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ORIGINAL ARTICLE

Lie-series for orbital elements: I. The planar case

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Abstract Lie-integration is one of the most efficient algorithms for numerical integration of ordinary differential equations if high precision is needed for longer terms. The method is based on the computation of the Taylor coefficients of the solution as a set of recurrence relations. In this paper, we present these recurrence formulae for orbital elements and other integrals of motion for the planar N-body problem. We show that if the reference frame is fixed to one of the bodies—for instance to the Sun in the case of the Solar System—the higher order coefficients for all orbital elements and integrals of motion depend only on the mutual terms corresponding to the orbiting bodies.

Keywords N-body problems · Numerical methods · Lie-integration · Planetary systems · Recurrence relations · Taylor coefficients

1 Introduction

Due to the lack of analytical solutions, numerical integration is required to solve the equations of motion of the gravitational N-body problem for almost any initial conditions for $3 \leq N$. There are many textbooks with algorithms related to general purpose numerical integration of ordinary differential equations (ODEs, see e.g., Press et al. 2002, for an introduction). In principle, if we have to solve the equation $\dot{x}_i = f_i(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_N)$, then the respective Lie-operator is defined as

$$L = \sum_{i=1}^{N} f_i \frac{\partial}{\partial x_i}. \tag{1}$$

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The solution of the equation after time Δt is then written in the form

$$x(t + \Delta t) = \exp(\Delta t \cdot L)\, x(t) = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} L^k x(t). \tag{2}$$

The finite approximation of the above sum is called Lie-integration (see also Gröbner and Knapp 1967). The higher order derivatives can efficiently be computed using recurrence relations where the derivatives $L^{k+1}x(t)$ are expressed as functions of $L^\ell x(t)$, where $0 \leq \ell \leq k$. The method has many advantages: It is one of the most efficient methods if we consider long-term and high precision computations, adaptive forms can be implemented without losing computation time, roundoff errors are smaller than other algorithms, etc. (see e.g., Pál and Süli 2007; Hanslmeier and Dvorak 1984). However, the need of derivations of the respective recurrence series for any new problem is a major drawback.

First, Hanslmeier and Dvorak (1984) have obtained the recurrence relations for the N-body problem, taking into account mutual and purely Newtonian gravitational forces. Soon after, the relations have been derived for the restricted three-body problem (Delva 1984). Many methods for stability analysis require the computation of linearized equations. The relations for the linearized N-body problem—including the equations where one of the bodies is fixed—have been presented by Pál and Süli (2007). The algorithm of Lie-integration has widely been applied for stability studies related to known planetary systems (see e.g., Asghari et al. 2004) or special resonant systems (see e.g., Funk et al. 2013). In addition, more sophisticated semi-numerical methods can be based on the Lie-series (see e.g., Pál 2010, about the numerical computation of partial derivatives of coordinates and velocities with respect to the initial conditions and the direct applications for exoplanetary analysis). Recently, Bancelin et al. (2012) published the relations extended with relativistic effects and some non-gravitational forces. It should be noted that Lie-integration does not handle regularization, i.e., equations are integrated in proper time by default. However, the method itself could be applied for regularized forms of the perturbed two-body problem (see e.g., Baù et al. 2013, for a review about recent methods). Due to its properties and implementation techniques, close encounters can be handled easily with Lie-series (see also Funk et al. 2013).

The aim of this paper is to present the recurrence relations for the osculating orbital elements and the mean longitude in the case of the planar N-body problem. Here we employ a reference frame where one of the bodies (i.e., the central body) has been fixed. Choosing this reference frame has the advantage that all of the bodies orbiting the center have constant osculating orbital elements *if* we neglect mutual interactions. As we show later on, all of the non-trivial terms depend purely on the mutual terms between the orbiting bodies. In other words, trivial cases yield constantly zero series for the Lie-coefficients. In Sect. 2, we summarize the relations for the fixed-center reference frame, following the notations of Hanslmeier and Dvorak (1984) and Pál and Süli (2007). The recurrence equations for constants of motion are derived in Sect. 3 while the relations for the mean longitude are obtained in Sect. 4. Our results and conclusions are summarized in Sect. 5.

2 Notations and Lie-series for the N-body problem

Throughout this paper, we follow the conventions used in Hanslmeier and Dvorak (1984) or Pál and Süli (2007). The Newtonian gravitational constant is denoted by G, the mass of the central body is M while the orbiting ones have a mass of m_i ($1 \leq i \leq N$, hence we deal with $1 + N$ bodies). Coordinates and velocities (with respect to the central body) are denoted by

$\mathbf{r}_i \equiv r_{ik}$ and $\mathbf{u}_i = u_{ik}$ (where $k = 1$ or 2) if we consider vector notations. The components of these vectors are denoted by $\mathbf{r}_i \equiv (x_i, y_i)$ and $\mathbf{u}_i \equiv (v_i, w_i)$. For simplicity, specific mass is denoted by $\mu_i \equiv G(M + m_i)$.

Based on Pál and Süli (2007), the relations for the fixed-center problem are the following series of equations. These are

$$L^{n+1}\mathbf{r}_i = L^n\mathbf{u}_i, \tag{3}$$

for the coordinates,

$$\begin{aligned} L^{n+1}\mathbf{u}_i = &-\mu_i \sum_{k=0}^{n} \binom{n}{k} L^k\phi_i L^{n-k}\mathbf{r}_i \\ &-G\sum_{j\neq i} m_j \sum_{k=0}^{n} \binom{n}{k} \left[L^k\phi_{ij}L^{n-k}(\mathbf{r}_i - \mathbf{r}_j) + L^k\phi_j L^{n-k}\mathbf{r}_j\right], \end{aligned} \tag{4}$$

for the velocities,

$$L^n\Lambda_i = \sum_{k=0}^{n} \binom{n}{k} L^k\mathbf{r}_i L^{n-k}\mathbf{u}_i, \tag{5}$$

$$L^n\Lambda_{ij} = \sum_{k=0}^{n} \binom{n}{k} L^k(\mathbf{r}_i - \mathbf{r}_j)L^{n-k}(\mathbf{u}_i - \mathbf{u}_j), \tag{6}$$

for the auxiliary quantities $\Lambda_i = \mathbf{r}_i\mathbf{u}_i$ and $\Lambda_{ij} = (\mathbf{r}_i - \mathbf{r}_j)(\mathbf{u}_i - \mathbf{u}_j)$, and

$$L^{n+1}\phi_i = \rho_i^{-2} \sum_{k=0}^{n} F_{nk}^{(-3)} L^{n-k}\phi_i L^k\Lambda_i, \tag{7}$$

$$L^{n+1}\phi_{ij} = \rho_{ij}^{-2} \sum_{k=0}^{n} F_{nk}^{(-3)} L^{n-k}\phi_{ij} L^k\Lambda_{ij}. \tag{8}$$

for the distances $\rho_i = |\mathbf{r}_i|$, the mutual distances $\rho_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ and the reciprocal cubic distances $\phi_i \equiv \rho_i^{-3}$, $\phi_{ij} \equiv \rho_{ij}^{-3}$. Here

$$F_{nk}^{(-3)} = -3\binom{n}{k} - 2\binom{n}{k+1}. \tag{9}$$

If we evaluate the above relations in the order of equations (3)–(8), for all values of $1 \le i \le N$ and then increase n by one in each step (thus starting over with $i = 1$, etc.), we obtain the Lie-terms for the coordinates and the velocities. The solution of the original ODE after Δt time can be approximated as

$$\mathbf{r}_i(t + \Delta t) \approx \sum_{n=0}^{n_{\max}} \frac{\Delta t^n}{n!} L^n\mathbf{r}_i(t), \tag{10}$$

$$\mathbf{u}_i(t + \Delta t) \approx \sum_{n=0}^{n_{\max}} \frac{\Delta t^n}{n!} L^n\mathbf{u}_i(t). \tag{11}$$

Note that for the last value of $n = n_{\max}$, we need only to evaluate equations (3) and (4). In order to bootstrap these relations, one could consider the fact that for any quantity Q, $L^0 Q \equiv Q$. Hence, the above definitions and relations for Λ_i and Λ_{ij} are self-explanatory.

In the following, we derive the relations for the integrals of motion, the orbital elements and the mean longitude.

3 Relations for the orbital elements

In order to introduce the features of the Lie-series for the classical Keplerian orbital elements, first, we compute the relations for the specific angular momentum,

$$C_i = \mathbf{r}_i \wedge \mathbf{u}_i = x_i \dot{y}_i - y_i \dot{x}_i = x_i w_i - y_i v_i. \tag{12}$$

Since the definition of C_i is similar to the relations for Λ_i (both are second-order *and* bilinear functions of the coordinates and velocities), one could expect a similar type of relations like equation (5). Indeed, the relations for the $L^n C_i$ terms can be written as

$$L^n C_i = \sum_{k=0}^{n} \binom{n}{k} L^k \mathbf{r}_i \wedge L^{n-k} \mathbf{u}_i = \sum_{k=0}^{n} \binom{n}{k} \left[L^k x_i L^{n-k} w_i - L^k y_i L^{n-k} v_i \right]. \tag{13}$$

Here, equations for the coordinates and velocities should be computed using Eqs. (3)–(8) up to some order of $n \leq n_{\max}$. In the case of $N = 1$, $L^n C_i$ must be equal to 0 for any $1 \leq n$ since $C_i \equiv C_1$ is an integral of motion. However, Eq. (13) does not imply this property. In order to obtain the values for $L^n C_i$, first we compute $L^1 C_i$:

$$L^1 C_i = L C_i = L(x_i w_i - y_i v_i) = (L x_i) w_i + x_i L w_i - (L y_i) v_i - y_i L v_i. \tag{14}$$

Since $L x_i = v_i$ and $L y_i = w_i$, we get

$$L C_i = v_i w_i + x_i L w_i - w_i v_i - y_i L v_i = x_i L w_i - y_i L v_i. \tag{15}$$

Now, Eq. (4) is substituted for $n = 1$:

$$\begin{aligned} L C_i = &+ x_i \left[-\mu_i \phi_i y_i - G \sum_{i \neq j} m_j [\phi_{ij}(y_i - y_j) + \phi_j y_j] \right] \\ &- y_i \left[-\mu_i \phi_i x_i - G \sum_{i \neq j} m_j [\phi_{ij}(x_i - x_j) + \phi_j x_j] \right]. \end{aligned} \tag{16}$$

By expanding the above summations and multiplications, the following can easily be seen. In addition to the Keplerian terms (the first ones, proportional to $\mu_i \phi_i$), one part of the terms corresponding to the direct perturbations also cancels. Therefore,

$$L C_i = G \sum_{i \neq j} m_j (\phi_{ij} - \phi_j)(x_i y_j - x_j y_i). \tag{17}$$

For higher orders, the set of relations can be written as

$$L^n S_{ij} = \sum_{k=0}^{n} \binom{n}{k} (L^k x_i L^{n-k} y_j - L^k x_j L^{n-k} y_i), \tag{18}$$

$$L^{n+1} C_i = G \sum_{i \neq j} m_j \sum_{k=0}^{n} \binom{n}{k} L^k \hat{\phi}_{ij} L^{n-k} S_{ij}, \tag{19}$$

where we introduce $S_{ij} = x_i y_j - x_j y_i$ and $\hat{\phi}_{ij} = \phi_{ij} - \phi_j$ for simplicity.

3.1 Eccentricity and longitude of pericenter

In the following, we compute the recurrence relations for the Lagrangian orbital elements $k = e\cos\varpi$ and $h = e\sin\varpi$. These are widely used as an equivalent alternative in astrodynamics studies instead of eccentricity, e and longitude of pericenter, ϖ. In the planar case, k and h are the components of the Laplace–Runge–Lenz vector:

$$\begin{pmatrix} k_i \\ h_i \end{pmatrix} = \frac{C_i}{\mu_i}\begin{pmatrix} +w_i \\ -v_i \end{pmatrix} - \frac{1}{\rho_i}\begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{20}$$

Due to the properties of the Lie-operator (linearity and Leibniz' product rule), the components of the above equation can easily be expanded once $L\rho_i^{-1}$ is known. Indeed, similarly to $\phi_i = \rho_i^{-3}$, it can be shown that

$$L\rho_i^{-1} = L\left[(\rho_i^2)^{-1/2}\right] = (-1/2)(\rho_i^2)^{-3/2}L(\rho_i^2) = -1/2\phi_i 2\Lambda_i = -\phi_i\Lambda_i, \tag{21}$$

see also Hanslmeier and Dvorak (1984) or Pál and Süli (2007). Now, our goal is to obtain a relation for k_i and h_i like equation (17) that contains only mutual terms. Right after multiplying equation (20) by μ_i, we got the relation

$$\mu_i Lk_i = (LC_i)w_i + C_i Lw_i - \mu_i\rho_i^{-1}Lx_i - \mu_i L(\rho_i^{-1})x_i. \tag{22}$$

Then, we have to substitute equations (17), (4), (21), w_i and $\phi_i(x_i^2 + y_i^2)$ for LC_i, Lw_i, $L(\rho_i^{-1})$, Lx_i and ρ_i^{-1}, respectively, and then perform a full expansion on Eq. (22). The Keplerian terms indeed cancel and the remaining parts can be written as

$$\mu_i Lk_i = G\sum_{i\neq j} m_j\left[\hat{\phi}_{ij}(w_i S_{ij} + C_i y_j) - C_i y_i \phi_{ij}\right] \tag{23}$$

Lh_i can be computed in a similar manner, thus the relations for $L(k_i, h_i)$ are

$$L\begin{pmatrix} k_i \\ h_i \end{pmatrix} = \sum_{i\neq j}\frac{Gm_j}{\mu_i}\left[\hat{\phi}_{ij}\begin{pmatrix} +w_i S_{ij} + C_i y_j \\ -v_i S_{ij} - C_i x_j \end{pmatrix} - C_i\phi_{ij}\begin{pmatrix} +y_i \\ -x_i \end{pmatrix}\right]. \tag{24}$$

In order to obtain higher order Lie-derivatives, $L^{n+1}(k_i, h_i)$, we should use Leibniz' product rule for the multilinear expressions appearing in the above relation. This can either be done directly using the multilinear form

$$L^n(Q_1 Q_2 \ldots Q_m) = \sum_{k_1+k_2+\cdots+k_m=n}\frac{n!}{k_1!k_2!\ldots k_m!}L^{k_1}Q_1 L^{k_2}Q_2\ldots L^{k_m}Q_m \tag{25}$$

or by introducing auxiliary quantities (e.g., $C_i y_j$, $w_i S_{ij}$) and subsequently apply the bilinear Leibniz' product rule for these ones.

3.2 Specific energy and semimajor axis

The specific energy is defined as

$$\varepsilon_i = \frac{U_i^2}{2} - \frac{\mu_i}{\rho_i}, \tag{26}$$

where $U_i = |\mathbf{u}_i| = \sqrt{v_i^2 + w_i^2}$. The semimajor axis can then be computed as $a_i = -\mu_i/(2\varepsilon_i)$. For simplicity, in the following, we compute relations for the quantity $H_i := -2\varepsilon_i = \mu_i/a_i$.

Using the relations for ρ_i^{-1} and the velocities (see Eq. 4), derivation schemes presented above yields

$$LH_i = 2\sum_{i\neq j} Gm_j \left[\phi_{ij}\Lambda_i - \hat{\phi}_{ij}\hat{\Lambda}_{ji}\right], \tag{27}$$

where we introduce $\hat{\Lambda}_{ji} = x_j v_i + y_j w_i$. The higher order Lie-derivatives are then obtained as it is described at the end of the previous section.

4 Relations for the mean longitude

The previously obtained relations for the orbital elements can applied not only for closed (circular or elliptic) orbits but for parabolic and hyperbolic orbits, as well. In the following, due to its relevance, we handle only closed orbits. Hence, eccentricity $e = \sqrt{k^2 + h^2}$ is expected to be smaller than unity for all orbits and the reciprocal semimajor axis $\mu/a = -2\varepsilon = H$ is also positive.

The mean longitude is the only related quantity which is defined for both circular and elliptical orbits *and* which is an analytic function of the coordinates and velocities (see e.g., Pál 2009). Therefore, in the following, we ignore the eccentric, mean and true anomalies from the computations. It should be noted that some quantities like $e \sin E$ or $e \cos E$ also behaves analytically in the $e \to 0$ limit, hence Lie-series can also be defined for these (where E denotes the eccentric anomaly, see e.g., Pál 2009).

4.1 Full expansion of the mean longitude

The mean longitude λ_i can be computed using the analytic equation

$$\lambda_i = \arg\left[+\hat{\rho}_i w_i + h_i\Lambda_i, -\hat{\rho}_i v_i - k_i\Lambda_i\right] - \frac{\Lambda_i}{C_i}J_i. \tag{28}$$

Here, we introduced $J_i = \sqrt{1 - e_i^2} = b_i/a_i$, the oblateness of the orbit and $\hat{\rho}_i = \rho_i(1 + J_i)$. Regarding to the differentiation, the $\arg(x, y)$ function behaves like the arc tangent function, $\mathrm{arc\,tg}(y/x)$:

$$\mathrm{d}\left[\arg(x, y)\right] = \mathrm{d}\left[\mathrm{arc\,tg}\left(\frac{y}{x}\right)\right] = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}. \tag{29}$$

The first-order Lie-derivative of λ_i is then

$$L\lambda_i = \frac{(\hat{\rho}_i v_i + k_i\Lambda_i)L(\hat{\rho}_i w_i + h_i\Lambda_i) - (\hat{\rho}_i w_i + h_i\Lambda_i)L(\hat{\rho}_i v_i + k_i\Lambda_i)}{(\hat{\rho}_i w_i + h_i\Lambda_i)^2 + (\hat{\rho}_i v_i + k_i\Lambda_i)^2} - L\left(\frac{\Lambda_i}{C_i}J_i\right). \tag{30}$$

The denominator of the first (apparently large) fraction can significantly be simplified to the form $(1 + J_i)^2 C_i^2$. Now, one has to simplify the above equation in order to depend mostly on the mutual interactions. Since $L\lambda_i = \dot{\lambda}_i = n_i \neq 0$ even for non-perturbed orbits, this simplification cannot be homogeneous with respect to Gm_j. In the following, we deal with the perturbed and non-perturbed terms separately and expand the above equation into two parts. The expansion of the numerator in the first fraction of Eq. (30) yields

$$\begin{aligned}&(\hat{\rho}_i v_i + k_i \Lambda_i) L(\hat{\rho}_i w_i + h_i \Lambda_i) - (\hat{\rho}_i w_i + h_i \Lambda_i) L(\hat{\rho}_i v_i + k_i \Lambda_i)\\ &\quad = \hat{\rho}_i^2 (v_i L w_i - w_i L v_i) + (\Lambda_i L\hat{\rho}_i - \hat{\rho}_i L \Lambda_i)(w_i k_i - v_i h_i)\\ &\qquad + \hat{\rho}_i \Lambda_i (v_i L h_i - w_i L k_i + k_i L w_i - h_i L v_i) + \Lambda_i^2 (k_i L h_i - h_i L k_i).\end{aligned} \tag{31}$$

The terms appearing above can be expanded as:

$$v_i L w_i - w_i L v_i = \mu_i \phi_i C_i + G \sum_{i \neq j} m_j \left[\phi_{ij} C_i - \hat{\phi}_{ij} \hat{C}_{ji}\right], \tag{32}$$

$$v_i h_i - w_i k_i = C_i \left(\frac{1}{\rho_i} - \frac{U_i^2}{\mu_i}\right), \tag{33}$$

$$v_i L h_i - w_i L k_i + k_i L w_i - h_i L v_i = -C_i \phi_i \Lambda_i - \sum_{i \neq j} G m_j \hat{\phi}_{ij} S_{ij} \left(\frac{1}{\rho_i} + \frac{U_i^2}{\mu_i}\right), \tag{34}$$

$$\begin{aligned}k_i L h_i - h_i L k_i = \sum_{i \neq j} \frac{G m_j}{\mu_i} \Bigg[&-\frac{C_i^2}{\mu_i} \hat{\phi}_{ij} \hat{C}_{ji} + \frac{C_i^3}{\mu_i} \phi_{ij}\\ &+ \frac{\hat{\phi}_{ij}}{\rho_i} (\Lambda_i S_{ij} + C_i R_{ij}) - C_i \rho_i \phi_{ij}\Bigg],\end{aligned} \tag{35}$$

$$\Lambda_i L\hat{\rho}_i - \rho_i L \Lambda_i = (1 + J_i)(\Lambda_i^2 \rho_i^{-1} - \rho_i L \Lambda_i) + \Lambda_i \rho_i L J_i \tag{36}$$

and

$$L \Lambda_i = \left(U_i^2 - \frac{\mu_i}{\rho_i}\right) + \sum_{i \neq j} G m_j \left[\hat{\phi}_{ij} R_{ij} - \phi_{ij} \rho_i^2\right]. \tag{37}$$

where $\hat{C}_{ji} = x_j w_i - y_j v_i$ and $R_{ij} = \mathbf{r}_i \mathbf{r}_j = x_i x_j + y_i y_j$.

Using the well-known relations from classical celestial mechanics, it can be shown that the double-negative specific energy, H_i relates to the oblateness J_i and the specific angular momentum C_i as $C_i^2 H_i = J_i^2 \mu_i^2$. From this relation, by taking the Lie-derivative of both sides, we got

$$L J_i = J_i \left(\frac{L C_i}{C_i} + \frac{L H_i}{2 H_i}\right). \tag{38}$$

Therefore, the last term in Eq. (30) can be written as

$$\begin{aligned}L\left(\frac{\Lambda_i}{C_i} J_i\right) &= -\frac{L C_i}{C_i^2} J_i \Lambda_i + \frac{J_i}{C_i} L \Lambda_i + \frac{\Lambda_i}{C_i} L J_i\\ &= -\frac{L C_i}{C_i^2} J_i \Lambda_i + \frac{J_i}{C_i} L \Lambda_i + \frac{\Lambda_i}{C_i} \frac{J_i}{C_i} L C_i + \frac{\Lambda_i}{C_i} \frac{J_i}{2 H_i} L H_i.\end{aligned} \tag{39}$$

Here, the first and third terms cancel each other, thus

$$L\left(\frac{\Lambda_i}{C_i} J_i\right) = \frac{J_i}{C_i} L \Lambda_i + \frac{J_i \Lambda_i}{2 C_i H_i} L H_i. \tag{40}$$

4.2 The non-perturbed part

From the above series of equations, we collect those where terms after the summation $\sum_{i\neq j} Gm_j(\cdot)$ do not occur. This part, denoted as $L\lambda_i|_0$ is

$$L\lambda_i|_0 = -\frac{J_i}{C_i}\left(U_i^2 - \frac{\mu_i}{\rho_i}\right) + \frac{1}{C_i^2(1+J_i)^2}\left\{\hat{\rho}_i^2\mu_i\phi_i C_i - \hat{\rho}_i\Lambda_i^2 C_i\phi_i - \left[\Lambda_i^2\rho_i^{-1}(1+J_i) - \hat{\rho}_i\left(U_i^2 - \frac{\mu_i}{\rho_i}\right)\right]C_i\left(\frac{1}{\rho_i} - \frac{U_i^2}{\mu_i}\right)\right\}. \tag{41}$$

By substituting the relations $U_i^2 - \mu_i/\rho_i = \mu_i/\rho_i - H_i$, $C_i^2 H_i = J_i^2\mu_i^2$ and $\Lambda_i^2 + C_i^2 = U_i^2\rho_i^2$, Eq. (41) can greatly be simplified to obtain Kepler's third law:

$$L\lambda_i|_0 = \frac{\mu_i^2 J_i^3}{C_i^3} = \frac{1}{\mu_i}H_i^{3/2} = \sqrt{\frac{\mu_i}{a_i^3}}. \tag{42}$$

4.3 The perturbed part

Let us write the full Lie-derivative of $L\lambda_i$ in the form

$$L\lambda_i = \frac{1}{\mu_i}H_i^{3/2} + \sum_{i\neq j} Gm_j[L\lambda]_{ij}. \tag{43}$$

This is similar to the forms obtained for the angular momentum, specific energy and Lagrangian orbital elements, with the exception of the presence of the term related to Kepler's third law. The goal now is to compute the terms $[L\lambda]_{ij}$ as simple as possible. It can be shown that this term is

$$[L\lambda]_{ij} = +\frac{\hat{\phi}_{ij}}{1+J_i}\left[\left(-\frac{2J_i(1+J_i)}{C_i} + \frac{2C_i}{\mu_i\rho_i}\right)R_{ij} - \left(\frac{\rho_i}{\mu_i} + \frac{C_i^2}{\mu_i^2}\right)\hat{C}_{ji}\right] + \frac{\phi_{ij}}{1+J_i}\left[\frac{C_i^3}{\mu_i^2} - \frac{C_i}{\mu_i}\rho_i + \frac{2J_i(1+J_i)}{C_i}\rho_i^2\right]. \tag{44}$$

The deduction of the above equation has the following steps. First, one should fully expand equation (31) while keeping only the terms $\sum Gm_j(\cdot)$. Then, it is divided by $(1+J_i)^2C_i^2$ after which we add the expansion of Eq. (40), still keeping only the terms $\sum Gm_j(\cdot)$. This Eq. (44) can be simplified in terms of computation implementation by introducing the dimensionless quantity $g_i = \mu_i\rho_i C_i^{-2}$:

$$[L\lambda]_{ij} = +\frac{\hat{\phi}_{ij}}{1+J_i}\left[\frac{2}{C_i}\left(g_i^{-1} - J_i(1+J_i)\right)R_{ij} - \frac{\rho_i}{\mu_i}\left(g_i^{-1}+1\right)\hat{C}_{ji}\right] + \frac{\phi_{ij}}{1+J_i}\frac{\rho_i^2}{C_i}\left[g_i^{-2} - g_i^{-1} + 2J_i(1+J_i)\right]. \tag{45}$$

Therefore, the first Lie-derivative of λ_i can be written as

$$L\lambda_i = \frac{1}{\mu_i}H_i^{3/2} + \sum_{i\neq j} Gm_j\left[\hat{\phi}_{ij}(A_{\mathrm{R}}R_{ij} + A_{\mathrm{C}}\hat{C}_{ji}) + \phi_{ij}A_0R_{ii}\right] \tag{46}$$

where

$$A_\mathrm{R} = \frac{2}{C_i}\left(\frac{g_i^{-1}}{1+J_i} - J_i\right), \tag{47}$$

$$A_\mathrm{C} = \frac{\rho_i}{\mu_i}\left(\frac{1+g_i^{-1}}{1+J_i}\right), \text{ and} \tag{48}$$

$$A_0 = \frac{1}{C_i}\left(\frac{g_i^{-2}-g_i^{-1}}{1+J_i} + 2J_i\right). \tag{49}$$

Higher order derivatives can then be computed using the relation

$$L^{n+1}\lambda_i = \frac{1}{\mu_i}L^n\left(H_i^{3/2}\right) = +\sum_{i\neq j} Gm_j \sum_{k+p+q=n} \frac{n!}{k!p!q!} \times \left[L^k\hat{\phi}_{ij}\left(L^p A_\mathrm{R} L^q R_{ij} + L^p A_\mathrm{C} L^q \hat{C}_{ji}\right) + L^k \phi_{ij} L^p A_0 L^q R_{ii}\right] \tag{50}$$

Let us suppose that the Lie-derivatives of the arbitrary quantity Q are known up to the order of $n+1$. It can be shown by mathematical induction that the $(n+1)$th Lie-derivative of Q^p can be computed using the relation

$$L^{n+1}Q^p = Q^{-1}\sum_{k=0}^{n}\left[p\binom{n}{k} - \binom{n}{k+1}\right]L^{n-k}\left(Q^p\right)L^{k+1}Q \tag{51}$$

By substituting $p = 3/2$, this relation can be used to compute $L^n H_i^{3/2}$ if higher order derivatives of H_i are known. In addition, Eq. (51) can be exploited in order to compute $(1+J_i)^{-1}$, C_i^{-1}, C_i^2 and g_i^{-2}. The additional terms A_R, A_C and A_0 depend only on the ith orbit. Hence, the relatively complex equations (47)–(49) are only computed N times in a single iteration, instead of $N^2/2$. Therefore, these calculations do not significantly increase the total computing time for larger number of bodies.

5 Conclusions and summary

In this paper, we presented recurrence formulae of the orbital elements related to the planar N-body problem. As we showed, the structure of these formulae depends only on the terms related to the mutual interactions. Therefore, the relations for the two-body problem reduces to a constant motion that can be integrated with arbitrary step size. It should be noted that although the presented procedure still requires the computation of higher order derivatives of coordinates and velocities, these relations are exploited as auxiliary equations for computing the mutual terms and these are not integrated directly.

In order to estimate the merits of using the orbital elements instead of the coordinates and velocities, we can compare, for instance, the magnitude of the terms $L^k C_i$ when these are computed using Eq. (13) or Eq. (19). In the unperturbed case, the latter one yields exactly zero while roundoff errors initiate an exponential growth in the higher order derivatives yielded by naive computation. Using double-precision arithmetic and bootstrapping with unity specific mass and angular momentum, the roundoff errors accumulate to unity around the order of $k \approx 19\ldots21$, depending on the initial eccentricity and orbital phase. In addition, for a given step size and desired precision, employing orbital elements instead of coordinate components decrease the integration order n_max. For weakly perturbed systems (like the inner

Solar System), this decrement can be a factor of $\sim$2. This would naively yield a gain of $\sim$4 in computing time due its $\mathcal{O}(n_{\max}^2)$ dependence. However, the additional computations needed by the orbital elements make a practical implementation less efficient. Our initial analysis also showed that the higher the perturbations, the less the gain in the integration order. In the case of the outer Solar System (where $m_i/M \lesssim 10^{-3}$), this gain in the decrease of the maximum of derivative order is less prominent.

Following studies could investigate the relations for the spatial problem. In some cases, this extension could be straightforward for scalar quantities like the specific energy. Care must be taken in the cases where pseudo-scalars (like C_i) or explicit coordinates occur. Another interesting point can be the elimination of the need for computing the recurrence formulae for coordinates and velocities and employ directly the orbital elements.

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References

Asghari, N., et al.: Stability of terrestrial planets in the habitable zone of Gl 777 A, HD 72659, Gl 614, 47 UMa and HD 4208. Astron. Astrophys. **426**, 353–365 (2004)

Bancelin, D., Hestroffer, D., Thuillot, W.: Numerical integration of dynamical systems with Lie series. Relativistic acceleration and non-gravitational forces. Celest. Mech. Dyn. Astron. **112**, 221–234 (2012)

Baù, G., Bombardelli, C., Peláez, J.: A new set of integrals of motion to propagate the perturbed two-body problem. Celest. Mech. Dyn. Astron. **116**, 53–78 (2013)

Delva, M.: Integration of the elliptic restricted three-body problem with Lie series. Celest. Mech. **34**, 145–154 (1984)

Funk, B., Dvorak, R., Schwarz, R.: Exchange orbits: an interesting case of co-orbital motion. Celest. Mech. Dyn. Astron. **117**, 41–58 (2013)

Gröbner, W., Knapp, H.: Contributions to the Method of Lie-Series. Bibliographisches Institut, Mannheim (1967)

Hanslmeier, A., Dvorak, R.: Numerical integration with Lie series. Astron. Astrophys. **132**, 203–207 (1984)

Pál, A.: An analytical solution for Kepler's problem. Mon. Not. R. Astron. Soc. **396**, 1737–1742 (2009)

Pál, A.: Analysis of radial velocity variations in multiple planetary systems. Mon. Not. R. Astron. Soc. **409**, 975–980 (2010)

Pál, A., Süli, Á.: Solving linearized equations of the N-body problem using the Lie-integration method. Mon. Not. R. Astron. Soc. **381**, 1515–1526 (2007)

Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P.: Numerical Recipes in C++: the Art of Scientific Computing, 3rd edn. Cambridge University Press, Cambridge (2002)