilon\ U\ (t,x,u) + 0(\varepsilon^2)$$

The $O(\varepsilon^2)$ terms are generally dropped and the result is what is called the infinitesimal transformation.

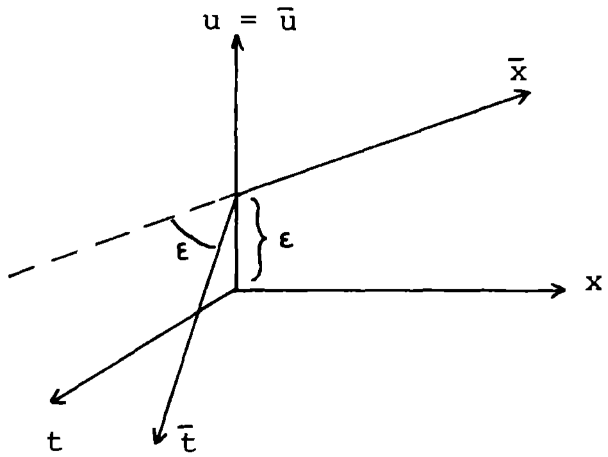



Figure 1. Coordinate transformation

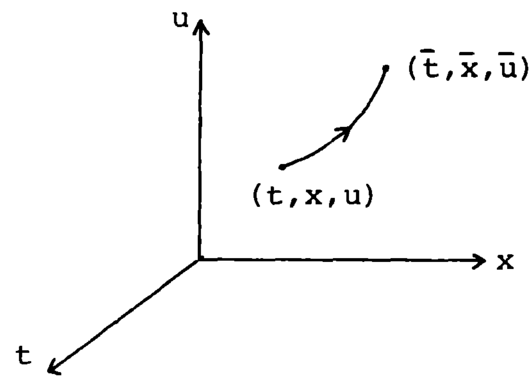



Figure 2. Point transformation

Also, there is associated a natural vector field (T,X,U) and operator $= T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u}$ with the group. T, X, and U are called the generators of the group. The integral curves of the vector field (T,X,U) are the orbits of the group. As it turns out, they define special coordinates which make the group a simple translation. It is these coordinates which give the similarity variable and similarity solution.

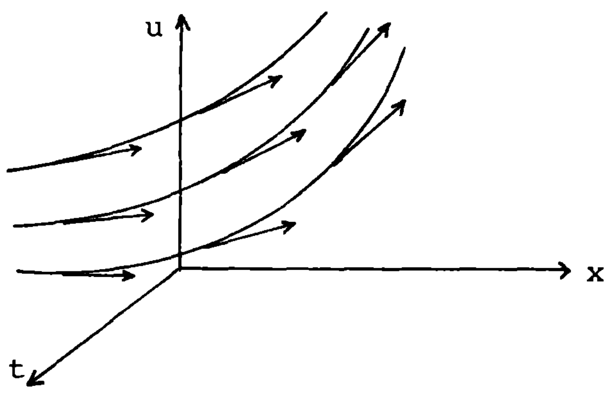



Figure 3. The orbits or integral curves of the vector field (T,X,U).

Note that the orbits are found from the equations

$$\frac{dt}{d\varepsilon} = T, \ \frac{dx}{d\varepsilon} = X, \ \frac{du}{d\varepsilon} = U.$$

Example (Heat equation) $u_t - u_{xx} = 0$.

Earlier, we calculated the group of transformations $\bar{t} = \varepsilon^2 t$, $\bar{x} = \varepsilon x$, $\bar{u} = u$. Here, the infinitesimal generators are (note $\varepsilon = 1$ gives the identity)

$$T = 2t, \ X = x, \ U = 0.$$

Hence the orbits are found from the equations

$$\frac{dt}{d\varepsilon} = 2t, \ \frac{dx}{d\varepsilon} = x, \ \frac{du}{d\varepsilon} = 0,$$

or

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{0} = d\varepsilon.$$

The first pair of equations gives

$$\frac{x}{\sqrt{t}} \equiv s = \text{constant}.$$

Also, $u = \text{constant} \equiv w$ Also, from $\frac{dx}{x} = d\varepsilon$ we get $\ln x = \varepsilon + c$. Pick $r = \ln x$. So, take a new coordinate system to be (canonical variables)

$$s = \frac{x}{\sqrt{t}} \qquad r = \ln x \qquad w = u.$$

In this new coordinate system the transformation is a translation. This is easily checked as follows:

$$\bar{s} = \frac{\bar{x}}{\sqrt{\bar{t}}} = \frac{\varepsilon x}{\sqrt{\varepsilon^2 t}} = \frac{x}{\sqrt{t}} = s$$

$$\bar{r} = \ln \bar{x} = \ln \varepsilon x = \ln \varepsilon + \ln x = \ln \varepsilon + r$$

$$\bar{w} = \bar{u} = u.$$

Writing out the heat equation in the canonical variables r, s, and w, we obtain

$$-\frac{s}{2}\frac{\partial w}{\partial s} - \frac{1}{s^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{s^2}\frac{\partial w}{\partial r} - \frac{2}{s}\frac{\partial^2 w}{\partial r \partial s} - \frac{\partial^2 w}{\partial s^2} = 0.$$

Note that there are no r's in the coefficients. Therefore we notice that there is a solution of the form

$$w = f(s).$$

If we substitute this into the last equation, we obtain an ordinary differential equation for f(s), namely,

$$f'' + \frac{s}{2} f' = 0$$

This can be solved to obtain

$$f(s) = a \int_0^s e^{-z^2} dz + D$$

$$= b \operatorname{erf}(s) + D$$

Therefore, $u(x,t) = b \operatorname{erf}(\frac{x}{\sqrt{t}}) + D,$

and we have obtained a solution to our problem. The constants b and D can be evaluated from initial and boundary data.

## 3. Summary and Algorithm

The following schematic summarizes an algorithm for determining similarity solutions via canonical coordinates given the fact that the problem is invariant under a known group with generators T, X, and U:

Given a PDE

$$H(t,x,u,u_t,u_x,\ldots) = 0$$

invariant under

a group with generators

T, X, U

Solve the characteristic system

$$\frac{dt}{T}\frac{dx}{X}\frac{du}{U} = d\varepsilon$$

to find the canonical coordinates r, s, and w as first integrals of this system.

Rewrite the PDE in the new coordinate system as

$$\tilde{H}(r,s,w,w_r,w_s,\ldots) = 0,$$

and solve.

In the canonical system the partial differential equation $\tilde{H}(r,s,w,w_r,w_s,\ldots) = 0$ will not have explicit dependent on one of the independent variables r or s. As a result, a solution which is just a function of one variable can be expected.

The reason for this simplification is indicated in the following schematic. In the new coordinate system the group is a translation in one of the variables, and so the PDE in that system cannot depend explicitly on that variable; hence we have a reduction of the number of independent variables.

| PDE | | GROUP |
|---|---|---|
| $H(t,x,u,u_t,u_x,\ldots) = 0$ | invariant under | $\bar{t} = \phi(t,x,u,\varepsilon)$ <br> $\bar{x} = \psi(t,x,u,\varepsilon)$ <br> $\bar{u} = \Omega(t,x,u,\varepsilon)$ |
| change of variables to r,s,w ⇓ | | ⇓ change of variables to r,s,w |
| $\tilde{H}(r,s,w,w_r,w_s,\ldots) = 0$ | invariant under | $\bar{r} = r + \varepsilon$ <br> $\bar{s} = s$ <br> $\bar{w} = w$ |

4. Exercise

A. The equation $u_y u_{xy} - u_x u_{yy} - u_{yyy} = 0$ occurs in boundary layer theory for flow along a flat plate.

a. Verify that the equation is invariant under the group

$$\bar{x} = \varepsilon^2 x,\ \bar{y} = \varepsilon y,\ \bar{u} = \varepsilon u.$$

b. Determine the generators of the group.

c. Show that the canonical coordinates are given by

$$r = y/\sqrt{x},\ s = \frac{1}{2}\ln x$$

and

$$w = u/\sqrt{x}$$

d. Show that in canonical coordinates the equation becomes

$$w_{rrr} + \frac{1}{2} w_{rr} w - \frac{1}{2} w_r w_{rs} + \frac{1}{2} w_{rr} w_s = 0$$

and then obtain an ordinary differential equation for a solution of the form

$$w = f(r).$$

LECTURE 4

THE INVARIANT SURFACE CONDITION

## 1. Introduction

Let us assume that the boundary-value problem

$$(S)\qquad \begin{aligned} &H(t,x,u,u_t,u_x) = 0 \\ &B(t,x,u) = 0 \text{ on } w(t,x) = 0 \end{aligned}$$

is invariant under a 1-parameter local Lie group of transformations

$$(\Gamma_\varepsilon)\qquad \begin{aligned} \bar{t} &= \phi(t,x,u,\varepsilon) \\ \bar{x} &= \psi(t,x,u,\varepsilon) \\ \bar{u} &= \Omega(t,x,u,\varepsilon) \end{aligned}$$

In the last lecture we determined a new set of canoncial or preferred coordinates in which the group became a simple translation and the PDE could be simplified so as to depend on one less independent variable. Now, let us take an alternate viewpoint and derive a condition on the solution $u = g(t,x)$ of (S) which must hold when (S) is invariant under the group $(\Gamma_\varepsilon)$.

## 2. Transformation of Solutions

Let $u = g(t,x)$ be a solution to (S). We ask how the surface representing this solution changes under the transformation $\Gamma_\varepsilon$. The set of points $(t,x,u)$ on the solution surface gets mapped for each $\varepsilon$ to a set of points $(\bar{t},\bar{x},\bar{u})$ in $\bar{t}\bar{x}\bar{u}$-space via (also see figure 1)

$$\bar{t} = \phi(t,x,g(t,x),\varepsilon)$$

$$\bar{x} = \psi(t,x,g(t,x),\varepsilon)$$

$$\bar{u} = \Omega(t,x,g(t,x),\varepsilon)$$

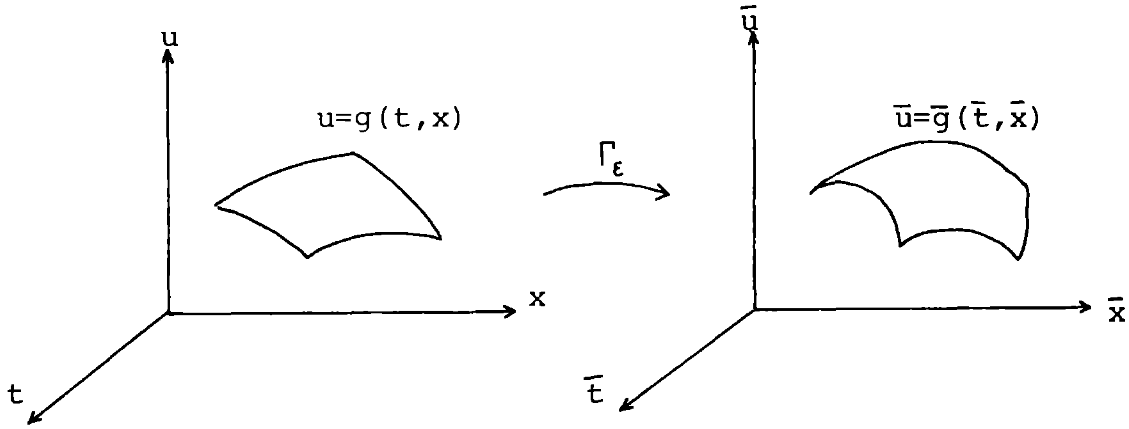



Figure 1

Solving the first pair of equations for t and x in terms of $\bar{t}$ and $\bar{x}$ (it can be shown that these two equations can be inverted provided $\varepsilon$ is sufficiently small), we get

$$t = t(\bar{t},\bar{x}), \quad x = x(\bar{t},\bar{x})$$

Substituting into the third equation gives

$$\bar{u} = \Omega(t(\bar{t},\bar{x}),\ x(\bar{t},\bar{x}),\ g(t(\bar{t},\bar{x}),\ x(\bar{t},\bar{x})),\varepsilon) \equiv \bar{g}(\bar{t},\bar{x})$$

This is the equation that defines the transformed surface $\bar{u} = g(\bar{t},\bar{x})$.

## 3. The Invariant Surface Condition

Now consider the same system (S) in barred or transformed coordinates:

$$H(\bar t,\bar x,\bar u,\bar u_{\bar t},\bar u_{\bar x}) = 0$$

$(\bar S)$

$$B(\bar t,\bar x,\bar u) = 0 \text{ on } w(\bar t,\bar x) = 0$$

Note that H, B, and w have the <u>same</u> functional form. Obviously, a solution of $(\bar S)$ is $\bar u = g(\bar t,\bar x)$, since we have only renamed the variables. For <u>invariance</u>, we require that $\bar u = \bar g(\bar t,\bar x)$ satisfy $(\bar S)$. That is, we say (S) is invariant under $\Gamma_\varepsilon$ if $\bar u = \bar g(\bar t,\bar x)$ is a solution to $(\bar S)$ whenever $u = g(t,x)$ is a solution to (S).

Since both $g(\bar t,\bar x)$ and $\bar g(\bar t,\bar x)$ satisfy $(\bar S)$, which we can assume has unique solutions, we must have $g(\bar t,\bar x) = \bar g(\bar t,\bar x)$ or

$$g(\bar t,\bar x) = \Omega(t,x,g(t,x),\varepsilon)$$

where $t = t(\bar t,\bar x)$ and $x = x(\bar t,\bar x)$ on the right-hand side. Differentiating this equation with respect to $\varepsilon$ at $\varepsilon = 0$ gives, using the chain rule,

$$\frac{\partial g}{\partial t}\left(\frac{\partial \bar t}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial g}{\partial x}\left(\frac{\partial \bar x}{\partial \varepsilon}\right)_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\,\Omega(t,x,g(t,x),\varepsilon)\Big|_{\varepsilon=0}$$

or

$$T\,\frac{\partial g}{\partial t} + X\,\frac{\partial g}{\partial x} = U$$

where T, X, and U are the generators of the group $\Gamma_\varepsilon$. Now, there is no need to distinguish g and u so we write

$$T\,\frac{\partial u}{\partial t} + X\,\frac{\partial u}{\partial x} = U. \tag{1}$$

This is the invariant surface condition. It is a condition on a solution surface u(x,t) in order that the problem be invariant under a local, Lie group $\Gamma_\varepsilon$. We recognize it as a first order, quasi-linear, partial differential equation. The method of characteristics allows us to find the general solution. We form the characteristic system

$$\frac{dt}{T(t,x,u)} = \frac{dx}{X(t,x,u)} = \frac{du}{U(t,x,u)} .$$

If $w_1(t,x,u)$ = constant and $w_2(t,x,u)$ = constant are two independent first integrals, then the general solution of (1) is

$$F(w_1,w_2) = 0$$

where F is an arbitrary function.

We now observe that the characteristic system for the condition that a surface be invariant is the same as the equations used to determine the canonical coordinates.

An example will now illustrate how the invariant surface condition leads to self-similar solutions.

Example $u_t + uu_x = 0$.

This PDE is invariant under the local, Lie group with generators

$$T = (\beta-\alpha)\, t + \delta$$

$$X = (2\beta-\alpha)\, x + \gamma t + \eta$$

$$U = \beta u + \gamma , \qquad \alpha,\beta,\gamma,\delta,\eta \text{ constants.}$$

We will calculate this group later. For simplicity we take

$$\alpha = \gamma = 0.$$

Then

$$T = \beta t + \delta$$

$$X = 2\beta x + \eta$$

$$U = \beta u.$$

The invariant surface condition is

$$(\beta t+\delta) \frac{\partial u}{\partial t} + (2\beta x+\eta) \frac{\partial u}{\partial x} = \beta u,$$

and the characteristic system is

$$\frac{dt}{\beta t+\delta} = \frac{dx}{2\beta x+\eta} = \frac{du}{\beta u}.$$

Solving the first two we obtain the similarity variable s:

$$\frac{\left(t+\frac{\delta}{\beta}\right)^2}{x+\frac{\eta}{2\beta}} = \text{constant} \equiv s.$$

Now, from

$$\frac{dt}{\beta t + \delta} = \frac{du}{\beta u},$$

we get

$$\frac{u}{t + \delta/\beta} = \text{constant}$$

Hence

$$u = (t + \delta/\beta)\, f(s).$$

To determine f we substitute into the partial differential equation:

$$u_t = \frac{2(t + \delta/\beta)^2}{x + \eta/2\beta} \quad f'(s) + f(s),$$

$$u_x = - \frac{(t + \delta/\beta)^3}{(x + \eta/2\beta)^2} \quad f'(s).$$

Hence

$$u_t + uu_x = 0 \text{ becomes } 2f'(s)s + f(s) - s^2 f'(s) f(s) = 0$$

or

$$\frac{df}{ds} = \frac{-f}{2s - s^2 f}$$

This can be solved by introducing $\phi = sf$. We obtain

$$\frac{2 - \phi}{\phi(1-\phi)} \, d\phi = \frac{ds}{s} .$$

Integrating gives

$$\frac{\phi^2}{\phi - 1} = sc, \qquad c \text{ a constant.}$$

Hence

$$\frac{s^2 f^2}{sf - 1} = sc,$$

or

$$f(s) = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{c}{s}} .$$

Therefore

$$u(x,t) = (t + \delta/\beta)\left[\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{c(x + \eta/2\beta)}{(t + \delta/\beta)^2}}\;\right],$$

and we have obtained similarity solutions to our problem.

## 4. Exercises

1. Consider the wave equation $u_{xx} - u_{tt} = 0$.

   a. Determine a 1-parameter local, Lie group of transformations (stretchings) of the form

   $$\bar{t} = \varepsilon^a t, \quad \bar{x} = \varepsilon^b x, \quad \bar{u} = u$$

   under which the PDE is invariant.

   b. Show that the canonical coordinates are

   $$r = x/t, \quad s = \ln t, \quad w = u.$$

   c. In canonical coordinates show that the wave equation becomes

   $$(1-r^2)w_{rr} - w_{ss} + 2rw_{rs} + w_s - 2rw_r = 0.$$

   d. Reduce this equation to an ODE in $w(r,s) = f(r)$ and find a similarity solution.

2. Show that if the invariance group for a PDE is given by

   $$\bar{t} = \phi(t,x,\varepsilon)$$

   $$\bar{x} = \psi(t,x,\varepsilon)$$

   $$\bar{u} = g(t,x)u\varepsilon + u,$$

then the general solution for the invariant surface condition equation is

$$u(t,x) = F(x,t)f(s)$$

where f is an arbitrary functions of s, F is a known function of x and t, and s is a known function of t and x.

3. Consider the nonlinear diffusion equation

$$u_t = \frac{\partial}{\partial x}\left[K(u)\frac{\partial u}{\partial x}\right], \quad 0 \le x < \infty, \; t > 0$$

with boundary conditions

$$u(0,t) = u_0, \quad u(\infty,t) = u_1$$

and initial condition

$$u(x,0) = u_2$$

where $u_0$, $u_1$, and $u_2$ are constants. Using the Boltzman transformation

$$s = x^{\alpha}t^{\beta} ,$$

show that for proper choice of $\alpha$ and $\beta$ the PDE is reduced to an ODE. Assuming $u_1 = u_2$, find the boundary conditions for the ODE.

4. Consider the PDE

$$(S) \qquad u_{tt} + u_{xx} = 0$$

with solution $u = g(t,x) = e^{t}\cos x$. Write down the system $(\bar{S})$ and show that $\bar{u} = g(\bar{t},\bar{x})$ satisfies the equation. Compute $\bar{u} = \bar{g}(\bar{t},\bar{x})$ where

$$\bar{t} = \varepsilon t, \quad \bar{x} = \varepsilon x, \quad \bar{u} = u$$

and show that it satisfies $(\bar{S})$.

LECTURES 5 AND 6

CALCULATION OF THE GROUP

## 1. Invariance Conditions

Let us now ask again what invariance means. We are given a problem

$$(S)\qquad \begin{aligned} &H(t,x,u,p,q) = 0, \quad p = u_t, \quad q = u_x \\ &B(t,x,u) = 0 \text{ on } w(t,x) = 0. \end{aligned}$$

We seek a 1-parameter, local, Lie group

$$(\Gamma_\varepsilon)\qquad \begin{aligned} \bar{t} &= \phi(t,x,u,\varepsilon) \\ \bar{x} &= \psi(t,x,u,\varepsilon) \\ \bar{u} &= \Omega(t,x,u,\varepsilon) \end{aligned}$$

under which (S) is invariant. We recall that our definition stated that (S) is <u>invariant</u> under $(\Gamma_\varepsilon)$ is $u = u(x,t)$ is a solution to (S) implies $\bar{u} = \bar{u}(\bar{t},\bar{x})$ is a solution to $(\bar{S})$, where $(\bar{S})$ is the system

$$(\bar{S})\qquad \begin{aligned} &H(\bar{t},\bar{x},\bar{u},\bar{p},\bar{q}) = 0, \quad \bar{p} = \bar{u}_{\bar{t}}, \quad \bar{q} = \bar{u}_{\bar{x}} \\ &B(\bar{t},\bar{x},\bar{u}) = 0 \quad \text{on} \quad w(\bar{t},\bar{x}) = 0 \end{aligned}$$

We desire to make a more general definition of invariance in terms of the partial differential equation and boundary condition directly which will imply our previous definition given above. Clearly, the above definition is satisfied if, for example,

$$H(\bar{t},\bar{x},\bar{u},\bar{p},\bar{q}) = H(t,x,u,p,q)$$

$$B(\bar{t},\bar{x},\bar{u}) = B(t,x,u)$$

and

$$w(\bar{t},\bar{x}) = w(t,x)$$

for all $\varepsilon$ sufficiently small. In this case we say (S) is <u>absolutely invariant</u> under $(\Gamma_\varepsilon)$. Taking the derivative of the first equation above with respect to $\varepsilon$ at $\varepsilon = 0$ gives

$$\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial x} X + \frac{\partial H}{\partial u} U + \frac{\partial H}{\partial p} P + \frac{\partial H}{\partial q} Q = 0$$

where

$$P = \left(\frac{\partial \bar{p}}{\partial \varepsilon}\right)_{\varepsilon=0}, \quad Q = \left(\frac{\partial \bar{q}}{\partial \varepsilon}\right)_{\varepsilon=0}$$

Similar equations hold for the auxiliary conditions. To compute P and Q we must know how the derivatives p and q transform. The group $\Gamma_\varepsilon : (t,x,u) \to (\bar{t},\bar{x},\bar{u})$ automatically induces a transformation on the derivatives; that is, the group $\Gamma_\varepsilon$ can be extended to a larger group $\tilde{\Gamma}_\varepsilon : (t,x,u,p,q) \to (\bar{t},\bar{x},\bar{u},\bar{p},\bar{q})$, called the extended group. The formula for the generators P and Q are

$$P = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial u} p - \frac{\partial T}{\partial t} p - \frac{\partial X}{\partial t} q - \frac{\partial T}{\partial u} p^2 - \frac{\partial X}{\partial u} pq$$

$$Q = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial u} q - \frac{\partial T}{\partial x} p - \frac{\partial X}{\partial x} q - \frac{\partial T}{\partial u} pq - \frac{\partial X}{\partial u} q^2$$

So, our condition for invariance is, for absolute invariance,

$$\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial x} X + \frac{\partial H}{\partial u} U + \frac{\partial H}{\partial p} P + \frac{\partial H}{\partial q} Q = 0,$$

with similar equations holding for the auxiliary condition.

But this is still not quite what we want. If we recall our example of the heat equation $u_t - u_{xx} = 0$ from Lecture 3, we found that under the transformation $\bar{t} = \varepsilon^2 t$, $\bar{x} = \varepsilon x$, $\bar{u} = u$, the invariance condition was

$$\bar{u}_{\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = \varepsilon^2 (u_t - u_{xx}).$$

In this case there was some "conformal" or "stretching" factor $\varepsilon^2$ connecting the barred and unbarred equations. So, we say in general that (S) is <u>constant conformally invariant</u> under $(\Gamma_\varepsilon)$ if there is a constant $\alpha$ such that

$$\frac{\partial}{\partial \varepsilon} H(\bar{t},\bar{x},\bar{u},\bar{p},\bar{q})\Big|_{\varepsilon=0} = \alpha H(t,x,u,p,q) \quad \text{for all } t,x,u,p,q,$$

and similarly for B and w. The left-hand side is the Lie derivative of H in the direction of the vector field (T,X,U,P,Q); written out, we get

$$\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial x} X + \frac{\partial H}{\partial u} U + \frac{\partial H}{\partial p} P + \frac{\partial H}{\partial q} Q = \alpha H(t,x,u,p,q) \qquad (*)$$

Further generalizations can be made be permitting $\alpha$ to be a function $\alpha = \alpha(t,x,u)$.

The algorithm for finding the generators is to substitute P and Q into (*), set the coefficients of $p, q, p^2$, $q^2$, and pq to zero, thereby obtaining a system of linear PDEs for the generators T, X, U. These equations are called the <u>determining equations for the group</u>. In practice we can usually solve them.

## 2. Example

We now perform a detailed calculation of the group for the PDE

$$u_t + uu_x = 0$$

We write the equation as

$$H(t,x,u,p,q) \equiv p + uq = 0, \quad p = u_t, \quad q = u_x$$

Then the invariance condition (*) becomes

$$H_t T + H_x X + H_u U + H_p [U_t + U_u p - T_t p - X_t q - T_u p^2 - X_u pq]$$
$$+ H_q [U_x + U_u q - T_x p - X_x q - T_u pq - X_u q^2] = \alpha(p + uq)$$

But

$$H_t = H_x = 0, \quad H_u = q, \quad H_p = 1, \quad H_q = u.$$

Substituting these quantities we obtain

$$qU + U_t + U_u p - T_t p - X_t q - T_u p^2 - X_u pq$$
$$+ u[U_x + U_u q - T_x p - X_x q - T_u pq - X_u q^2] = \alpha(p + uq)$$

Now set the coefficients of 1, p, q, $p^2$, $q^2$, pq to zero to get the determining equations of the group:

$$U_t + uU_x = 0$$

$$U_u - T_t - uT_x = \alpha$$

$$U - X_t + uU_u - uX_x = \alpha u$$

$$- T_u = 0$$

$$- uX_u = 0$$

$$- X_u - uT_u = 0.$$

We can conclude that

$$T = T(t,x), \quad X = X(t,x),$$

and

$$U_t + uU_x = 0 \tag{1}$$

$$T_t + uT_x = U_u - \alpha \tag{2}$$

$$X_t + uX_x = U + uU_u - \alpha u \tag{3}$$

It is these equations we must solve to obtain the generators T, X, and U.

From (2),

$$U_u = \alpha + T_t + uT_x.$$

Therefore,

$$U_u = \alpha(t,x)\ u + \beta(t,x),$$

and

$$U = \alpha(t,x)\ \frac{u^2}{2} + \beta(t,x)u + \gamma(t,x),$$

where

$$\alpha(t,x) = T_x, \quad T_t = \beta(t,x) - \alpha \tag{4}$$

From (3),

$$X_t + uX_x = \alpha(t,x)\frac{u^2}{2} + \beta(t,x)u + \gamma(t,x)$$

$$+ u[\alpha(t,x)u + \beta(t,x)] - \alpha u$$

$$= \frac{3}{2}\alpha(t,x)u^2 + (2\beta(t,x) - \alpha)u + \gamma(t,x).$$

Equating coefficients of u we get

$$X_t = \gamma(t,x)$$

$$\alpha(t,x) = 0 \tag{5}$$

$$X_x = 2\beta(t,x) - \alpha.$$

So, we have $T = T(t)$ and $\beta = \beta(t)$ [from (4)].

From (5),

$$X_{tx} = \frac{\partial\gamma}{\partial x} = X_{xt} = 2\beta'(t)$$

or

$$2\beta'(t) - \frac{\partial\gamma}{\partial x} = 0 \tag{6}$$

Now we know

$$U = \beta(t)u + \gamma(t,x)$$

Plug into (1):

$$\beta'(t)u + \frac{\partial\gamma}{\partial t} + u\frac{\partial\gamma}{\partial x} = 0$$

Hence

(7) $$\frac{\partial\gamma}{\partial t} = 0 \quad \text{and} \quad \beta'(t) + \frac{\partial\gamma}{\partial x} = 0.$$

But, adding (6) and (7) gives

$$\beta'(t) = 0$$

$$\beta = \text{constant}, \quad \gamma = \text{constant}.$$

Thus,

$$T_t = \beta - \alpha$$

$$T = (\beta - \alpha)t + \delta.$$

Also,

$$X_x = 2\beta - \alpha, \quad X_t = \gamma$$

$$X = (2\beta - \alpha)x + \gamma t + \eta.$$

Hence,

$$U = \beta u + \gamma$$

Therefore, the generators are

$$T = (\beta-\alpha)t + \delta$$

$$X = (2\beta-\alpha)x + \gamma t + \eta$$

$$U = \beta u + \gamma.$$

## 3. The General Theory

In this section we shall indicate the general procedure.

Suppose we are given a system of N partial differential equations

$$H_n(x,u,p) = 0, \quad n=1,\ldots N \tag{1}$$

where $x = (x^1, x^2)$, $u = (u^1, \ldots, u^m)$, and $P = (p_j^{\ i}) = (\partial u^i / \partial x^j)$, with boundary conditions

$$B_\ell(x,u) = 0, \quad \ell=1,\ldots L, \tag{2}$$

holding on the curve(s) $w(x) = 0$. We consider a one-parameter local, Lie group acting on xu-space given in infinitesimal form by

$$\Gamma_\varepsilon : \quad \begin{aligned} \bar{x}^i &= x^i + \varepsilon X^i(u) \\ \bar{u}^j &= u^j + \varepsilon U^j(x,u). \end{aligned} \tag{3}$$

We explicitly assume that the generators $X^i$ do not depend on u. Although this is not the most general case, it is sufficient for most applications. The transformation $\Gamma_\varepsilon$ induces a transformation on the derivatives p given by

$$\bar{p}_j^{\ i} = p_j^{\ i} + \varepsilon P_j^{\ i}(x,u,p), \tag{4}$$

where the generators $P_j^{\ i}$ are given by

$$P_j^{\ i} = \frac{\partial U^i}{\partial x^j} + \frac{\partial U^i}{\partial u^k} p_j^{\ k} - \frac{\partial X^k}{\partial x^j} p_k^{\ i}. \tag{5}$$

If the generators $X^i$ depend on u, then there is an additional term

$$- \frac{\partial X^k}{\partial u^h} p_k^{\ i} p_j^{\ h}$$

in the last equation for the generator $P_j^{\ i}$.

We say that partial differential equation $H(x,u,p) = 0$ is <u>constant conformally invariant</u> under (3) and (4) if there is a constant $\alpha$ such that

$$\frac{\partial}{\partial \varepsilon} H(\bar{x},\bar{u},\bar{p})\Big|_{\varepsilon=0} = \alpha H(x,u,p) \tag{6}$$

for all x, u, and p. Given the group (3), we say $u^j = g^j(x)$ satisfies the invariant surface condition if $\bar{g}^j = g^j$, that is, if the group transforms g into itself. It is not difficult to prove that if $H(x,u,p) = 0$ is constant conformally invariant under (3) and (4), and if $u^j = g^j(x)$ is a solution to $H(x,u,p) = 0$, then $\bar{u}^j = \bar{g}^j(\bar{x})$ is also a solution. By differentiating $\bar{u}^j = \bar{g}^j(\bar{x})$ with respect to $\varepsilon$ at $\varepsilon = 0$ we get

$$\frac{\partial g^j}{\partial x^i}(x)\; X^i(x) = U^j(x,g(x)), \tag{7}$$

which is the analytic form of the invariant surface condition.

To determine the group (3) and (4) under which $H(x,u,p) = 0$ is constant conformally invariant we write down condition (6) explicitly and solve to determine the generators $X^i$ and $U^j$.

Once the group is known, the similarity variable s is determined as a first integral of the vector field $X^i$, i.e., it is a constant of

$$\frac{dx^1}{X^1} = \frac{dx^2}{X^2}$$

Exercise

A. (Computationally involved) Consider the system of equations

$$u_t - v_x = 0$$

$$v_t - uu_x = 0$$

Calculate the group

$$\bar{t} = t + \varepsilon T, \quad \bar{x} = x + \varepsilon X, \quad \bar{u} = u + \varepsilon U, \quad \bar{v} = v + \varepsilon V$$

under which the system is absolutely invariant.

[Hint: The determining equations are

$$U_t - V_x = 0$$

$$uU_v + X_t + V_u + uT_x = 0$$

$$U_u - T_t - V_v + X_u = 0$$

$$X_v - T_u = 0$$

$$V_t + uU_x = 0$$

$$U - uV_v + uT_t + uU_u - uX_x = 0$$

$$V_u - X_t + uU_v - uT_x = 0$$

$$X_u + uT_v = 0]$$

The details of this calculation are in Ovsjannikov (see references).

LECTURE 7

DETONATION WAVES

## 1. Self-Similar Solutions

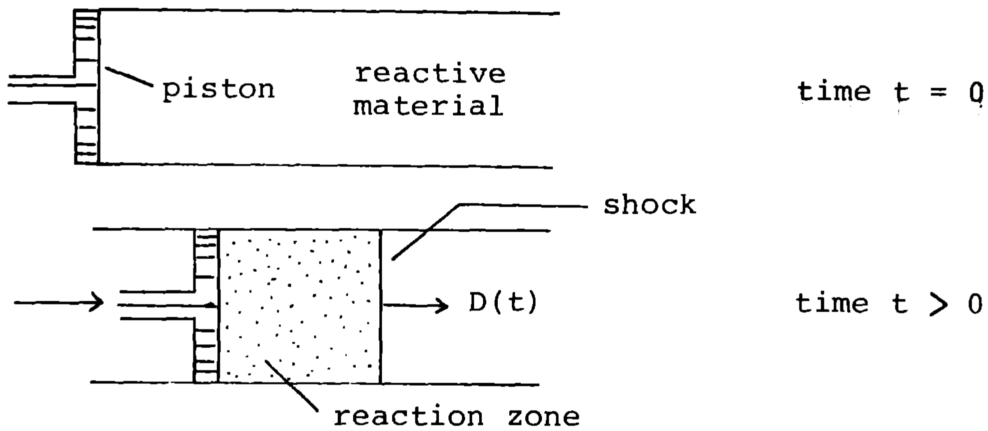



Figure 1. Schematic of the physical problem.

At time t = 0 a piston impacts a reactive material, thereby driving a shock wave into the material with velocity D = D(t). The shock wave initiates a chemical reaction A → B measured by a progress variable $\lambda$. $\lambda = 0$ at the shock front and $\lambda = 1$ at the end of the reaction zone. The shock path is not known a priori, but must be calculated.

The variables to be determined in the reaction zone are p (pressure), u (particle velocity), v (specific volume), and $\lambda$. All are functions of time t and Lagrangian position h.

We assume an equation of state of the reactant-product mixture of the form

$$E = \frac{pv}{\gamma-1} - \lambda q,$$

where q is the heat of detonation. In Lagrangian coordinates, the governing equations are the conservation laws

$$v_t - v_0 u_h = 0 \qquad \text{(mass)} \qquad (1)$$

$$u_t + v_0 p_h = 0 \qquad \text{(momentum)} \qquad (2)$$

$$p_t + \frac{\gamma p}{v} v_t - \frac{(\gamma-1)q}{v} \lambda_t = 0 \qquad \text{(energy)} \qquad (3)$$

and the chemical reaction equation

$$\lambda_t = Q(p,u,v,\lambda) \qquad (4)$$

where Q is the reaction rate. Q is presently treated as an unknown; in the following calculation we characterize the possible functional forms for Q in order that self-similar solutions exist. Boundary conditions are given along the shock path by the Rankine-Hugoniot jump conditions. Assuming the strong shock condition, the jump conditions may be written

$$D = \frac{\gamma + 1}{2} u_1,$$

$$\frac{v_1}{v_0} = \frac{\gamma - 1}{\gamma + 1} \qquad (5)$$

$$p_1 = \frac{\gamma + 1}{2v_0} u_1^2 \qquad (6)$$

$$E_1 - E_0 = \frac{1}{\gamma + 1} v_0 p_1 \qquad (7)$$

$$\lambda_1 - \lambda_0 = 0, \qquad (8)$$

where 0 and 1 (subscripts) denote values just in front of and just behind the shock, respectively.

The problem now is to find a local, Lie group of transformations

$$\bar{t} = t + \varepsilon T, \quad \bar{h} = h + \varepsilon H, \quad \bar{u} = u + \varepsilon U$$

$$\bar{p} = p + \varepsilon P, \quad \bar{v} = v + \varepsilon V, \quad \bar{\lambda} = \lambda + \varepsilon \Lambda$$

under which the equations and boundary conditions are constantly conformally invariant. The details of the calculation of T, H, U, P, V, and $\Lambda$ are found in the paper by J. D. Logan and J. Pérez, "Similarity Solutions for Reactive Shock Hydrodynamics", listed in the references. The following theorem results.

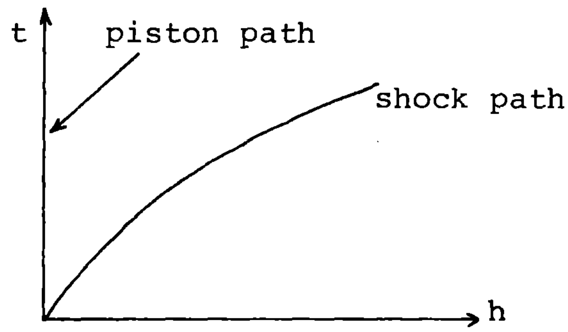



Figure 2. Spacetime Diagram

<u>Theorem</u> Under the polytropic gas and strong shock assumptions, the equations for reactive flow (1) - (4) and the Rankine-Hugoniot jump conditions (5) - (8) are constantly conformally invariant under the local, Lie group with generators given by

$$\begin{aligned} T &= at + c, \quad H = bh + d, \quad U = (b-a)u \\ P &= 2(b-a)p, \quad V = 0, \quad \Lambda = 2(b-a)\lambda, \end{aligned} \tag{9}$$

provided the reaction rate $Q = Q(u,p,v,\lambda)$ satisfies the partial differential equation

$$\frac{u}{2}Q_u + pQ_p + \lambda Q_\lambda = \beta Q \tag{10}$$

where

$$\beta = (2b-3a)/(2b-2a).$$

Equation (10) comes from one of the invariance conditions; it can be solved to determine the possible reaction rates for which similarity solutions exist. The characteristic system for (10) is

$$\frac{dv}{0} = \frac{du}{u/2} = \frac{dp}{p} = \frac{d\lambda}{\lambda} = \frac{dQ}{\beta Q},$$

and has first integrals given by

$$v,\ u^2/p,\ \lambda/p,\ Q/p^\beta.$$

Hence, the general solution of (10) is given by

$$Q = p^\beta\, f\left(v, \frac{\lambda}{p}, \frac{u^2}{p}\right) \quad . \tag{11}$$

Therefore, we have characterized all possible reaction rates for which there exists a similarity solution. We are now ready to find the self-similar solutions. The invariant surface conditions are

$$Tu_t + Hu_h = U,$$

$$Tp_t + Hp_h = P,$$

$$Tv_t + Hv_h = V,$$

$$T\lambda_t + H\lambda_h = \Lambda.$$

The characteristic system for the first equation is

$$\frac{dt}{at + c} = \frac{dh}{bh + d} = \frac{du}{(b-a)u}$$

The first pair gives the similarity variable:

$$s = \frac{c_4 h + 1}{(c_3 t+1)^{c_2}} \quad , \tag{12}$$

where

$$c_2 = b/a, \quad c_3 = a/c, \quad c_4 = b/d.$$

The equation

$$\frac{dt}{at + c} = \frac{du}{(b-a)u}$$

integrates to give the following self-similar form for u:

$$u(t,h) = (c_3 t+1)^{c_2 - 1} \; u_i \hat{u}(s) \tag{13}$$

Similarly,

$$p(t,h) = (c_3 t+1)^{2(c_2 - 1)} \; p_i \hat{p}(s) \tag{14}$$

$$v(t,h) = v_i \hat{v}(s) \tag{15}$$

$$\lambda(t,h) = (c_3 t+1)^{2(c_2 - 1)} \; \hat{\lambda}(s). \tag{16}$$

These are the similarity-solutions. The functions $\hat{u}, \hat{p}, \hat{v}$, and $\hat{\lambda}$ are yet to be determined by solving the set of ordinary differential equations to which the PDEs reduce. These ODEs are found by substituting the self-similar forms (13)-(16) into the PDEs (1)-(4); we obtain

$$\begin{bmatrix} 1 & 0 & \frac{(\gamma+1)s}{2} & 0 \\ s & -1 & 0 & 0 \\ 0 & s & \frac{\lambda s\hat{p}}{\hat{v}} & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} \frac{d\hat{u}}{ds} \\ \frac{d\hat{p}}{ds} \\ \frac{d\hat{v}}{ds} \\ \frac{d\hat{\lambda}}{ds} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{u}/(3-2\beta) \\ \frac{2p}{3-2\beta} - \frac{4kq\hat{p}^{\beta}F}{(\gamma+1)c_4 u_i^3 \hat{v}} \\ \frac{2\hat{\lambda}}{3-2\beta} - \frac{2k\hat{p}^{\beta}F}{(\gamma+1)c_4 u_i} \end{bmatrix}$$

where we have taken the reaction rate Q to be

$$Q = k\left(\frac{p}{p_i}\right)^{\beta} F\left(\frac{v}{v_i}, \frac{p_i\lambda}{p}, \frac{p_i u^2}{p u_i^2}\right),$$

consistent with the general form (11).

We recall that the similarity variable s is

$$s = \frac{c_4 h + 1}{(c_3 t+1)^{c_2}}$$

Since h = t = 0 is on the shock path, and the shock path must be a similarity curve, it is given by s = 1 or

$$c_4 h + 1 = (c_3 t+1)^{c_2} \qquad \text{[shock path]}.$$

The shock velocity is

$$D = \frac{dh}{dt} = \frac{c_2 c_3}{c_4} (c_3 t+1)^{c_2 - 1} = \frac{\gamma+1}{2} u_i \; (c_3 t+1)^{c_2 - 1}$$

The initial conditions for the system of ODEs are then

$$\hat{u}(1) = \hat{p}(1) = \hat{v}(1) = 1, \quad \hat{\lambda}(1) = 0.$$

Also,

$$c_2 = \frac{2\beta - 3}{2\beta - 2} .$$

We note that there is only one freee constant, namely $c_4$. The remaining constants are assumed to be given:

(i) k and β from the rate law (and the function F)

(ii) γ and q from the reactive material

(iii) $u_i$ from the initial piston energy.

2. <u>A Particular Case</u>

Let us examine closely the case $\beta > 3/2$, $c_4 < 0$. This forces $c_3 < 0$ and $0 < c_2 < 1$.

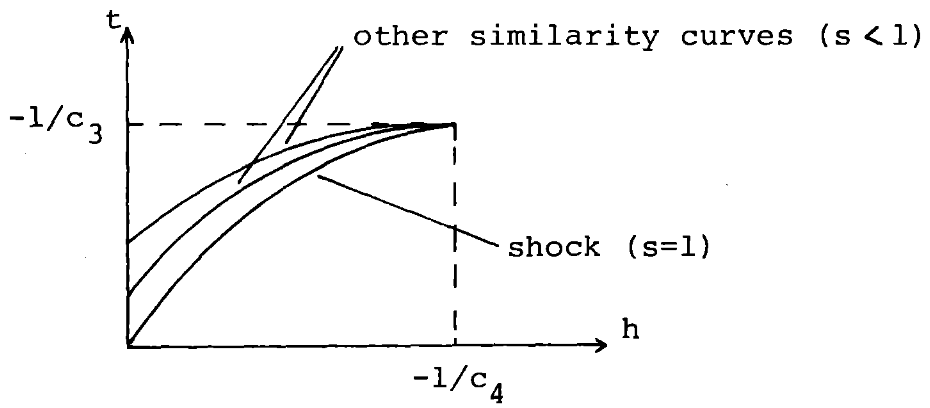



Figure 2. Similarity curves for the case $\beta > 3/2$, $c_4 < 0$.

The pressure at the front is $p_s = p_i(c_3 t+1)^{2(c_2-1)}$, which is increasing, and the shock is accelerating. The similarity curves meet at the singular point $(-1/c_4, -1/c_3)$ which can be considered as a transition-to-detonation point, a detonation being considered as a steady-state solution when viewed from the shock front.

With a rate law of the form

$$\lambda_t = k \left(\frac{p}{p_i}\right)^{\beta} \left(1 - c_1 \frac{p_i \lambda}{p}\right)$$

the system of ODEs was numerically integrated to give the qualitative pressure profiles shown in Figure 3.

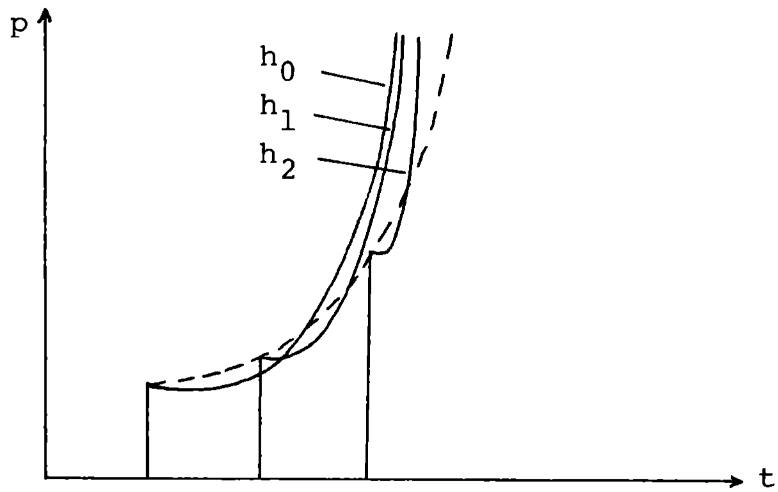



Figure 3. Pressure profiles of Lagrangian particles $h_0$, $h_1$, $h_2$, $h_0 < h_1 < h_2$. The dotted line shows the pressure $p_s$ at the shock.

These profiles are qualitatively the same as profiles obtained experimentally using Lagranian pressure gages by several investigators.

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