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INFINITESIMAL LORENTZ TRANSFORMATIONS AND THE STATES OF POLARIZATION OF FREE PHOTONS

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Applying the tangent dynamics technique as generated by infinitesimal Lorentz transformations, it is shown that the components of an electromagnetic field correspond to the Lie-Cartan parameters of the group SO(3,1). Thus the maximum number of non-null components of the magnetic field (*i.e.*, the number of the states of polarization) is not decided by the number of group generators.

Key words: Lie-Cartan parameters, electromagnetic field, states of polarizations.

1. ACTIVE LORENTZ TRANSFORMATIONS AND TANGENT KINEMATICS

An accelerated motion of a test particle (generally, on a curved trajectory) can be considered as generated by a succession of infinitesimal active Lorentz boosts and rotations (see, *e.g.*, [1]). The evolution of the four-velocity unit vector $u^\alpha(x^\beta) \equiv u^\alpha(\tau)$ $(\alpha = 0, 1, 2, 3)$, which is tangent to the world line and is defined by the relations

$$u^\alpha = \frac{dx^\alpha}{d\tau}, \quad u^\alpha u_\alpha = 1, \tag{1}$$

is given by

$$u^\alpha(\tau + d\tau) = M^\alpha{}_\beta u^\beta(\tau), \tag{2}$$

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where

$$M = e^{L} \approx I + L. \tag{3}$$

I is the 4×4 unit matrix and L is an infinitesimal Lorentz transformation matrix *i.e.*, an element of the SO(3,1) group. The transformation (2) can be called an *active Lorentz transformation* since we observe here a mapping of the four-vector $u^\alpha(\tau)$, defined at a point x, into the four-vector $u^\alpha(\tau + d\tau)$, defined at a point $x + dx$, both points being in the spacetime of a single observer.

The matrix L has the form

$$L \equiv L^\alpha{}_\beta = \delta v \cdot k - \delta\phi \cdot s = \begin{pmatrix} 0 & \delta v_1 & \delta v_2 & \delta v_3 \\ \delta v_1 & 0 & \delta\phi_3 & -\delta\phi_2 \\ \delta v_2 & -\delta\phi_3 & 0 & \delta\phi_1 \\ \delta v_3 & \delta\phi_2 & -\delta\phi_1 & 0 \end{pmatrix}, \tag{4}$$

where the parameter vectors

$$\delta v = (\delta v_1, \delta v_2, \delta v_3) \tag{5}$$

and

$$\delta\phi = (\delta\phi_1, \delta\phi_2, \delta\phi_3) \tag{6}$$

yield the six Lie-Cartan parameters of L. The matrices

$$k = (k_1,\ k_2,\ k_3) \quad \text{(generators of Lorentz boosts)} \tag{7}$$

and

$$s = (s_1,\ s_2,\ s_3) \quad \text{(generators of spatial rotations)} \tag{8}$$

are the six generators of the orthochronous proper Lorentz group SO(3,1) (see, *e.g.*, [2, 3]), *i.e.*,

$$k_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$k_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{10}$$

$$s_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

with the commutation relations

$$\left[s_i,\, s_j\right] = \varepsilon_{ijk} s_k, \quad \left[k_i,\, k_j\right] = -\varepsilon_{ijk} s_k, \tag{12}$$

$$\left[s_i,\, k_j\right] = \varepsilon_{ijk} k_k, \quad (i,\, j\; k = 1,\, 2,\, 3); \tag{13}$$

ε_{ijk} is the real three-dimensional alternating symbol (purely antisymmetric Levi-Civita tensor) and plays the role of commutation coefficients (numbers) for the basis of generators or structure constants of the restricted Lorentz group.

It is important to remember that the infinitesimal Lorentz transformation matrix $\boldsymbol{L}$ defines the *tangent kinematics* which describes the displacement motion which the particle traces out on its world line. We have next to define the *tangent dynamics*, *i.e.*, the determination of the forces required to generate the tangent kinematics.

2. TANGENT DYNAMICS AND THE STATES OF POLARIZATION OF FREE PHOTONS

Let us assume that a change $\delta \boldsymbol{v}$ of the velocity of a particle is produced by an external force field $\boldsymbol{e}(x)$ and a rotation $\delta\boldsymbol{\phi}$ is caused by another field $\boldsymbol{b}(x)$. Then we can write

$$\delta \boldsymbol{v} = C\boldsymbol{e}(x) d\tau, \tag{14}$$

$$\delta \boldsymbol{\phi} = C\boldsymbol{b}(x) d\tau, \tag{15}$$

where C is a constant to be determined. In terms of these fields, we derive, from the Lorentz transformation (2),

$$\begin{pmatrix} du^0 \\ du^1 \\ du^2 \\ du^3 \end{pmatrix} \approx C \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ e_1 & 0 & b_3 & -b_2 \\ e_2 & -b_3 & 0 & b_1 \\ e_3 & b_2 & -b_1 & 0 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} d\tau, \tag{16}$$

which, in a $3+1$ spacetime representation, can be written as

$$\frac{du^0}{d\tau} = C\boldsymbol{u} \cdot \boldsymbol{e}, \tag{17}$$

$$\frac{d\boldsymbol{u}}{d\tau} = C(\boldsymbol{e}u^0 + \boldsymbol{u} \times \boldsymbol{b}). \tag{18}$$

In a four-dimensional tensorial form these relations may be expressed as

$$\frac{du^\mu}{d\tau} = C\eta^{\mu\nu}\mathcal{F}_{\nu\sigma}u^\sigma \tag{19}$$

where $\mathcal{F}_{\nu\sigma}$ is given by

$$\mathcal{F}_{\nu\sigma} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -b_3 & b_2 \\ -e_2 & b_3 & 0 & -b_1 \\ -e_3 & -b_2 & b_1 & 0 \end{pmatrix} \tag{20}$$

and $\eta^{\mu\nu}$ is the Minkowskian metric tensor. Equations (19) reproduce the equations of motion of a test particle in a field generated by an antisymmetric tensor $\mathcal{F}_{\nu\sigma}$. If

$$C = \frac{q}{m} \tag{21}$$

and $\mathcal{F}_{\nu\sigma}$ is the electromagnetic field tensor

$$F_{\nu\sigma} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{22}$$

then Eqs. (19) are precisely the Lorentz equations of motion of a charged test particle under the action of an electromagnetic field.

The electromagnetic field tensor can be expressed in a similar form to the Lorentz matrix (4), namely

$$\boldsymbol{F} \equiv F^\alpha{}_\beta = \boldsymbol{E}\cdot\boldsymbol{k} - \boldsymbol{B}\cdot\boldsymbol{s} \tag{23}$$

where the six components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are now assigned the role of the Lie-Cartan parameters.

Thus, the magnetic field components correspond naturally to the Lie-Cartan parameters (associated with the generators of spatial rotations) and not to the three-dimensional basis (generators) of a linear vector space which may be defined here by the Lie algebra sO(3). Only if we were in a position to associate the magnetic field components with the three generators $\boldsymbol{s}$ of the spatial rotations would we be forced to admit the existence of a longitudinal polarization of photons.

We observe, when operating within the realm of the quantum theory, that the angular momentum does satisfy of course precisely the commutation relations of the generators of the rotation group, but there exists a problem of measurability of its components. It is well known that the only restrictions upon the measurability of the angular momentum of an atomic object are those which result from the commutation relations [4]

$$\left[J_i,\, J_j\right] = \mathrm{i}\hbar\varepsilon_{ijk} J_k, \tag{24}$$

$$\left[J_i,\, \phi_j\right] = -\mathrm{i}\hbar\delta_{ij}. \tag{25}$$

Relation (24) defines the domain of comparison of the eigenvalues of the Casimir operator

$$\boldsymbol{J}^2 = (J_1)^2 + (J_2)^2 + (J_3)^2 \tag{26}$$

having the classical magnitude of an angular momentum. As is well known, only in the limit of the principle of correspondence for large values of J with respect to $\hbar$ do the two quantities coincide, whereas in the quantum limit when $J = \hbar/2$, for example for an electron, they differ by a factor of $\sqrt{3}$.

Relation (25) indicates that the angular-momentum operator is the generator of infinitesimal rotations of the dynamical angle variables in a 3-dimensional Euclidean (physical) space. Furthermore, it represents a generalization of the classical Poisson bracket and leads to the uncertainty relation

$$\Delta J_i \Delta\phi_i \gtrsim \hbar. \tag{27}$$

It is important to note that this uncertainty relation when applied to an intrinsic spin along a given axis leads to the conclusion that it is impossible to associate a component of the spin with a rotation of the matter around the axis in question. This does not preclude by itself an observation of an angular momentum of the order of $\hbar$. The definition of the angular momentum leads also to the conclusion that the (longitudinal) component of an orbital angular momentum along the direction of motion is zero since

$$\boldsymbol{\mathcal{L}} \cdot \boldsymbol{p} = 0. \tag{28}$$

Thus, for longitudinal components, there exists a distinction between effects of the orbital angular momentum and those of the intrinsic spin which yield, in fact, the particle helicity. This elementary discussion indicates that the problem of measurability of longitudinal and transverse components which satisfy the angular momentum commutation relations, should be considered with caution.

We are faced here with two different physical quantities, the magnetic field and, respectively, the angular momentum which both can be associated with the rotation group SO(3) and its Lie algebra sO(3). However, there exists a clear difference: whereas the components of the angular momentum operator are associated with the matrices which form a basis of Lie algebra (or a representation of the infinitesimal generators of the group) the components of the magnetic field are associated with three Lie-Cartan parameters of the group of transformations. In the present case of the group SO(3) the number of generators of spatial rotations is precisely three, but the number of non-null Lie-Cartan parameters (and thus the number of the components of the magnetic field, and also the number of states of polarizations) depends on other physical constraints.

The final conclusion is that the states of polarizations of free photons correspond to the Lie-Cartan parameters and thus their maximum number is not decided by the number of generators of the group. This conclusion is in agreement with the transverse character of free photons which is a well-established result obtained by experiments and other theoretical deductions.

3. DISCUSSION

First let us mention that our contribution refers to the motion of a charged *test* particle (not of a charged macroscopic body as a plasma, a dielectric medium etc) which, by definition, does not change the configuration of an external (free) electromagnetic field. We can always couple the Lorentz equation for a charged test particle with Maxwell's equations for a free electromagnetic field (in a vacuum) in order to study the properties of the latter. Hence, our result refers to a free electromagnetic field.

Our result (association of electromagnetic field components with the Lie-Cartan parameters) is obtained (via the technique of tangent dynamics) applying Lorentz transformations and the Lorentz force equation. Since Lorentz equation is postulated separately and independently from the Maxwell field equations, we can assert that our result is also independent of Maxwell equations. We established that the magnetic field components correspond to the Lie-Cartan parameters (associated with the generators of spatial rotations) and not to the generators themselves whose number is fixed and equal to three. Thus, as mentioned also in the abstract, the (maximum) number of non-null components of the magnetic field (i.e., the number of the states of polarization) is not fixed by the number of group generators but by the number of Lie-Cartan parameters (associated with the generators of spatial rotations) which can generally be 0, 1, 2, 3, depending on the type of the field defined by the skew-symmetric second rank tensor which enters into the Lorentz equation. At this point, and for an electromagnetic field, the Maxwell equations are becoming relevant to the argument and determine what number of the sequence 0, 1, 2, 3, the nature chooses. And, as we infer from Maxwell's equations, the number is 2, that is free photons have a transverse character, a fact confirmed by experiments and other theoretical deductions. For the time being there are no known experiments which contradict transversality.

A longitudinal photon would possess a very small mass. The best available upper limit on the rest mass of a longitudinal photon is 10^{-53} gm or 10^{-60}gm (see, *e.g.*, [5-7]); we assert that there does not exist any laboratory equipment to confirm such a mass. "A small photon mass may lead to a catastrophic emission of longitudinal photons" [8]. We emphasize that some experiments refering to longitudinal components of electromagnetic waves are not related to free photons but to nonlinear interactions between a medium and electromagnetic waves in which a photon may achieve an "effective" mass due to the interaction. (We parenthetically note that a similar situation arises in the case of "free" electrons in metals where the electrons have an effective mass due to electron-electron and electron-phonon interactions.) And if indeed a free longitudinal photon will emerge in experiments, a theory is ready to explain this, namely the Proca theory. We note that the result of our work does not forbid the existence of the third polarization of the photon but shows that we are not obliged to admit the existence of a longitudinal photon within the framework of relativistic electrodynamics.

Our result can be obtained also independently of the Lorentz force. The electromagnetic field tensor has the same structure as the matrix of the Lie-Cartan parameters which correspond to infinitesimal Lorentz transformations. The classification of the infinitesimal Lorentz transformations (dual, screw-like etc) coincides with the classification of the electromagnetic field tensor. We can assert that there exists a mapping between vectors $(\boldsymbol{E}+i\boldsymbol{B})$ of the carrier space of the SO(3,C) representation matrices and the second rank skew-symmetric tensors $(F_{\alpha\beta})$ of the Minkowski spacetime. In all these analogies the components of the electromagnetic field are associated with the Lie-Cartan parameters.

When we study the electromagnetic field in term of vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, the number of the states of polarisation is fixed by the number of independent components of one of these vectors. Alternatively, if we study the electromagnetic field in terms of the four-potential A^α, the number of states of polarization is given by the number of components of A^α. Maxwell's equations and quantum theory decide how many of these states of polarization are physically realizable. In the case of a plane wave, the non-null components of the magnetic field vector $\boldsymbol{B}$ (as also of the electric field vector $\boldsymbol{E}$) are proportional to the non-null components of the four-potential A^α (or, equivalently, to the non-null components of the polarization vector, see, *e.g.*, Ref. [9], p. 105, eq. 3.42).

As is well known, Maxwell equations can be put in a manifestly covariant form by introducing the four-vector potential A^α which leads to four orthogonal polarization unit vectors (two are spacelike-transverse, one is spacelike-longitudinal, and one is timelike-scalar). There arise complications when quantising the electromagnetic field because this possesses only two independent components, but is covariantly described by the four components of A^α. In choosing two of these components as the physical ones, and thence quantising them, we loose evidently the covariance. If, on the other hand, we wish to keep covariance, we have two redundant components. Hence, at this point, we require a constraint condition by virtue of which unwanted photons (*e.g.* scalar photons which produce states with negative norms) in physical states are excluded. This constraint is represented by a gauge condition. We see that a gauge condition (such as the Coulomb gauge) *does not transversalize* the field but only expresses a physical and formal necessity. Gauge conditions are also currently applied in the non-Abelian gauge field theory.

We finally note that we are aware of the efforts of Evans and Vigier to find a magnetic field whose components are proportional to the SU(2)-group generators. Consequently this would signal the necessity of the existence of a longitudinal photon (see, *e.g.*, [10]). In simple models with groups of higher dimensions, the electromagnetic gauge symmetry $U(1)_{em}$ is always maintained as part of a larger symmetry since all known interactions conserve the charge. In general, however, this need not be the case and the $U(1)_{em}$ may temporarily be broken [11]. We appreciate the merits of the aforementioned authors in having introduced for the first time the idea of a field described by a light-like four-vector potential. This led us to the conclusion that such a field may be associated with a rotating body via the Kerr metric [12].

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