Part I of Nicole Oresme's Algorismus proportionum

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Part I of Nicole Oresme's *Algorismus proportionum*

Translated and Annotated by Edward Grant *

The importance of the first part of Nicole Oresme's *Algorismus proportionum* (*Algorism of Ratios*) [1] has long been recognized and appreciated.[2] Thus far it stands as the first known systematic attempt to present operational rules for multiplication and division (called addition and subtraction by Oresme) of ratios involving integral and fractional exponents. Since no translation of this work into a modern language has yet appeared, an obvious purpose of the present translation is to supply one. Furthermore, all who

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1 The *Algorismus proportionum* of Nicole Oresme (*c.* 1320–1382) consists of a prologue and three parts; only the prologue and first part — the most important — are translated here. In Part I, Oresme formulates a series of rules for operations with exponents and subsequently applies them to various physical problems in Parts II and III. All three parts — but not the prologue — were edited by Maximilian Curtze in his *Der Algorismus Proportionum des Nicolaus Oresme; zum ersten Male nach der Lesart der Handschrift R.4o.2. der königlichen Gymnasialbibliothek zu Thorn* (Berlin, 1868). Since Curtze used only the manuscript mentioned in the title of his edition, the present translation is made from my edition of Part I, based on thirteen manuscripts, which appears in *The Mathematical Theory of Proportionality of Nicole Oresme*, Ph.D. Dissertation (Univ. of Wisconsin, 1957), pp. 331–339.

In establishing the Latin text the following five manuscripts were collated through all of Part I: (1) Paris, Bibliothèque de l'Arsenal 522, fols. 121r–122v; (2) Bruges, MS. 530, fols. 25r–26v; (3) Florence, Biblioteca Medicea Laurenziana, MS. Ashburnham 210, fols. 172r–173v; (4) Vatican Library, Latin MS. 4082, fols. 109r–110r; (5) Thorn, Königlich Gymnasialbibliothek, R.4o.2., pp. 82–85. The remaining eight manuscripts were collated for all of the text prior to Rule One. (1) Basel, Université Bibliothèque, MS. F. II.33, fols. 95v–97v; (2) Florence, Biblioteca Nazionale, conv. soppr., J.IX.26, fols. 37r–39r; (3) Oxford, Bodleian Library, MS. St. John's College 188, fols. 104r–105v; (4) Venice, Biblioteca Nazionale Marciana, L.VI.133, fols. 65v–66v; (5) Paris, Bibliothèque Nationale, fonds Latin, 7197, fols. 74r–75v and (6) 7368, fols. 1r–4v; (7) Brussels, Bibliothèque Royale de Belgique, MS. 1043, fols. 217r–219r; (8) Utrecht, Bibliotheek der Rijksuniversiteit te Utrecht, fols. 165r–167v. The folio numbers for all manuscripts listed here apply only to Part I.

2 The most influential accounts based on Curtze's edition (*op. cit.*) and interpretation may be found in Moritz Cantor's *Vorlesungen über Geschichte der Mathematik* (2nd ed., Leipzig: Teubner, 1900), Vol. 2, pp. 133–135, and Johannes Tropfke's *Geschichte der Elementar-Mathematik in systematischen Darstellung* (Leipzig: Veit, 1902), Vol. 1, pp. 200, 206–207. A more critical appraisal of Curtze's interpretation of the first two rules appears in Heinrich Wieleitner's article "Zur Geschichte der gebrochenen Exponenten," *Isis*, 1924, *6*: 509–522. For a somewhat different approach to the significance of Oresme's treatise, see Carl Boyer, "Fractional Indices, Exponents, and Powers," *National Mathematics Magazine*, 1943, *18*:81–86.

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have discussed this treatise, with the partial exception of Heinrich Wieleitner, have relied on Maximilian Curtze's symbolic representation of Oresme's rules that accompanied his edition of the *Algorismus proportionum* in 1868. Where they are not completely false, these representations are quite unsatisfactory, for Curtze often omitted some of the cumbersome, and perhaps repetitious, intermediate steps and thereby left the impression of a greater conciseness of expression than is actually the case. The second major objective, then, is to annotate this important text as faithfully and accurately as possible.

Despite the apparent popularity of the *Algorismus* (judging from the rather large number of extant manuscripts — at present I know of twenty-five — it appears to have been copied frequently and to have been widely known), its influence on the subsequent history of mathematics was probably minimal. Tropfke, who traced the development of exponents and their various modes of representation, found little to suggest Oresme's influence.[3] Nevertheless, until a more intensive investigation is made, the role played by the *Algorismus* must remain an open question.

* * *

The Prologue[4] of Master Nicole Oresme

On the Treatise

Algorism of Ratios

To the Most Excellent Reverend Phillip of Meaux,[5] whom I would call Pythagoras if it were possible to believe in the opinion about the return of souls, I present this *Algorism of Ratios* so that if it is agreeable to your Excellency you may correct that which I put before you. For should it be approved by the authority of so great a man and corrected after his examination [of it], everything that has been revised by your correction would be an improvement. Then, if a disparager should open his mouth and set his teeth to rend [my work] into pieces, he would not find [what he seeks].

Part I

One half is written as 1/2, one third as 1/3, and two thirds as 2/3, and so on. The number above the crossbar is called the numerator, that below the crossbar the denominator.

[3] Tropfke, *op. cit.*, p. 206.

[4] Because the Thorn manuscript lacked the prologue, it is missing from Curtze's edition.

[5] This is the eminent Phillip de Vitri (1291–1361), renowned musical scholar and friend of Petrarch, who was Bishop of Meaux from 3 January 1351 to his death in Paris on 9 June 1361. Since Phillip did not have an ecclesiastical post at Meaux prior to 1351, Oresme's reference to him as " Most Excellent Reverend Phillip of Meaux " (*Reverende Presul Meldensis Phillipe, . . .*) indicates that Oresme composed the *Algorismus* sometime between 1351 and 1361, the period during which Phillip was Bishop of Meaux.

A double ratio [i.e. 2/1] is written as 2^p, a triple ratio [i.e. 3/1] as 3^p, and so forth. A sesquialterate ratio [i.e. 3/2] is written as 1^p 1/2, and a sesquitertian ratio [i.e. 4/3] as 1^p 1/3. A superpartient two-thirds ratio [i.e. 5/3] is written as 1^p 2/3, a double superpartient three-fourths [i.e. 11/4] as 2^p 3/4, and so on. Half of a double ratio [i.e. $(2/1)^{1/2}$] is written as 1/2 2^p and a fourth part of a double sesquialterate [i.e. $(5/2)^{1/4}$] as 1/4 2^p 1/2, and so on.[6] But sometimes a rational ratio is written in its least terms or numbers just as a ratio of 13 to 9, which is called a superpartient four-ninths [i.e. 1 4/9]. Similarly, an irrational ratio such as half of a superpartient two-thirds [i.e. $(5/3)^{1/2}$] is written as half of a ratio of 5 to 3.

Every irrational ratio — and these shall now be considered — is denominated by a rational ratio in such a manner that it is said to be a part or parts of the rational ratio, as [for example], half of a double [i.e. $(2/1)^{1/2}$], a third part of a quadruple [i.e. $(4/1)^{1/3}$], or two thirds of a quadruple [i.e. $(4/1)^{2/3}$].[7] It is clear that there are three things [or elements] in the denomination of such an irrational ratio: [1] a numerator, [2] a denominator, and [3] a rational ratio by which the irrational ratio is denominated, that is, a rational ratio of which that irrational ratio is said to be a part or parts as [for example] in half of a double ratio [i.e. $(2/1)^{1/2}$] the unit is the numerator, or represents the numerator, two is the denominator,[8] and the double ratio is that by which the irrational ratio is denominated.[9] And this can easily be shown for other ratios.

[6] In writing irrational ratios, Oresme places the exponent first, followed by the ratio that expresses the rational base. Since the manuscripts reveal that Oresme followed the pattern consistently, we have here an early attempt at a mathematical notation. However, the rational bases were written with considerable variation. Thus 1/4 2^p 1/2 might appear as 1/4 2 1/2, 1/4 p2/12, 1/4 2^{p1a} 1/2, and so on. Often enough, Oresme followed the usual medieval practice and verbalized the entire expression. Both numerical and verbal modes are illustrated in Fig. 1, which accompanies this translation. The plate includes the Latin text of the concluding lines of Rule Four, all of Rule Five, and the enunciation of Rule Six. With the exception of exponents, the first part of the *Algorismus* is concerned exclusively with ratios of greater inequality — i.e., with ratios of the form A/B, where $A>B$.

[7] For Oresme the irrational ratio $(4/1)^{1/3}$ is a *part* of the rational ratio (4/1) because $(4/1)^{1/3} < (4/1)$ *and* the exponent, 1/3, is a unit fraction. The irrational ratio $(4/1)^{2/3}$ is *parts* of (4/1) because $(4/1)^{2/3} < (4/1)$ *and* the exponent is a proper fraction in its lowest terms where both integers are greater than 1. Oresme relates the concept of exponential parts with that of commensurability in a later work, the *De proportionibus proportionum* (*On Ratios of Ratios*; my edition and translation of this treatise will appear soon in the University of Wisconsin Press series Publications in Medieval Science under the title *Nicole Oresme: De proportionibus proportionum and Ad pauca respicientes*).

[8] Oresme is referring here to the numerator and denominator of the exponent.

[9] When an irrational ratio has a rational base the latter is said to denominate the former. Thus if $(A/B)^{p/q}$ is irrational, (A/B) rational, and p, q are integers with $p < q$, then (A/B) will denominate $(A/B)^{p/q}$. In his *De proportionibus proportionum* Oresme says that such an irrational ratio is *immediately denominated* by the rational ratio (A/B) and *mediately denominated* by a number — i.e., by the exponent p/q when it is a ratio of numbers. Irrational ratios in the form cited above were always expressed by Oresme as exact parts of some rational ratio. The bare distinction, without any elaboration, between immediate and mediate denomination appears as early as 1328 in Thomas Bradwardine's *Tractatus proportionum seu de proportionibus velocitatum in motibus* (*Thomas of Bradwardine His Tractatus de proportionibus*, ed. and trans. by H. Lamar Crosby, Jr. [Madison: Univ. of Wisconsin Press, 1955]).

Rule One. [*How*] *to add a rational ratio to a rational ratio.*

Assuming that each ratio is in its lowest terms, multiply the smaller term, or number, of one ratio by the smaller number of the other ratio; and [then] multiply the greater [number of one ratio] by the greater [number of the other], thereby producing the numbers or terms of the ratio composed of the two given ratios. In this way, three or [indeed] any number [of ratios] can be added by adding two of them at a time and then adding a third to the whole composed of those two; then, if you wish, add a fourth ratio, and so on.

For example, I wish to add sesquitertian [i.e. 4/3] and quintuple [i.e. 5/1] ratios. The prime numbers of a sesquitertian are 4 and 3, of the other 5 and 1. And so, as already stated, I shall multiply 3 by 1 and 4 by 5 obtaining 20 and 3, which is a sextuple superpartient two-thirds ratio [i.e. 6 2/3]. In this way a ratio can be doubled, tripled, and quadrupled, as many times as you please.[10] And this can be demonstrated and is [indeed] adequately shown in the sixth proposition of the fifth book of the *Arithmetic* of Jordanus [de Nemore].[11]

[10] If we have a rational ratio $(A)^n$, where n is an integer, Oresme would expand this ratio by stages. Thus $(A) \cdot (A) = (A)^2$; $(A)^2 \cdot (A) = (A)^3$; $(A)^3 \cdot (A) = (A)^4$, and so on until $(A)^{n-1}$ $(A) = (A)^n$. Note that Oresme says a ratio is "to be doubled" (*duplari*) and "tripled" (*triplari*) when he obviously means "squared" and "cubed." This ambiguous terminology was quite common, but usually not troublesome since the context would almost always reveal an author's meaning.

[11] Bk. V, Prop. 6, of the *Arithmetica* of Jordanus de Nemore (*fl.* in the first half of the 13th century) is not relevant to the "addition" of ratios in this rule, for in that proposition Jordanus considers how to reduce proportions to their lowest terms (Cambridge, Peterhouse 277 [the codex is in Magdalene College], Bibliotheca Pepysiana 2329, fol. 13v, c. 1–2). But Jacques Le Fèvre d'Estaples (Jacobus Faber Stapulensis [1455–1536]) explains the addition of two ratios in his edition of Jordanus' *Arithmetica* (Paris, 1496; this includes Jordanus' enunciations and Le Fèvre's demonstrations and comments; for a description of this edition see David Eugene Smith, *Rara Arithmetica* [Boston/London: Ginn, 1908], pp. 62–63). Before actually commencing the demonstration of Bk. V, Prop. 3, Le Fèvre says (sig. b6, c. 1; the folios are unnumbered):

> Before we demonstrate what has been proposed, I wish to show how one ratio is added to another. Now I say that the ratio of the products of the first term of one ratio by the first term of the other, and of the second [term of one] by the second [term of the other] is composed of these two ratios. For let A to B and C to D be the two ratios that I wish to add and to show what is composed from them. I multiply C by A and get E; and I multiply D by B and get G. I say that ratio E to G is composed of ratios A to B and C to D. Now, again, I multiply C by B and get F, and by the seventh [proposition] of the second [book] ratio A to B is as E to F; and by the eighth [proposition] of the same [book] ratio C to D is as F to G and ratio E to G is composed of ratios E to F and F to G, and, therefore, ratio E to G is composed of ratios A to B and C to D, which was asserted. (Priusquam ad propositum demonstrandum veniamus, volo demonstrare quo pacto proportio proportioni addenda sit. Et dico proportionem productorum ex primo termino unius proportionis in primum alterius; et ex secundo in secundum esse ex duabus illis proportionibus compositam. Sint enim A ad B et C ad D due proportiones quas volo simul addere. Atque ex ipsis compositam constare. Duco C in A et proveniat E; et D in B et proveniat G. Dico proportionem E ad G esse compositam ex proportionibus A ad B et C ad D. Duco enim iterum C in B et proveniat F, et per septimam secundi que proportio A ad B ea est E ad F; et per octavam eiusdem que proportio C ad D ea est F ad G. Et proportio E ad G componitur ex proportionibus E ad F et F ad G, quare et ex proportionibus A ad B et C ad D, quod intendebatur.)

We see that "addition" of ratios is simply multiplication. Why, or when, the multiplication of such ratios as 5/1 and 4/3 came to be designated "addition" is unknown to me (e.g., *Rule Two. [How] to subtract a rational ratio from [another] rational ratio.*

As before, assume any [two] rational ratios in their lowest numbers. The smaller number of one ratio is then cross-multiplied (*ducantur contradictorie*) by the greater number of the other ratio, and the same is done with the remaining numbers, thus producing the terms of a ratio in which a greater term exceeds a smaller term. The greater of the given ratios was that whose greater term when multiplied by the smaller term of the other given ratio produces the greater number [of the resultant ratio].

For example, let a sesquitertian [i.e. 4/3] be subtracted from a sesquialterate ratio [i.e. 3/2]. The prime numbers, or terms, of a sesquitertian are 4 and 3, of a sesquialterate 3 and 2. I shall multiply 4 by 2 and obtain 8, and then multiply 3 by 3 to obtain 9. A sesquialterate ratio is, therefore, greater than a sesquitertian by a ratio of 9 to 8, that is, greater by a sesquioctavan ratio.[12] This can be shown by the twenty-seventh proposition of the second book of the *Arithmetic* of Jordanus.[13]

it does not appear in Boethius' *Arithmetica*, a very widely used treatise). Where Oresme applies this terminology to exponents a plausible explanation can be offered (see below). In the *De proportionibus proportionum* (Ch. I, lines 75–83, of my forthcoming edition), Oresme speaks briefly of adding (i.e., multiplying) ratios of greater inequality. There, however, the procedure is couched exclusively in terms of continuous proportionality where extreme terms are assigned and the ratios composed. In the *Algorismus*, however, the addition of ratios is effected by multiplication performed directly on the prime numbers, or numerical denominations, of the ratios; extreme terms are not assigned.

In his eagerness to interpret all of Oresme's rules exponentially, Curtze mistakenly represents the first rule as $a^m \cdot a^n = a^{m+n}$. (All references to Curtze's representations of Oresme's rules are to p. 10 of his edition, *op. cit.*) H. Wieleitner correctly observed (*op. cit.*, p. 511) that this has no connection with Oresme's rule.

[12] $3/2 : 4/3 = 9/8$.

[13] The enunciation of this proposition in Jordanus' *Arithmetica* reads (Bibliotheca Pepysiana 2329, fol. 6r., c. 1):

> If the ratio of the first to the second term is greater than the ratio of the third term to the fourth, then the product of the first and fourth terms is greater than the product of the second and third terms. And if the product is greater the ratio of the first term to the second will be greater. (Si fuerit proportio primi ad secundum maior quam tertii ad quartum qui ex primo in quartum productum maior est producto ex secundo in tertium. Quod si productus maior fuerit etiam proportio primi ad secundum maior erit.)

In his proof Jordanus assumes that $A/B > C/D$, so that $AD = E$, $CB = F$, and $BD = G$. Since $E/G = A/B$ and $F/G = C/D$, it follows that $E/G > F/G$ and, consequently, that $E > F$ (i.e., $AD > CB$).

Although this proposition follows the steps outlined by Oresme, nowhere does Jordanus speak of subtracting one ratio from another. But in Bk. V, Prop. 1, he approximates rather closely the ideas expressed by Oresme, again without using any form of the term *subtrahere*. The enunciation of this proposition reads (Bibliotheca Pepysiana 2329, fol. 13r., c. 1):

> That ratio by which the ratio of the first to the second term exceeds the ratio of the third to the fourth term is the ratio formed by the product of the first and fourth terms and the product of the second and third terms. (Quod addit proportio primi ad secundum super proportionem tertii ad quartam est proportio que est inter productum ex primo in quartum et productum ex secundo in tertium.)

Jordanus assumes that A, B, C, and D are four successive terms where $A/B > C/D$. Thus if $DA = E$ and $BC = F$, then $A/B : C/D = E/F$, and Jordanus would say that A/B exceeds C/D by ratio E/F just as Oresme says that $3/2$ exceeds $4/3$ by $9/8$, a sesquioctavan ratio. Although Jordanus did not refer to this as subtraction of ratios, Jacques Le Fèvre, in commenting on this proposition, says (*op. cit.*, sig. b6, c. 1):

> This [proposition] shows how to subtract a ratio from a ratio. And after the lesser ratio has been subtracted from the greater ratio, the ratio that is left, howsoever much it be, is what we call here the difference between

Rule Three. If an irrational ratio is parts of any rational ratio, it is possible to designate it as a part of yet another rational ratio so that it might more appropriately be called a part rather than parts.

Let B, an irrational ratio, be parts of A, a rational ratio. Without changing the denominator [of the exponent], I say that B will be a part of some ratio [obtained by expanding A by the numerator of the exponent] and this [expanded] ratio will be multiple to A. Any irrational — and it is of these we speak — will be a part of some rational ratio, since a multiple ratio can be found for every rational ratio.[14]

one ratio and another. (Hec docet subtrahere proportionem a proportione et que quantaque proportio minore proportione a maiore subtracta relicta sit quam hic differentiam proportionis a proportione nominamus.)

For Oresme, as for Jacques Le Fèvre and others, "subtraction" of ratios of greater inequality is actually division by cross-multiplication. But why was the quotient of $3/2 : 4/3$ called a "difference" and the whole process of division designated "subtraction"? In our concluding note we shall see that where exponents are concerned a certain rationale may be supplied for this terminology. But as with "addition" of ratios, its origin is unknown to me and can only be conjectured (see final note, below). In the *De proportionibus* (Ch. I, lines 72–74), Oresme mentions briefly the subtraction (i.e., division) of one ratio of greater inequality from another. In that treatise, subtraction is performed by assigning a mean term between the terms of the greater ratio and subsequently composing the ratios. Thus, if ratio E is to be subtracted from ratio B/C, it follows that $B/C > E$. Assign a mean term, D, between B and C so that either (1) $D > C$ and $D/C = E$ in which event $B/C : D/C = B/D$; or (2) $D < B$ and $B/D = E$, so that $B/C : B/D = B/C \cdot D/B = D/B \cdot B/C = D/C$. Here the *modus operandi* is to produce continuous proportionality by assigning mean terms and composing the two ratios. This is a wholly different emphasis than in the *Algorismus* where the operation is effected directly by cross-multiplying the prime numbers of the given rational ratios; hence no means need be assigned.

Perhaps with his own *Algorismus* in mind, Oresme informs the readers of the *De proportionibus* that ratios may be added and subtracted in a manner different than that which is appropriate to the latter treatise (*De proportionibus proportionum*, Ch. 1, lines 84–89):

If, however, you wish to add a ratio of greater inequality to another by means of algorism (*per artem*), it is necessary to multiply the denomination of one ratio by the denomination of the other. And if you wish to subtract one ratio from another, you do this by dividing the denomination of one ratio by the denomination of the other. The [method] of finding denominations will be taught afterward. Multiplication and division of denominations are done by algorism. (Si autem volueris per artem proportionem maioris inequalitatis alteri addere tunc oportet denominationem unius per denominationem alterius multiplicare. Et si volueris unam ab altera subtrahere hoc facies denominationem unius per denominationem alterius dividendo. Denominationum inventio postea docebitur; quarum multiplicatio atque divisio habetur per algorismum.)

Although we are told here that the addition of ratios can be performed by multiplication of denominations (i.e., multiplication of the prime numbers of the two ratios) and that subtraction of ratios can be performed by division of denominations (i.e., division of the prime numbers of one ratio into the prime numbers of the greater ratio), Oresme, in the ninth rule of his *Algorismus*, emphasizes that the multiplication of the denominations of ratios must, strictly speaking, be called addition, not multiplication. Similarly, the division of the numerical denominations of two ratios must be called subtraction, not division (see the final note, below, for a possible explanation of this).

The second rule of the *Algorismus* was also misrepresented by Curtze, who treated it as a case of subtraction of exponents where $a^m/a^n = a^{m-n}$ (also see Wieleitner, *op. cit.*, p. 511).

[14] If the irrational ratio is $B = (A)^{m/n}$, where A is a rational ratio and m/n is a ratio of integers in its lowest terms with $n > m > 1$, then $(A)^{m/n}$ can be expanded to $(D)^{1/n}$ so that $D = (A)^m$. Initially, then, $(A)^{m/n}$, or B, is an irrational ratio that is exponentially *parts* of rational ratio A, since the exponent m/n is a proper fraction whose numerator is an integer greater than 1. By expanding $(A)^m$ we obtain a rational ratio D which is multiple to A in an exponential sense, since $D = (A)^m$. (Thus when Oresme says that "a multiple ratio can be

As an example, let us take a ratio which is two-thirds of a quadruple [i.e. $(4/1)^{2/3}$]. Since 2 is the numerator [of the exponent], we shall have one-third of a quadruple ratio squared [i.e. $[(4/1)^2]^{1/3}$], namely [one-third of] a sedecuple ratio [i.e. $(16/1)^{1/3}$]. The same applies to other ratios of this kind. The justification for this lies in the general truth that one-third of a whole [i.e. $(A)^{1/3}$] equals two-thirds of its half or subdouble [15] [i.e. $(A^{1/2})^{2/3}$], and, conversely, two-thirds of a subdouble [i.e. $(A^{1/2})^{2/3}$] equals one-third of a double [i.e. $(A)^{1/3}$]. The same reasoning is applicable to any other parts.

Rule Four. [How] to assign the most appropriate denomination of an irrational ratio.

Here it must be understood that a rational ratio is called *primary* when it cannot be divided into equal rational ratios, and no mean proportional number or numbers can be assigned between its least numbers, as is the case with double [i.e. 2/1], triple [i.e. 3/1], or sesquialterate [i.e. 3/2] ratios. But a rational ratio is called *secondary* when it can be so divided and a mean proportional number or numbers can be assigned between its [least] numbers.[16] As examples, take a quadruple ratio [i.e. 4/1] which is divisible into two doubles [i.e. $4/2 \cdot 2/1$], an octuple ratio [i.e. 8/1] divisible into three doubles [i.e. $8/4 \cdot 4/2 \cdot 2/1$], a nonacuple ratio [i.e. 9/1] into two triples [i.e. $9/3 \cdot 3/1$], and so on.

From all this we see that if any proposed irrational ratio is denominated by parts, that ratio, by the preceding rule, could be transformed and called a part. However, it must [first] be seen if the rational ratio denominating the irrational is a *primary* ratio. If it is, let it stand, for then the irrational ratio, which is our topic of discussion, is most appropriately denominated, as in one-third of a sextuple [i.e. $(6/1)^{1/3}$] or double ratio [i.e. $(2/1)^{1/3}$], and so forth.

But if the rational ratio that denominates the irrational ratio is *secondary*, one must determine how many primary rational ratios are contained by it, where each primary is an equal part of it. Should the number representing

found for every rational ratio" he means that any rational ratio A can be expanded by any integral exponent m, and the rational ratio which is the product of that expansion, namely D, is called the multiple of A.) Hence $(A)^{m/n} = (D)^{1/n}$ and has been transformed from an irrational ratio that was *parts* of rational ratio A, to one that is a *part* — i.e., where the exponent is a unit fraction — of another rational ratio D. The most appropriate form of irrational ratio $(A)^{m/n}$ is, therefore, $(D)^{1/n}$. This is illustrated by the example in the very next paragraph of the text: $(4/1)^{2/3} = [(4/1)^2]^{1/3} = (16/1)^{1/3}$.

15 *Medietas* and *subduplus* are the Latin terms rendered here as "half" and "subdouble" and clearly mean square root. More commonly, however, they signified "half" in the arithmetic sense. The context usually reveals the meaning without ambiguity.

16 In the *De proportionibus proportionum*, Oresme does not use the terms "primary" (*primaria*) and "secondary" (*secundaria*) but, at the conclusion of the first chapter, these concepts constitute the first two of seven conceivable ways in which rational ratios are divisible (see Ch. I, lines 385–389, of my forthcoming edition of the *De proportionibus*). In the first way rational ratios are divisible into smaller equal rational ratios, which corresponds to secondary ratios in the *Algorismus*; all rational ratios not divisible in the first way constitute the second group, which makes this category equivalent to the primary ratios of the *Algorismus*.

these parts [or primary ratios] be incommunicant or prime to the denominator of the proposed irrational ratio, the [initial] denomination must be left as it is. The denomination of half of an octuple [i.e. $(8/1)^{1/2}$], for example, is proper because an octuple ratio [i.e. (8/1)] has three equal rational parts, namely three doubles [i.e. $8/4 \cdot 4/2 \cdot 2/1$ or $(2/1)^3$], and 2 is the denominator of the proposed irrational ratio. But 3 and 2 are incommunicant [17] [or prime to one another] so that half of an octuple is not a part of any rational ratio smaller than an octuple, although it could certainly be parts since half of an octuple is three-fourths of a quadruple [i.e. $(4/1)^{3/4}$] — but this would not constitute an appropriate denomination.

However, if the number representing the smallest, or primary, parts of such a secondary rational ratio by which an irrational ratio is denominated *and* [the number representing] the denominator [of the exponent] of this irrational ratio were communicant [or mutually nonprime] numbers, the greatest number [or common factor] in which they are communicant must be taken and each of them divided by it. By dividing the number representing the parts of the secondary ratio we arrive at the number of parts or [primary] ratios composing the rational ratio that will most suitably denominate the [proposed] irrational ratio; and by dividing the denominator of the [initially] proposed [irrational ratio] by the same greatest number [or common factor], the most appropriate denominator of the irrational ratio is found.

For example, let a ratio of three-fourths of a quadruple [i.e. $(4/1)^{3/4}$] be proposed. By utilizing the third rule it is clear that it equals one-fourth of 64/1 [i.e. $(64/1)^{1/4}$]. But 64/1 is composed of six double ratios [i.e. $64/1 = (2/1)^6$] where 6, which signifies the number of primary parts in 64/1, and 4, which represents the denominator of the proposed ratio,[18] are communicant [19] [or have a common measure] in 2, so that dividing 2 into 6 gives 3, signifying that the proposed ratio is a part of three doubles, namely part of 8/1.[20] Dividing 2 into 4 yields 2, so that the proposed ratio is one-half — i.e., the rule shows that the proposed ratio is one-half of 8/1 which is written as $1/2\ 8^p$ [i.e. $(8/1)^{1/2}$], and this is its most proper denomination. In the same way, one-twelfth of four triple ratios [i.e. $[(3/1)^4]^{1/12}$], namely 81/1 [i.e. $(81/1)^{1/12}$], is one-third of 3/1 [i.e. $(3/1)^{1/3}$]; and similarly one-fourth of six triple ratios [i.e. $[(3/1)^6]^{1/4}$] is one-half of three triple ratios [i.e. $[(3/1)^3]^{1/2}$], namely [one-third of] 27/1 [i.e. $(27/1)^{1/3}$], and so on.[21]

17 The numbers 3 and 2 refer to the exponent in $(2/1)^{3/2}$.

18 Oresme refers here to the denominator of the exponent in the ratio $(2/1)^{6/4}$.

19 In Campanus of Novara's 13th-century edition of Euclid's *Elements*, Bk. VII, Def. 8, defines "communicant" numbers as follows:

> Numbers are said to be mutually composite or communicant which are measured by a number other than unity, and none of them is prime to any other of them. (Numeri adinvicem compositi sive communicantes dicuntur quos alius numerus quam unitas metitur, nullusque eorum est ad alium primus.) (*Euclidis . . . Elementorum geometricorum libri XV cum expositione Theonis in priores XIII a Bartholomaeo Veneto Latinitate donata, Campani in omnes* . . . [Basel, 1546], p. 168.)

20 That is, $[(2/1)^3]^{1/2} = (8/1)^{1/2}$.

21 In the fourth rule, Oresme presents criteria for determining the final form of an irrational ratio. If possible, all irrational ratios should have a primary rational ratio as base

Rule Five. [*How*] *to add* [22] *an irrational ratio to a rational ratio.*

Let B, an irrational ratio, be added to A, a rational ratio, and assume that B is a part of rational ratio D [23] and, in accordance with previous rules, this constitutes the most proper denomination of B.[24] Then, by the first rule, add A to ratio D a number of times equal to the value of the denominator of B, and let C be the total result.[25] I say, therefore, that the ratio composed of [ratios] B and A will be part of C and [it] will be denominated by the same denominator which denominated [or denoted] the part which B was of its denominating ratio, namely D.[26]

For example, let one-third of a double ratio [i.e. $(2/1)^{1/3}$] be added to a sesquialterate ratio [i.e. 3/2]. Now join three sesquialterate ratios with a double ratio to yield a sextuple superpartient three-fourths ratio [i.e. 6 3/4], which is a ratio of 27 to 4. The ratio produced from [the addition of] one-third of a double ratio and a sesquialterate ratio is one-third of the ratio of 27 to 4 [i.e. $(27/4)^{1/3}$] which is written as 1/3 6^p 3/4.[27]

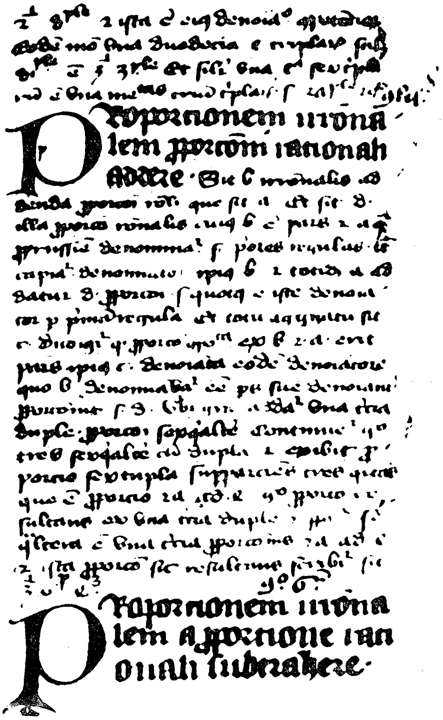

FIGURE 1. A page of an *Algorismus* manuscript which shows the closing lines of Rule Four, all of Rule Five, and the enunciation of Rule Six (Florence, Laurentian Library, Ashburnham MS. 210, fol. 173r, c. 1).

and a unit fraction for an exponent, as in ratios $(6/1)^{1/3}$ and $(3/1)^{1/4}$. If, however, the base is a secondary rational ratio, Oresme outlines the procedure for determining its most proper denomination or representation, i.e., the most proper rational ratio that will serve as base. If $(B)^{1/q}$ is a given irrational ratio, where B is a secondary rational ratio, one must decompose B into its constituent primary ratios. Let A represent the primary ratios so that $B=(A)^m$, where m is an integer. Therefore, $(B)^{1/q}=(A^m)^{1/q}$. Should m and q be unequal mutually prime numbers greater than 1, the original form of the ratio, namely $(B)^{1/q}$, will be the most appropriate denomination, since it expresses the irrational ratio as a part of rational ratio B. For example, $(8/1)^{1/2}$ is most properly denominated because the alternatives, $(2/1)^{3/2}$ and $(4/1)^{3/4}$, fail to represent the relationship in terms of a single part, or unit fraction.

But if m and q are not mutually prime, they must then be reduced to their lowest terms. Then, either (1) $m/q=1/n$ and the final form is $(A)^{1/n}$; or (2) $m/q=p/n$, where p and n are greater than 1, and the final form is $(D)^{1/n}$ (rather than $(A)^{p/n}$), where $D=(A)^p$. An example of (1) is $(81/1)^{1/12}=(3/1)^{4/12}=(3/1)^{1/3}$, the last expression being the most suitable final form; an example of (2) is $(64/1)^{1/4}$ (given initially as $(4/1)^{3/4}$ but expanded to $(64/1)^{1/4}$ by the third rule) which equals $(2/1)^{6/4}=(2/1)^{3/2}=(8/1)^{1/2}$, the last expression being the most appropriate. The two examples given by Oresme in which $(8/1)^{1/2}$ is the final form are both represented here.

The ways in which Oresme may have expressed the ratios mentioned in the last six lines of the translation of Rule Four may be seen in Fig. 1. Observe that some ratios are

Rule Six. [*How*] *to subtract an irrational from a rational ratio, or the converse.*

Let A be a rational ratio and B an irrational ratio denominated by rational ratio D. Then, whether A is greater or smaller than B, let it be multiplied by the denominator of B. This is the same thing as taking a ratio composed of a number of A's equivalent to [the number representing] B's denominator. The method for accomplishing this is shown in the first rule. Now let this composed, or produced, ratio be called C. By the second rule one can determine whether C is greater or smaller than D. If it is greater, then A was greater than B; but if D is greater than C, B was greater than A. Whatever the given ratios, namely C and D, the lesser must be subtracted from the greater. This is done by the second rule. Let the remainder be F. And so I say that if A is subtracted from B, or conversely, the remainder will be the same total part of F as B was of D.[28]

For example, subtract a sesquitertian ratio [i.e. 4/3] from half of a double

given verbally and others numerically; indeed, sometimes the expression is partially verbal, partially numerical.

22 We have already seen that "to add" (*addere*) signifies multiplication in the context of this treatise.

23 Here $B = (D)^{1/q}$, where q is an integer.

24 That is, $(D)^{1/q}$ is the most proper denomination of B.

25 The denominator of B is q, so that $D \cdot (A)^q = C$. In accordance with the first rule, A is expanded in stages. Thus if $q = 3$, then $A \cdot A = (A)^2$ and $(A)^2 \cdot A = (A)^3$.

26 Since $D \cdot (A)^q = C$, $B < D$, and $A < (A)^q$, it follows that $(B \cdot A) < C$. Also, $(B \cdot A)$ is an exponential part of C, since $(B \cdot A) = (D \cdot A^q)^{1/q} = (C)^{1/q}$. But $B = (D)^{1/q}$ so that $(D)^{1/q}$ and $(C)^{1/q}$ have the same denominator in their exponents, namely q. It is in this sense that ratios B and $(B \cdot A)$ are denominated, or represented, by the same denominator.

27 The two given ratios are first cubed, then fully expanded, after which the cube root is taken. Thus $[(2/1)^{1/3} \cdot 3/2]^3 = 2/1 \cdot (3/2)^3 = 27/4$, whose cube root is $(27/4)^{1/3}$ but which Oresme writes as $(6\ 3/4)^{1/3}$. Since all of Rule Five is included in Fig. 1, the reader may see the variety of ways in which ratios might be written.

28 We are given rational ratio A and irrational ratio $B = (D)^{1/n}$, where n is an integer (in the enunciation of Rule Six Oresme does not specify that the numerator of the exponent must be 1; but the rule itself and the examples indicate this unmistakably). Now whether $A \gtrless B = (D)^{1/n}$, we must expand A exponentially by n, the denominator of the exponent of B in $(D)^{1/n}$. The expansion of A is carried out by the first rule, i.e., $A \cdot A = (A)^2$, $(A)^2 \cdot A = (A)^3$, $(A)^3 \cdot A = (A)^4$, . . . $(A)^{n-1} \cdot A = (A)^n$; and let $(A)^n = C$. We can now determine whether $C \gtrless D$. Let rational ratio $C = p/q$ (where $p > q$ and both are integers) and let rational ratio $D = s/t$ (where $s > t$ and both are integers). By the second rule, $p/q : s/t = pt/qs$; and if $pt > qs$, then $p/q > s/t$ and $C > D$; but if $qs > pt$, then $s/t > p/q$, and $D > C$. Furthermore, if $C > D$, then $A > B$; and if $D > C$, then $B > A$. This being determined, the lesser ratio must then be subtracted from (i.e. divided into) the greater — and this is also carried out by the second rule. If $C > D$, then let us assume that $C : D = F$ (or $A^n : B^n = F$); and since $A = (C)^{1/n}$ and $B = (D)^{1/n}$, it follows that $A : B = (F)^{1/n}$; but if $D > C$, then assume that $D : C = F$ and $B : A = (F)^{1/n}$. Hence, $A : B$, or $B : A$, $= (F)^{1/n}$ and $B = (D)^{1/n}$ which is what Oresme wished to show. Thus $A : B$, or $B : A$, is the same exponential part of F, namely $1/n$, as B is of D.

In general, the original ratios are expanded by a power equal to the denominator n of the exponent of the irrational ratio. The desired ratio is obtained by taking the nth root of the quotient resulting from the division of the expanded ratios. This is illustrated by the first example of the next paragraph, where Oresme subtracts (i.e. divides) $A = 4/3$ from (i.e. into) $B = (2/1)^{1/2}$. Here $D = 2/1$. Expanding both ratios by the denominator of the exponent, we obtain $[(2/1)^{1/2} : 4/3]^2 = 2/1 : 16/9$ where $16/9 = (4/3)^2 = C$; subtracting (i.e. dividing) the lesser from (i.e. into) the greater ratio by cross-multiplying (as we are told in Rule Two), we get $2/1 : 16/9 = 18/16 = 9/8$, which shows that $2/1 > 16/9$ (or $D > C$), since $2 \cdot 9 > 16 \cdot 1$, and indicates that the lesser ratio has been properly divided into the greater. Reverting to the original ratios we see that $(2/1)^{1/2} : 4/3 = (9/8)^{1/2}$.

[i.e. $(2/1)^{1/2}$]. I join [or combine] two sesquitertian ratios and obtain a ratio of 16 to 9, which is smaller than a double and must, therefore, be subtracted from [i.e. divided into] a double ratio leaving a sesquioctavan [i.e. 9/8] ratio. Hence, half of a sesquioctavan ratio [i.e. $(9/8)^{1/2}$] will remain after the subtraction of a sesquitertian from half of a double ratio.

Similarly, should you wish to subtract a third part of a double ratio [i.e. $(2/1)^{1/3}$] from a sesquialterate [i.e. 3/2], join [or combine] three sesquialterate ratios [i.e. $(3/2)^3$] to obtain a ratio of 27 to 8 from which the double ratio is [then] subtracted leaving a ratio of 27 to 16. Therefore, if a third part of a double is subtracted from a sesquialterate there will remain a third [part] of ratio 27 to 16 [i.e. $(27/16)^{1/3}$] which is a third part of what would remain if the whole double ratio were subtracted from three sesquialterate ratios.[29] This rule can be demonstrated easily since, generally, if a first ratio is subtracted from [i.e. divided into] a second, a third ratio is left, so that if a third part of the [same] first ratio is subtracted from [i.e. divided into] a third part of the [same] second ratio, there is left a third part of the [same] third ratio. This applies to half of any ratio, or a fourth part, or a fifth, and so forth.[30]

General Rule: In the addition of an irrational to an irrational ratio, and in the subtraction of an irrational from an irrational ratio, there are general rules applicable to any quantities whatever.

Let a known part of a known quantity be added to a known part of a known or measurable quantity. For example, let C be part of quantity A and D part of quantity B; and suppose that C is denominated by number E and D by number F. I then multiply A by F – i.e., I take A continuously F number of times – and produce G; in the same way I multiply B by E and obtain H. Hence C will be a part of G which is represented by [one part of] the number produced from the multiplication of E by F; and the same number will represent the part that D is of H. Therefore, as C is to G, so is D to H.[31]

Rule Seven. By addition [i.e. multiplication], it follows that as C is to G and D to H, so also is the total [or product] (*aggregatum*) of C and D to the total [or product] (*aggregatum*) of G and H. Consequently, the total

29 In this example we have $3/2 : (2/1)^{1/3}$. Now $[3/2 : (2/1)^{1/3}]^3 = 27/8 : 2/1 = 27/16$. Therefore $3/2 : (2/1)^{1/3} = (27/16)^{1/3}$.

30 Oresme concludes the sixth rule with a general statement that if A, B, and C are ratios and $A/B = C$, then $(A)^{1/n}/(B)^{1/n} = (C)^{1/n}$.

31 The steps outlined in Oresme's general rule are as follows. Let $C = (A)^{1/E}$, where A is a known quantity or ratio, and C is a known part of quantity or ratio A; and let $D = (B)^{1/F}$, where, similarly, B is a known quantity or ratio, and D is a known part of quantity or ratio B. Next take the integral exponent F and expand A so that $(A)^F = G$; in the same manner take integral exponent E and expand B so that $(B)^E = H$. Since $A = (G)^{1/F}$, it follows that $C = (G^{1/F})^{1/E}$, or $C = (G)^{1/EF}$, where C is an exponential part of ratio G. Similarly, since $B = (H)^{1/E}$ it follows that $D = (H^{1/E})^{1/F}$, or $D = (H)^{1/EF}$, where D is an exponential part of H. Therefore, ratio C is the same exponential part of ratio G that ratio D is of H. In the seventh rule, this general rule is applied to addition – i.e., multiplication – of two irrational ratios.

[or product] CD is a part of the total [or product] GH and is represented by [one part of] the number produced by [the multiplication of] E by F.[32]

Rule Eight. By subtraction, however, it follows that if G is subtracted from H, or conversely, and C is subtracted from D, or conversely, the remainder will be the same part of the remainder as part C was of G, or D was of H, and this part is represented by [one part of] the number resulting from the multiplication of E by F.[33]

As an example in the addition of irrational ratios, let half of a double [i.e. $(2/1)^{1/2}$] be added to a third part of a triple ratio [i.e. $(3/1)^{1/3}$]. On the one hand, I join or multiply three double ratios just as shown by the first rule. I do this because the other ratio is denominated by a three and is called a third part. On the other hand, and for the same reason and in the same way, I join [or multiply] two triples and then multiply the denominators of the parts, namely 2 by 3, and produce 6. Half of a double is, therefore, a sixth part of three doubles; similarly, a third part of a triple is a sixth part of two triples. Thus the total [or product] of half of a double and a third part of a triple is a sixth part of the total [or product] of three doubles and two triples. By the first rule it is obvious that such a product is a 72^{pla} ratio, namely 72 to 1. It follows, then, that from the addition [i.e. multiplication] of half of a double and a third part of a triple we obtain a sixth part of a 72^{ple} ratio [i.e. $(72/1)^{1/6}$].

As an example in subtraction [i.e. division] of irrational ratios, let half of a double ratio [i.e. $(2/1)^{1/2}$] be subtracted from a third part of a triple [i.e. $(3/1)^{1/3}$]. In the first place, and in accordance with the second rule, the product of three doubles [i.e. $(2/1)^3$] should be subtracted from [i.e. divided into] the product of two triple ratios [i.e. $(3/1)^2$], leaving a sesquioctavan ratio [i.e. 9/8]. Then, by subtracting a sixth part from a sixth part – i.e. half of a double from a third part of a triple ratio – there will be left a sixth part of a sesquioctavan ratio [i.e. $(9/8)^{1/6}$] since half of a double is a sixth part of three doubles [i.e. $[(2/1)^3]^{1/6}$] and a third part of a triple ratio is a sixth part of two triples [i.e. $[(3/1)^2]^{1/6}$]. Thus subtracting a sixth [part

[32] Since the general rule has shown that C is to G as D to H, by which Oresme means $C = (G)^{1/EF}$ and $D = (H)^{1/EF}$, in the seventh rule he multiplies C and D, obtaining the product $C \cdot D = (G \cdot H)^{1/EF}$. In an example directly following Rule Eight, Oresme adds (i.e., multiplies) two irrational ratios.

Let $C = (A)^{1/E} = (2/1)^{1/2}$ and $D = (B)^{1/F} = (3/1)^{1/3}$. Now $(A)^F = G = (2/1)^3$, so that $A = (G)^{1/F} = [(2/1)^3]^{1/3}$ and $C = (A)^{1/E} = (G^{1/F})^{1/E} = (2/1)^{1/2} = ([(2/1)^3]^{1/3})^{1/2} = [(2/1)^3]^{1/6}$. Similarly, $(B)^E = H = (3/1)^2$, so that $B = (H)^{1/E} = [(3/1)^2]^{1/2}$ and $D = (B)^{1/F} = (H^{1/E})^{1/F} = (3/1)^{1/3} = ([(3/1)^2]^{1/2})^{1/3} = [(3/1)^2]^{1/6}$. The product of $(2/1)^{1/2} \cdot (3/1)^{1/3} = [(2/1)^3]^{1/6} \cdot [(3/1)^2]^{1/6} = [(2/1)^3 \cdot (3/1)^2]^{1/6}$. Expanding each ratio by the first rule we obtain $8/1 \cdot 9/1 = 72/1$, which yields as a final product $(72/1)^{1/6}$. In brief, Oresme shows that $(A)^{1/E} \cdot (B)^{1/F} = [(A^F)^{1/EF} \cdot (B^E)^{1/EF}] = [(A)^F \cdot (B^E)^{1/EF}]$ or $[G \cdot H]^{1/EF}$.

In his representation of this rule, Curtze omits, or eliminates, the middle step, i.e. $[(A^F)^{1/EF} \cdot (B^E)^{1/EF}]$. This intermediate step was made quite explicit by Oresme.

[33] Here we have subtraction (i.e., division) of the ratios $D/C = (H/G)^{1/EF}$. In an example given immediately before the ninth rule, Oresme subtracts the same irrational ratios that were added in the previous example. Thus $(3/1)^{1/3} : (2/1)^{1/2} = [(3/1)^2]^{1/6} : [(2/1)^3]^{1/6} = (9/1)^{1/6} : (8/1)^{1/6} = (9/8)^{1/6}$. Generally, the eighth rule asserts that $(B)^{1/F} : (A)^{1/E} = (B^E)^{1/EF} : (A^F)^{1/EF} = (B^E : A^F)^{1/EF}$ or $(H : G)^{1/EF}$. As in the seventh rule, Curtze omits, or eliminates, the middle step, $(B^E)^{1/EF} : (A^F)^{1/EF}$.

of a ratio] from a sixth [part of another ratio] leaves a sixth [part] of what remains after the subtraction of [one] whole [ratio] from [the other] whole [ratio] — and this can be demonstrated easily.

Rule Nine. Now if the parts should have the same denomination, then it is easier to posit a special rule apart from the general rule already stated so that if a third part of A [i.e. $(A)^{1/3}$] is added to a third part of B [i.e. $(B)^{1/3}$] we obtain a third part of the result produced by the addition [i.e. multiplication] of A and B.[34] In a similar manner if a third part of A is subtracted from [i.e. divided into] a third part of B there will remain a third part of the remainder that was left after the subtraction of A from B.[35] Thus [for example] if a double ratio is added to a triple ratio, a sextuple ratio is produced, so that if half of a double [i.e. $(2/1)^{1/2}$] is added to [i.e. multiplied by] half of a triple [i.e. $(3/1)^{1/2}$] the result will be half of a sextuple [i.e. $(6/1)^{1/2}$].[36] Similarly, if a double ratio is subtracted from [i.e. divided into] a triple, a sesquialterate ratio is left, so that if a third part of a double ratio [i.e. $(2/1)^{1/3}$] is subtracted from [i.e. divided into] a third part of a triple [i.e. $(3/1)^{1/3}$], there remains a third part of a sesquialterate ratio [i.e. $(3/2)^{1/3}$]; [37] and the same holds in other cases.

Moreover, just as in other things, addition always proves subtraction, and conversely.[38] A ratio can be doubled, tripled, and multiplied — even sesquialterated — or increased proportionally as much as you wish by the addition [i.e. multiplication] of a ratio to a ratio. Thus if someone should want a ratio which is the sesquitertian of a double ratio, it would be necessary, by the fifth rule, to add a third part of a double ratio [i.e. $(2/1)^{1/3}$] to the double ratio so that a third part of a sedecuple [i.e. $(16/1)^{1/3}$] is obtained, which is a ratio that is the sesquitertian of a double ratio.[39] In the same way, subtraction allows a ratio to be subdoubled, subtripled, subsesquialterated,[40] and so on.

[34] $(A)^{1/3} \cdot (B)^{1/3} = (A \cdot B)^{1/3}$; or, generally, $(A)^{1/n} \cdot (B)^{1/n} = (A \cdot B)^{1/n}$. In effect, Oresme has eliminated the middle step enumerated at the conclusion of n. 32.

[35] $(B)^{1/3} : (A)^{1/3} = (B/A)^{1/3}$; or, generally, $(B)^{1/n} : (A)^{1/n} = (B : A)^{1/n}$. Here, again, Oresme has eliminated at least the middle step mentioned at the termination of n. 33. Curiously, in his representation of this, Curtze reverses the steps, giving $(a/b)^{1/n} = a^{1/n}/b^{1/n}$.

[36] Since $3/1 \cdot 2/1 = 6/1$, it follows that $(3/1)^{1/2} \cdot (2/1)^{1/2} = (6/1)^{1/2}$.

[37] Since $(3/1) : (2/1) = 3/2$, it follows that $(3/1)^{1/3} : (2/1)^{1/3} = (3/2)^{1/3}$.

[38] As shown in the next two notes, this converse relationship probably applies to addition and subtraction of exponents having the same base. In the *Algorismus de integris* ascribed to Master Gernardus, Rule 12 reads: "Addicio et subtractio se invicem probant." Gernardus tells us that if numbers $a + b = c$, and then if $c - b \neq a$, the addition was necessarily faulty; similarly, if $c - a = b$ and $a + b \neq c$, the subtraction was improperly performed. See G. Eneström, "Der 'Algorismus de integris' des Meisters Gernardus," *Bibliotheca Mathematica*, 1912/1913, *13*:301–302. Oresme's statement may be an application of this arithmetic relationship to addition and subtraction of exponents. Unfortunately, he offers no specific example to substantiate this interpretation.

[39] Here Oresme wishes to find ratio $(2/1)^{4/3}$ and invokes the fifth rule in which an irrational ratio is added to a rational ratio. Thus $(2/1) \cdot (2/1)^{1/3} = (2/1)^{4/3} = (16/1)^{1/3}$; or, generally, $(A)^{n/n} \cdot (A)^{1/n} = (A)^{(n+1)/n}$. Curtze summarizes this example as $a^m \cdot a^{1/n} = a^{m+1/n}$, but makes it more general than it is by failing to specify that m must equal 1. If we now wish to prove the addition we would subtract $(2/1)^{1/3}$ from $(2/1)^{4/3}$ and obtain $(2)^{4/3} : (2)^{1/3} = (2/1)$, the double ratio to which we initially added $(2/1)^{1/3}$

[40] For subtraction (i.e., division) $(A)^{n/n}$:

Indeed, one ratio cannot be multiplied (*multiplicatur*) or divided (*dividitur*) by another except improperly, as when two doubles are multiplied by two doubles to obtain four doubles. But this is nothing other than a multiplication of numbers, since the multiplication of two doubles [i.e. $(2/1)^2$] by two triples [i.e. $(3/1)^2$] comes to nothing, just as does the multiplication of a man by an ass. The same reasoning applies to division. Thus only addition and subtraction are appropriate types of algorism [i.e. operational procedures] for dealing with ratios [41] and we have discoursed sufficiently about them. The first tractate ends here.

$(A)^{m/n} = (A)^{(n-m)/n}$, where $m < n$ and both are integers. Addition proves this subtraction of ratios when we take $(A)^{m/n} \cdot (A)^{(n-m)/n} = (A)^{n/n}$. The terms "subdouble," "subtriple," and "subsesquialterate" are, in order, the fractional exponents 1/2, 1/3, and 2/3.

Curtze represents this as $a^m/a^{1/n} = a^{m-(1/n)}$. This fails in two ways: (1) it does not restrict m to the value 1; and (2) it restricts the divisor to $a^{1/n}$, contrary to Oresme's explicit statement that a ratio can be subtripled, which is possible only if $a^{2/3}$ is the divisor — i.e., $a : a^{2/3} = a^{1/3}$, where ratio $a^{1/3}$ is the subtriple of ratio a.

41 In this final paragraph, Oresme explains why he believes it proper to speak only of addition and subtraction, rather than multiplication and division, of ratios — and here he actually uses verb forms of *multiplicatio* and *divisio*. The distinction seems connected with the fact that exponents cannot be multiplied — except improperly, as in $(2/1)^2 \cdot (2/1)^2 = (2/1)^4$, where, coincidentally, the correct result is produced. But we cannot multiply the ratios $(2/1)^2$ and $(3/1)^2$ directly, for by analogy with $(2/1)^2 \cdot (2/1)^2$ we ought to obtain four of something. However, such a multiplication would yield something as unintelligible as the union of a man and an ass.

In adopting such terminology, perhaps Oresme was motivated by the fact that operations performed with exponents having the same base — as e.g., $(A)^m \cdot (A)^n = (A)^{m+n}$ and $(A)^m : (A)^n = (A)^{m-n}$ — required the addition and subtraction of exponents. It was then natural to think in terms of adding and subtracting ratios which are the very entities represented by the exponents. The altered terminology was then applied to cases involving exponents with different bases.

Multiplication, addition, subtraction, and division could be performed on numbers, whether or not those numbers denominated, or represented, ratios. But operations on ratios were restricted to multiplication and division, which are called addition and subtraction respectively. Although I have already confessed ignorance about the origins of this terminological switch, mention must be made of Wieleitner's unsuccessful attempt to conjecture such origins. He observes (*op. cit.*, p. 513) that in Euclid, Bk. V, Defs. 9 and 10, the ratio a^2/b^2 is called the "duplicate," and a^3/b^3 the "triplicate" ratio of a/b. From the use of such terminology, Wieleitner believes that the multiplication of two ratios came to be called "addition" and the division of two ratios "subtraction." (Years before, Tropfke offered this same historical explanation for Oresme's terminology [*op. cit.*, p. 207].) Admitting that no ancient sources reveal this shift, he notes that Gerard of Cremona, in his 12th-century translation of Anaritius' Euclid commentary, uses the terms *multiplicatio* and *aggregatio* synonymously. (Oresme uses *aggregatum* to mean a totality in the sense of a product.) Wieleitner does not pursue this line of thought, but obviously implies that *aggregatio* somehow became *additio*. But this would leave *additio* as a synonym, once removed, of *multiplicatio*, which, however, cannot be reconciled with Oresme's remark that "one ratio cannot be multiplied or divided by another except improperly." Since multiplication of ratios is not a proper operation in algorism of ratios, it can hardly be synonymous for addition insofar as the actual usage of the terms is concerned, even though the operations underlying both terms are identical. Furthermore, and even more important, if Oresme's terminology is derived ultimately from Euclid's *Elements*, why is such terminology wholly absent from Campanus of Novara's great 13th-century edition and commentary on Euclid's *Elements*, as well as from the Euclidean commentary literature generally?

An Arabic source is suggested when Wieleitner notes that Bassanus Politus mentions (*Questio de modalibus Bassani Politi* . . . [Venice, 1505]) that Al-Kindi, in a work called *De proportione et proportionalitate*, speaks of division of ratios as subtraction. This work, says Wieleitner, was probably available in Latin translation but is now lost. But perhaps, as appears more likely, Bassanus intended the *Epistola de proportione et proportionalitate* of Ametus Filius Iosephi, a widely known work available in Latin translation. If Ametus' work is meant, then Bassanus is mistaken since a careful examination of that treatise reveals no such usage (see the edition by Sister M. Walter Reginald Schrader, O. P., *The Epistola de proportione et proportionalitate of Ametus Filius Iosephi,* Ph.D. dissertation, Univ. of Wisconsin, 1961).

As the first work in which this terminology appears explicitly, Wieleitner offers (*op. cit.*, p. 513) the *Arithmetica* of Jordanus Nemorarius. Unfortunately, Wieleitner cites the *Arithmetica* from the 1496 edition of Jacques Le Fèvre in which, as we have already mentioned, the terms occur only in the demonstrations supplied by Le Fèvre, but are absent from the genuine *Arithmetica.* Wieleitner mistakenly assumed that Le Fèvre's demonstrations represented the actual text of the *Arithmetica.*

Indeed, it is by no means extravagant to suppose that Oresme himself may have originated this terminology and created a new genre of mathematical treatise, an *algorism of ratios* patterned after, but very different from, an *algorism of numbers.* Treatises of the latter kind, explaining and elaborating the four arithmetic operations as they are performed on numbers in the usual manner, were popular long before Oresme's time. But I am not aware of the existence of any algorism of ratios in the Oresmian sense prior to Oresme's *Algorismus proportionum.* The absence of such terminology in Jordanus' *Arithmetica* and its use by Jacques Le Fèvre in a much later commentary on that treatise point to a post-Jordanian origin, with Oresme himself as the possible originator. He may have derived his new, and peculiar, terminology from addition and subtraction of exponents, and then extended it to the simplest nonexponential cases (represented by the first two rules) in order to make the operational terminology for handling ratios as consistent as possible. If, on the other hand, Oresme inherited this terminology for the simplest cases, he may have extended it to operations involving integral and fractional exponents. The novel manner in which Oresme extended traditional Euclidean terminology to embrace exponential relations in the *De proportionibus* (see my forthcoming edition) renders such a conjecture perfectly credible.