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On the complete symmetry group of the classical Kepler system

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A rather strong concept of symmetry is introduced in classical mechanics, in the sense that some mechanical systems can be completely characterized by the symmetry laws they obey. Accordingly, a "complete symmetry group" realization in mechanics must be endowed with the following two features: (1) the group acts *freely* and *transitively* on the manifold of all allowed motions of the system; (2) the given equations of motion are the *only* ordinary differential equations that remain invariant under the specified action of the group. This program is applied successfully to the classical Kepler problem, since the complete symmetry group for this particular system is here obtained. The importance of this result for the quantum kinematic theory of the Kepler system is emphasized.

I. INTRODUCTION

In this paper, we introduce an extended notion of symmetry in mechanics, which is somehow stronger than the concept of symmetry that has been used hitherto. As we shall see, this new idea of symmetry demands the fulfilment of more tightened conditions than those required by the traditional Noether and Lie theories of symmetries in classical mechanics.[1] The main feature underlying our approach cherishes the idea of characterizing a classical system by the symmetry laws it obeys, in a strictly specified manner. As a matter of principle, this means that different mechanical systems cannot have exactly the same symmetry properties, and if they do, then the systems must have essentially the same mechanical nature. In this sense, the group of symmetries characterizing a given system would be *complete*.

This is not the place to dwell on the most general (i.e., philosophical) aspects of this idea. Here, will work out in full detail, a physically relevant model fulfilling this program. It seems interesting to see how a complete symmetry group shows up in the case of the classical Kepler problem, and study both the geometric and mechanical properties of such a complete group. This interest settles the contents of this paper.

Let us then precisely state our program. Given the equation of motion of the Kepler system in ordinary space E_3,

$$\ddot{\mathbf{x}}+Kr^{-3}\mathbf{x}=0, \tag{1}$$

where K is a constant and $r=|\mathbf{x}|$, we here search for a group G_K of transformations of the variables t and $\mathbf{x}$ meeting the following conditions:

K1. The mappings $t\to t'$ and $\mathbf{x}\to\mathbf{x}'$ have well defined geometric structures corresponding to faithful realizations of some overlapping subgroups whose union forms a finite dimensional Lie group G_K;

K2. These transformations leave invariant the equation of motion (1), and therefore the subgroups of G_K act on the manifold of solutions W_K, changing one solution to another;

K3. The action of G_K (through its covering subgroups) on the whole manifold W_K, however, is *free* and *transitive*, so that W_K is a homogeneous space of G_K; and furthermore,

K4. the desired realizations of G_K are *specific* to the equation of motion (1), in the sense that *no other* equation of motion, say

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$$\ddot{\mathbf{x}}=\mathbf{F}(t,\mathbf{x},\dot{\mathbf{x}}), \tag{2}$$

remains invariant under the specified action of the considered covering subgroups of G_K, unless the force is given by $\mathbf{F}=Cr^{-3}\mathbf{x}$, where C is an arbitrary constant.

As we shall see in this paper, the endeavour of finding such a *complete symmetry group* can be achieved for Eq. (1). We call G_K the *Kepler group*, even when $K<0$. Note that conditions K1 and K2 are those considered in the current approach leading to point symmetries in mechanics. They afford the basic definitions of symmetry in configuration space–time,[2] on which the standard Lie method is founded, for in such case one usually finds a group of transformations formed by just one covering subgroup (i.e., the group itself). However, in general, such group realizations in mechanics fail to be complete, because conditions K3 or K4 (or both) are not satisfied.[3]

Whether such a specific realization of a complete symmetry group exists for any given Newtonian system, is not known. This question is left here as an open problem. In the special case of all one-dimensional *linear* Newtonian systems, it is well known that there always exists a specific realization of the projective group SL(3,R) in the plane (t,x) that meets all four conditions stated above.[4] For *nonlinear* one-dimensional systems,[5] however, the problem remains as yet unsolved.[6]

II. POINT SYMMETRY GROUP OF THE KEPLER PROBLEM

So the question arises as to the general fitness of the standard Lie method of point symmetries of ordinary differential equations in order to solve this problem. As is well known, in the case of Eq. (1) this method yields the following global change of variables:[7]

$$t'=\rho^{3/2}t+q^0, \quad x'^j=\rho R_{jk}x^k, \tag{3}$$

where ρ is an isotropic dilation, $0<\rho<\infty$, q^0 corresponds to time translation, $-\infty<q^0<+\infty$, and the coefficients R_{jk} denote the entries of a 3×3 proper orthogonal matrix. In fact, these are the *most general space–time point transformations* that keep invariant Eq. (1). They form a realization of a five-dimensional noncompact, connected and simply connected, Lie group; henceforth denoted by G_L.[8] The group law for the parameters $\{q^0,\rho,R_{jk}\}$ of G_L is as follows:

$$q''^0=q'^0+\rho'^{3/2}q^0, \tag{4a}$$

$$\rho''=\rho'\rho, \quad R''_{jk}=R'_{jl}R_{lk}. \tag{4b}$$

In particular, note the semidirect product combination of the isotropic space dilation with time translation. The subgroup defined by $q^0\equiv0$ [i.e., obeying Eqs. (4b)] shall be denoted by $G_L^{(0)}$.

The exhaustive analysis of the point symmetries of the Kepler problem is rather successful because two vector constants of motion are obtained from that study.[7] These are quite familiar; one usually takes: the angular momentum,

$$\mathbf{J}=\mathbf{x}\times\dot{\mathbf{x}}=\sqrt{KL}\,\hat{\mathbf{m}}\times\hat{\mathbf{n}}, \tag{5}$$

and the Runge–Lenz vector,

$$\mathbf{M}=\dot{\mathbf{x}}\times\mathbf{J}-Kr^{-1}\mathbf{r}=K\epsilon\hat{\mathbf{m}}, \tag{6}$$

where L and ϵ denote the *semilatus rectum* and the *eccentricity*, respectively, of the conic trajectory in the fixed plane $(\hat{\mathbf{m}},\hat{\mathbf{n}})$, with $\hat{\mathbf{m}}\cdot\hat{\mathbf{n}}=0$. In this fashion, the geometric and dynamical characteristics of the Kepler motion become related in a rather simple way. Note that the other constants of motion (like the energy and the Hamilton vector[9]) do not appear here as independent quantities.

Nonetheless, from the point of view of our program, G_L is very distressing for it fails to meet conditions K3 and K4. First, it can be proven that under transformations (3) one has

$$L' = \rho L, \quad \epsilon' = \epsilon, \tag{7}$$

quite generally.[7] All the other features characterizing a given Kepler motion can be changed by transformations (3), but the eccentricity of the orbits remains the same. This means that the action of G_L on W_K is not transitive. Furthermore, if one assumes that an equation of motion of the general form (2) is invariant under (3), one easily obtains

$$\frac{\partial F^j}{\partial t} = 0, \tag{8}$$

$$2x^k \frac{\partial F^j}{\partial x^k} - \dot{x}^k \frac{\partial F^j}{\partial \dot{x}^k} + 4F^j(\mathbf{x},\dot{\mathbf{x}}) = 0. \tag{9}$$

These are necessary and sufficient conditions to that end; besides the obvious provision that under pure rotations (i.e., $t' = t$, $x'^j = R_{jk}x^k$) the components of the force behave as a Cartesian vector, wherefrom it follows:

$$\epsilon_{klm}\left(x^m \frac{\partial F^j}{\partial x^l} + \dot{x}^m \frac{\partial F^j}{\partial \dot{x}^l} + \delta_{mj} F^l \right) = 0. \tag{10}$$

It can be shown that there are infinitely many forces $\mathbf{F}(\mathbf{x},\dot{\mathbf{x}})$ satisfying Eqs. (9) and (10).[7] Hence G_L as realized in Eq. (3) is not a *specific* group of symmetries characterizing the Kepler system in a unique manner.

So we epitomize: since G_L is the *maximal* group of space–time point symmetries of Eq. (1), the usual Lie method, by itself, is unable to discover the group G_K. Different tools are needed to this end. As we hope to show through this particular example, these tools may throw new light into the role played by group theory in classical mechanics.

III. THE COMPLETE KEPLER GROUP

According to the outlined program, let us then consider the following active infinitesimal transformation of world-lines in Newtonian space–time:[10]

$$t'[\mathbf{x}] = t + \delta q \int_{t_0}^{t} d\tau \, \xi[\tau, \mathbf{x}(\tau)], \tag{11a}$$

$$\mathbf{x}'(t') = \mathbf{x}(t) + \delta q \, \eta[t, \mathbf{x}(t)], \tag{11b}$$

where $\mathbf{x}(t)$ denotes an arbitrary curve in E_3, parametrized by t, and $\mathbf{x}(\tau)$ is the same curve for $t_0 \leqslant \tau \leqslant t$. Here, δq denotes an arbitrary parameter of smallness (i.e., $0 < \delta q \ll 1$), and t_0 is also arbitrary. By definition, under transformations (11) one considers $\mathbf{x}'(t')$ as the image-curve parametrized by t'. Thus, one sets an infinitesimal mapping of "motions into motions" for a one-particle system in E_3, for the scheme defined in Eqs. (11) is invertible. To be sure, these equations do not define point transformations in space–time; they entail infinitesimal *functional transformations* of world-lines.

However, from these equations it follows:

$$\frac{dt'}{dt} = 1 + \delta q \, \xi(t,\mathbf{x}), \quad \mathbf{x}' = \mathbf{x} + \delta q \, \eta(t,\mathbf{x}), \tag{12}$$

which cast them into a more useful form. Here, one has well-defined transformations in every space–time point, since all world-curves intersecting at the event $(t,\mathbf{x})$ yield the same values for

dt'/dt and x'^j, $j=1,2,3$. Hence we can handle Eqs. (12) in the usual manner, notwithstanding the fact that dt'/dt is not a total time derivative (unless ξ does not depend on $\mathbf{x}$). Indeed, one may interpret dt'/dt as a relative-rate of ticking of two local clocks in a small neighborhood of the event $(t,\mathbf{x})$.

Hence the laws of transformation for the velocity ($\dot{\mathbf{x}}=d\mathbf{x}/dt\rightarrow\dot{\mathbf{x}}'=d\mathbf{x}'/dt'$) and the acceleration ($\ddot{\mathbf{x}}=d\dot{\mathbf{x}}/dt\rightarrow\ddot{\mathbf{x}}'=d\dot{\mathbf{x}}'/dt'$), induced by transformations (12), read:

$$\dot{\mathbf{x}}'=\dot{\mathbf{x}}+\delta q(\dot{\eta}-\xi\dot{\mathbf{x}}), \tag{13}$$

$$\ddot{\mathbf{x}}'=\ddot{\mathbf{x}}+\delta q(\ddot{\eta}-\dot{\xi}\dot{\mathbf{x}}-2\xi\ddot{\mathbf{x}}). \tag{14}$$

In the same way, for $r=|\mathbf{x}|\rightarrow r'=|\mathbf{x}'|$, one has

$$r'=r(1+\delta q r^{-2}\boldsymbol{\eta}\cdot\mathbf{x}). \tag{15}$$

So there is no hindrance to a straightforward examination for new symmetries of the equations of motion from this perspective.

Let us then demand that, in particular, transformations (12) are such that they map Kepler motions into Kepler motions. After a few typical steps, substituting from Eqs. (12)–(15) into (1) one obtains the following necessary and sufficient condition to this end:

$$\frac{\partial^2\eta^j}{\partial t^2}+2\dot{x}^k\frac{\partial^2\eta^j}{\partial x^k\,\partial t}+\dot{x}^k\dot{x}^l\frac{\partial^2\eta^j}{\partial x^k\,\partial x^l}-\dot{x}^j\left(\frac{\partial\xi}{\partial t}+\dot{x}^k\frac{\partial\xi}{\partial x^k}\right)+Kr^{-3}\left(2x^j\xi-x^k\frac{\partial\eta^j}{\partial x^k}+\eta^j-3r^{-2}x^jx^k\eta^k\right)$$
$$=0. \tag{16}$$

Therefore, equating to zero the coefficients of the different powers of $\dot{x}^j$, one obtains a system of linear homogeneous partial differential equations for the determination of the generating functions ξ and η^j:

$$2\frac{\partial^2\eta^j}{\partial x^k\,\partial x^l}-\left(\delta^j_k\frac{\partial\xi}{\partial x^l}+\delta^j_l\frac{\partial\xi}{\partial x^k}\right)=0, \tag{17a}$$

$$2\frac{\partial^2\eta^j}{\partial t\,\partial x^k}-\delta^j_k\frac{\partial\xi}{\partial t}=0, \tag{17b}$$

$$\frac{\partial^2\eta^j}{\delta t^2}+Kr^{-3}\left(2x^j\xi-x^k\frac{\partial\eta^j}{\partial x^k}+\eta^j-3r^{-2}x^jx^k\eta^k\right)=0. \tag{17c}$$

Let me remark that these equations exhibit some notable differences with the equivalent equations one obtains for the generating functions of the point symmetries (3) of Eq. (1). (Especially, we note that the second order partial derivatives of ξ are missing here.)

To proceed further, we have to integrate these equations. This can be done rather easily. (We present some details of the integration procedure in Appendix A.) The general solution to Eqs. (17) reads:

$$\xi(\mathbf{x})=\tfrac{3}{2}A-2C^kx^k, \tag{18a}$$

$$\eta^j(\mathbf{x})=Ax^j-B^l\epsilon_{ljk}x^k-C^kx^kx^j, \tag{18b}$$

where A,B^j,C^j are constants of integration. So, defining the parameters: $\delta\rho=A\,\delta q$, $k^j\,\delta\phi=B^j\,\delta q$, and $\delta q^j=C^j\,\delta q$, we have found:

$$\frac{dt'}{dt}=1+\frac{3}{2}\,\delta\rho-2\,\boldsymbol{\delta}\mathbf{q}\cdot\mathbf{x}, \tag{19a}$$

$$\mathbf{x}'=\mathbf{x}+\delta\rho\mathbf{x}+\delta\phi\hat{\mathbf{k}}\times\mathbf{x}-(\boldsymbol{\delta}\mathbf{q}\cdot\mathbf{x})\mathbf{x}. \tag{19b}$$

One interesting feature of these equations is that they are a generalization of the Lie method's result for the same problem, owing to the presence of the terms containing $\boldsymbol{\delta}\mathbf{q}\cdot\mathbf{x}$. If one sets $\boldsymbol{\delta}\mathbf{q}=\mathbf{0}$ in Eqs. (19) and integrates (19a), one recovers exactly the well-known infinitesimal point symmetry result corresponding to G_L. The reader can prove in a few lines that Eq. (1) remains invariant under the new transformations (19).

Of course, under a wide variety of circumstances Eqs. (19) can be integrated to yield well-defined transformations of any given world-curve $\mathbf{x}(t)$ in space–time:

$$t'[\mathbf{x}]=t+\frac{3}{2}\,\delta\rho t+\delta q^0-2\,\boldsymbol{\delta}\mathbf{q}\cdot\int_{t_0}^{t}d\tau\,\mathbf{x}(\tau), \tag{20a}$$

$$\mathbf{x}'(t')=\mathbf{x}(t)+\delta\rho\mathbf{x}(t)+\delta\phi\hat{\mathbf{k}}\times\mathbf{x}(t)-[\boldsymbol{\delta}\mathbf{q}\cdot\mathbf{x}(t)]\mathbf{x}(t). \tag{20b}$$

In particular, under these infinitesimal world-line transformations one has: $\mathbf{x}(t)\in W_k\Rightarrow\mathbf{x}'(t')\in W_k$. However, these are not point transformations and one does not obtain from them a set of differential operators satisfying a Lie algebra, as in the standard manner. This is not to say that they are useless.

The finite transformations are obtained by exponentiation of the corresponding infinitesimal ones, as usual. The exponentiations for getting $\delta\rho\rightarrow\rho$ and $\delta\phi\rightarrow\phi$ are immediate, because the respective infinitesimal transformations in Eqs. (19) are *linear* in x^j. In fact, we get Eqs. (4b) for the realization of $G_L^{(0)}$ in the present context, where $R_{jk}=R_{jk}(\phi,\hat{\mathbf{k}})$ has an obvious meaning. Thus one recovers $G_L^{(0)}$ as a subgroup of the complete group G_K we are looking for.

The exponentiation for obtaining $\delta q^k\rightarrow q^k$ is a little more elaborate, owing to the presence of the term $(\boldsymbol{\delta}\mathbf{q}\cdot\mathbf{x})\mathbf{x}$ in Eq. (19b) which is *bilinear* in x^j. (We briefly solve this problem in Appendix B.) The finite transformations generated by $\mathbf{q}$ are given by

$$\frac{dt'}{dt}=\frac{1}{(1+\mathbf{q}\cdot\mathbf{x})^2},\quad \mathbf{x}'=\frac{\mathbf{x}}{1+\mathbf{q}\cdot\mathbf{x}}. \tag{21}$$

These transformations constitute a realization of a Lie group $G_\Gamma^{(0)}$ acting on E_3. The parameters $\mathbf{q}=(q^1,q^2,q^3)$ are canonical, and the group law of $G_\Gamma^{(0)}$ is simply given by: $q''^j=q'^j+q^j$, $e^j=0$, $\bar{q}^j=-q^j$. $G_\Gamma^{(0)}$ is isomorphic to the group T_3 of rigid translations of a Cartesian scaffolding in E_3 (i.e., $\mathbf{x}\rightarrow\mathbf{x}'=\mathbf{x}+\mathbf{q}$), although the realization $\mathbf{x}\rightarrow\mathbf{x}'=(1+\mathbf{q}\cdot\mathbf{x})^{-1}\mathbf{x}$ defined in Eqs. (21) has a completely different geometric meaning. We further discuss this subject in Appendix C.

It can be proved easily that Eq. (1) is indeed invariant under (21). However, under transformations (3) one *preserves the eccentricity* of the orbits, $\epsilon'=\epsilon$ [cf. Eq. (6)], while under (21) *the eccentricity changes* $\epsilon'\neq\epsilon$ (cf. Appendix C). Let us then consider the *semidirect product* $G_\Gamma=G_L^{(0)}\times G_\Gamma^{(0)}$, which becomes realized as the following point transformations in E_3:

$$\frac{dt'}{dt}=\frac{\rho^{3/2}}{(1+\mathbf{q}\cdot\mathbf{x})^2},\quad x'^j=\frac{\rho R_{jk}x^k}{1+\mathbf{q}\cdot\mathbf{x}}. \tag{22}$$

The group law obeyed in $G_\Gamma=\{R_{jk},\rho,q^j\}$ reads as in Eqs. (4b), but one also has to add the following combination law

$$q''^j=\rho R_{kj}q'^k+q^j. \tag{23}$$

G_Γ is indeed the group of homofocal conics reviewed in Appendix C. The local relative rate of ticking dt'/dt assures that if a particle performs a Kepler motion $\mathbf{x}(t)$ along a given conic, in terms of time t, then the image particle performs a motion $\mathbf{x}'(t')$, in terms of time t', along the image conic, which is also a Kepler motion, and *vice versa*. Furthermore, it can be proven that if one performs two successive transformations (22), with parameters (R_{jk},ρ,q^j) and (R'_{jk},ρ',q'^j), one gets

$$\frac{dt''}{dt}=\frac{dt''}{dt'}\frac{dt'}{dt}, \tag{24}$$

as it must be.

The transformation laws for the velocity, $\dot{\mathbf{x}}\to\dot{\mathbf{x}}'$, and the acceleration, $\ddot{\mathbf{x}}\to\ddot{\mathbf{x}}'$, induced by Eqs. (22), are given by

$$\dot{x}'^j=\rho^{-1/2}R_{jk}[\dot{x}^k+q^l(x^l\dot{x}^k-\dot{x}^lx^k)], \tag{25}$$

$$\ddot{x}'^j=\rho^{-2}(1+\mathbf{q}\cdot\mathbf{x})^2R_{jk}[\ddot{x}^k+q^l(x^l\ddot{x}^k-\ddot{x}^lx^k)]. \tag{26}$$

Clearly, for $r=|\mathbf{x}|\to r'=|\mathbf{x}'|$ one has

$$r'=\frac{\rho r}{1+\mathbf{q}\cdot\mathbf{x}}. \tag{27}$$

In this fashion, whenever $\ddot{x}^j+Kr^{-3}x^j=0$, it follows:

$$\ddot{x}'^j=-K\rho^{-2}(1+\mathbf{q}\cdot\mathbf{x})^2r^{-3}R_{jk}x^k=-Kr'^{-3}x'^j, \tag{28}$$

as it was already remarked. This is of course one of our main results.

Hence, let us consider the following eight-dimensional Lie group

$$G_K=G_L\cup G_\Gamma=\{q^0,\rho,R_{jk},q^j\}, \tag{29}$$

with the group law given by

$$q''^0=q'^0+\rho'^{3/2}q^0, \tag{30a}$$

$$\rho''=\rho'\rho, \tag{30b}$$

$$R''_{jk}=R'_{jl}R_{lk}, \tag{30c}$$

and

$$q''^j=\rho R_{jk}q'^k+q^j. \tag{30d}$$

The group manifold is $M_K=\{-\infty<q^0<+\infty,\ 0<\rho<\infty,\ R_{jk}\in O_+(3),\ -\infty<q^j<+\infty\}$ and the identity is at the point $e=(0,1,\delta_{jk},\mathbf{0})\in M_K$. The reader can check Eqs. (30) against the group property. All the properties of G_K follow from this law. $G_L=\{q^0,\rho,R_{jk}\}$ and $G_\Gamma=\{\rho,R_{jk},q^j\}$ are two intersecting subgroups of G_K, whose union covers G_K completely (by definition). They are both noncompact, connected and simply connected, non-Abelian Lie groups. G_L acts as a group of point transformations in *Newtonian space–time*, as defined in Eqs. (3); G_Γ is a group of point transformations in ordinary *Euclidean space*, whose action is defined in Eqs. (C8), enhanced with the point transformation providing for the local rate dt'/dt in E_3 as shown in Eqs. (22). In this way, both subgroups keep invariant Eq. (1). It will be shown further that, given these point symmetry realizations of the two covering subgroups, G_K is indeed the Kepler group.

One can formally integrate Eqs. (22) along a given curve $\mathbf{x}(t)$, to read:

$$t'[\mathbf{x}]=\rho^{3/2}\int_{t_0}^{t}\frac{d\tau}{[1+\mathbf{q}\cdot\mathbf{x}(\tau)]^2}, \tag{31a}$$

$$x'^j(t')=\frac{\rho R_{jk}x^k(t)}{1+\mathbf{q}\cdot\mathbf{x}(t)}. \tag{31b}$$

These formulas entail transformations of worldlines in Newtonian space–time, in general. They are endowed with the following special properties: (a) $\mathbf{x}(t)\in W_K \Rightarrow \mathbf{x}'(t')\in W_K$; moreover, (b) given $\mathbf{x}(t)$, $\mathbf{x}'(t')\in W_K$, there always exist a transformation (31) such that $\mathbf{x}(t)\rightarrow\mathbf{x}'(t')$, and also an "inverse" one such that $\mathbf{x}'(t')\rightarrow\mathbf{x}(t)$; furthermore (c) if one sets $\mathbf{x}(t)\rightarrow\mathbf{x}'(t')\rightarrow\mathbf{x}''(t'')$ under two successive transformations (31), starting with $\mathbf{x}(t)\in W_K$, there always exists a transformation (31) that yields $\mathbf{x}(t)\rightarrow\mathbf{x}''(t'')$ directly. So these transformations afford a *functional realization* of G_K acting freely and transitively on the manifold W_K of all solutions to Eq. (1). However, to handle the group property of G_K directly in terms of this functional realization [in particular, to study the Lie algebra (cf. below)] becomes rather cumbersome; for these matters, we prefer to use the nonintegrated point symmetries shown in Eqs. (22).

Nevertheless, from the point of view of the Kepler kinematics, Eqs. (31) are interesting. For instance, starting with a Kepler uniform circular motion, of radius L and angular frequency $\omega = \sqrt{KL^{-3}}$ [see Eq. (C5), with $\alpha=\omega t$], one can evaluate the integral

$$t'=\rho^{3/2}\int_{0}^{t}\frac{d\tau}{(1+qL\cos\omega\tau)^2} \tag{32a}$$

(here we take $\mathbf{q}=q\hat{\mathbf{m}}$), in order to obtain the resulting motion

$$\mathbf{x}'(t')=\frac{\rho L(\hat{\mathbf{m}}\cos\omega t+\hat{\mathbf{n}}\sin\omega t)}{1+qL\cos\omega t} \tag{32b}$$

(let us omit the rotation), which turns out to be a Kepler motion in terms of t', with an elliptic, parabolic, or hyperbolic orbit (with $\epsilon=qL$).[11] We left this problem as an exercise to the interested reader. The transformations into rectilinear Kepler motions ($J=0$) are much more difficult to analyze, because one must use limiting processes which force the group to its most extreme singular elements.

We now come to the heart of the matter (i.e., condition K4). Let us look for the most general equation of motion

$$\ddot{x}^j=F^j(\mathbf{x},\dot{\mathbf{x}}) \tag{33}$$

[cf. Eq. (8)] that remains invariant under both covering subgroups G_L *and* G_Γ, acting according to their special realizations defined in Eqs. (3) and (22), respectively. To begin with, since $G_L\cap G_\Gamma\subset G_\Gamma$, we need to discuss the invariance of Eq. (33) under $G_\Gamma^{(0)}$ [i.e., Eq. (21)]. Thus we set:

$$\dot{x}'^j=\dot{x}^j+q^k(x^k\dot{x}^j-\dot{x}^kx^j), \tag{34}$$

$$\ddot{x}'^j=(1+\mathbf{q}\cdot\mathbf{x})^2[\ddot{x}^j+q^k(x^k\ddot{x}^j-\ddot{x}^kx^j)], \tag{35}$$

and we require

$$(1+\mathbf{q}\cdot\mathbf{x})^2[\ddot{x}^j+q^k(x^k\ddot{x}^j-\ddot{x}^kx^j)]=F^j[(1+\mathbf{q}\cdot\mathbf{x})^{-1}\mathbf{x},\dot{\mathbf{x}}-\mathbf{q}\times(\mathbf{x}\times\dot{\mathbf{x}})]. \tag{36}$$

In this way, taking $\lim_{\mathbf{q}\to 0}(\partial/\partial q^j)$ and assuming (33), one obtains:

$$x^k x^l \frac{\partial F^j}{\partial x^l} + \epsilon_{klm} J^l \frac{\partial F^j}{\partial \dot{x}^m} + 3x^k F^j - x^j F^k = 0. \tag{37}$$

This means that the necessary and sufficient conditions for Eq. (33) to be invariant under G_L and G_Γ is that the force $\mathbf{F}(\mathbf{x},\dot{\mathbf{x}})$ must satisfy Eqs. (9), (10), and (37) simultaneously. We solve these equations in Appendix D. The final solution is given in Eqs. (D2), (D9), and (D14). This finishes our task, for we have proven that G_K is indeed the desired *complete Kepler group.*

Finally, some few comments on the Lie algebra $\mathbf{g}_K$ of G_K briefly follow. According to Eqs. (3) and (C8), we introduce the operators generating the respective point transformations. These are Lie's vector fields given by

$$T \equiv \frac{\partial}{\partial t}, \quad D \equiv \frac{3}{2}\, t \frac{\partial}{\partial t} + x^k \frac{\partial}{\partial x^k},$$
$$L_j \equiv \epsilon_{jkl} x^k \frac{\partial}{\partial x^l}, \quad \Gamma_j \equiv -x^j x^k \frac{\partial}{\partial x^k}. \tag{38}$$

Thus, one obtains:

$$[T,D] = \tfrac{3}{2}T, \quad [T,L_j] = 0 \quad [T,\Gamma_j] = 0,$$
$$[D,L_j] = 0, \quad [D,\Gamma_j] = -\Gamma_j, \tag{39}$$
$$[L_j,L_k] = \epsilon_{jkl} L_l, \quad [L_j,\Gamma_k] = \epsilon_{jkl}\Gamma_l, \quad [\Gamma_j,\Gamma_k] = 0.$$

This closed algebra seems interesting, owing to the following feature. The operator

$$\mathbf{\Gamma}^2 = \Gamma_k \Gamma_k \tag{40}$$

yields

$$[D,\mathbf{\Gamma}^2] = -2\mathbf{\Gamma}^2, \tag{41}$$

and commutes with all the other operators of the algebra. If one then defines the operators

$$A_j = \epsilon_{jkl} L_k \Gamma_l - \Gamma_j \tag{42}$$

the following closed commutation relations can be proved by means of Eq. (39):

$$[L_j,L_k] = \epsilon_{jkl} L_l, \quad [L_j,A_k] = \epsilon_{jkl} A_l, \quad [A_j,A_k] = \mathbf{\Gamma}^2 \epsilon_{jkl} L_l, \tag{43}$$

where

$$[\mathbf{\Gamma}^2,L_j] = [\mathbf{\Gamma}^2,A_j] = 0. \tag{44}$$

This result is very reassuring in the present context, because in any *irreducible representation* of the Lie algebra [Eqs. (43)], $\mathbf{\Gamma}^2$ becomes a *multiple of the identity* and Eqs. (43) become essentially into the Lie algebra of the four-dimensional rotation group $O_+(4)$.[12] These and other interesting details of the Lie algebra $\mathbf{g}_K$ (as well as the invariants and conservation laws) associated with the complete Kepler group G_K shall be discussed elsewhere.

IV. CONCLUDING REMARKS

Perhaps the main interest of this paper lies on the *method* introduced to obtain the complete symmetry group of a system. One complements the familiar Lie method (for obtaining the point symmetries of the equations of motion) with a local "dt'/dt transformation" that is manifestly *not* a total time derivative,[13] and which can be integrated only along given world curves. By the same reason, it seems rather natural to conjecture that the method (as it stands) does not work for nonconservative systems acted on by time-dependent applied forces. However, *conservative systems* seem worthwhile to be analyzed by this approach in order to find their complete symmetry groups (if any). For instance, it would be interesting to see how a complete symmetry group shows up in the case of the three-dimensional harmonic oscillator (both, isotropic and anisotropic), in the case of the classical Helium-atom system, or even in other more complicated physically relevant conservative mechanical systems.

To finish this work, I would like to remark that my own interest in *complete symmetry groups* stems from *quantum kinematics*.[14] Non-Abelian quantum kinematics is a "group-theoretic quantization" program in which one builds a quantum model of a system directly by means of its characteristic symmetries, without recourse to any thought out prequantized classical analog.[15] Thus, the familiar Schrödinger equation, as well as the respective propagator kernel, have been *deduced* in this way from the complete symmetry group of the simple harmonic oscillator,[16] and also from the Galilei symmetry of a Newtonian free particle.[17] In the present example, it is rather clear that if one quantizes the group G_L by no means would one produce a quantum model of the Kepler system, because G_L as defined in Eqs. (3) is not complete. On the other hand, the quantum kinematic theory of the complete group G_K of the Kepler system will certainly produce a quantum model of *that* system, and of nothing else. This seems to be an intriguing endeavour worthy to be done. Work is in progress concerning this issue. We expect to tackle the problem set by G_K quantum kinematics in a forthcoming paper.

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APPENDIX A: SOLVING FOR THE INFINITESIMAL SYMMETRY TRANSFORMATIONS

Although the task is not difficult, for the sake of completeness we here append the integration of Eqs. (17).

Equations (17a) can be easily cast in the form

$$2\epsilon_{jlm}\eta_{l,mk}=\epsilon_{jkl}\xi_{,l}, \tag{A1}$$

which yields

$$2\epsilon_{jlm}\eta_{l,mkn}=\epsilon_{jkl}\xi_{,ln}=\epsilon_{jnl}\xi_{,lk}, \tag{A2}$$

wherefrom one obtains

$$\xi_{,jk}+\delta_{jk}\xi_{,ll}=0. \tag{A3}$$

Substituting back from (A3) into (A2), it follows:

$$2\epsilon_{jlm}\eta_{l,m(nk)}=\epsilon_{jkl}\xi_{,ln}=-\xi_{j[kn]}\xi_{,ll}\equiv 0. \tag{A4}$$

Thus we get $\xi_{,jk}=0$, and we write

$$\xi(t,\mathbf{x})=a(t)+c_j(t)x^j. \tag{A5}$$

But then Eq. (17a) becomes

$$\eta_{j,kl}+\eta_{j,lk}=\delta_{jk}c_l(t)+\delta_{jl}c_k(t), \tag{A6}$$

which one readily integrates, to read

$$\eta_j(t,\mathbf{x})=\tfrac{1}{2}c_k(t)x^kx^j+b_{jk}(t)x^k+d_j(t). \tag{A7}$$

Thus far, all functions of t remain arbitrary. This finishes the integration of Eq. (17a).

Next, substitution from (A5) and (A7) into Eq. (17b) yields

$$x^j\dot{c}_k(t)+2\dot{b}_{jk}(t)-\delta_{jk}\dot{a}(t)=0; \tag{A8}$$

namely: $c_j=-2C_j$ and $b_{jk}(t)=\frac{1}{2}\delta_{jk}a(t)+B_{jk}$. So, up to here, the solution reads:

$$\xi(t,\mathbf{x})=a(t)-2C_kx^k, \tag{A9}$$

$$\eta_j(t,\mathbf{x})=\tfrac{1}{2}a(t)x^j+B_{jk}x^k-C_kx^kx^j+d_j(t). \tag{A10}$$

The capital letters denote constants of integration.

Finally, one brings (A9) and (A10) into Eq. (17c). As the reader can verify, all terms containing C_kx^k cancel out, and one obtains

$$r^4\ddot{a}-2r^3\ddot{d}_j\hat{r}^j=Kr(6B_{jk}-a\,\delta_{jk})\hat{r}^j\hat{r}^k+4Kd_j\hat{r}^j, \tag{A11}$$

where we have written $x^j=r\hat{r}^j$. Hence, it follows: $a=\frac{3}{2}A$ (for future convenience), $B_{(jk)}=\frac{1}{4}A\,\delta_{jk}$, $B_{[jk]}-=-\epsilon_{jkl}B^l$ (say), and d_j=0. After substituting these constants in (A9) and (A10) one gets Eqs. (18).

APPENDIX B: EXPONENTIATION OF THE INFINITESIMAL TRANSFORMATIONS

Here we prove that Eqs. (21) are in fact the finite transformations generated by

$$\frac{dt'}{dt}=1-2\,\delta\mathbf{q}\cdot\mathbf{x},\quad \mathbf{x}'=\mathbf{x}-(\delta\mathbf{q}\cdot\mathbf{x})\mathbf{x}, \tag{B1}$$

cf. Eqs. (19). We discuss this subject in a rather sketchy way.

Let N be a very large positive integer ($N\gg1$), so that $|\mathbf{q}\cdot\mathbf{x}|\ll N$. We write $\mu=\mathbf{q}\cdot\mathbf{x}$, and we consider a sequence of N infinitesimal transformations generated by $\delta\mathbf{q}=N^{-1}\mathbf{q}$:

$$\mathbf{x}_{(1)}=\left(1-\frac{\mu}{N}\right)\mathbf{x}\equiv w_{(1)}\mathbf{x},$$

$$\mathbf{x}_{(2)}=\left(1-\frac{\mathbf{q}\cdot\mathbf{x}_{(1)}}{N}\right)\mathbf{x}_{(1)}=\left(w_{(1)}-\frac{\mu}{N}\,w_{(1)}^2\right)\mathbf{x}\equiv w_{(2)}\mathbf{x},$$

$$\mathbf{x}_{(3)}=\left(1-\frac{\mathbf{q}\cdot\mathbf{x}_{(2)}}{N}\right)\mathbf{x}_{(2)}=\left(w_{(2)}-\frac{\mu}{N}\,w_{(2)}^2\right)\mathbf{x}\equiv w_{(3)}\mathbf{x}, \tag{B2}$$

$$\dots$$

$$\mathbf{x}_{(N)}=\left(w_{(N-1)}-\frac{\mu}{N}w_{(N-1)}^{2}\right)\mathbf{x}\equiv w_{(N)}\mathbf{x}.$$

We take the limit $N\rightarrow\infty$ and define

$$\mathbf{x}'=\lim_{N\rightarrow\infty}\mathbf{x}_{(N)}=\lim_{N\rightarrow\infty}w_{(N)}\mathbf{x}=w(\mu)\mathbf{x}. \tag{B3}$$

In order to find $w(\mu)$, let us set

$$\Delta w_{(N)}=w_{(N)}-w_{(N-1)}=-\frac{\mu}{N}w_{(N-1)}^{2}\rightarrow dw=-d\mu\ w^{2}. \tag{B4}$$

We then integrate this equation, from the identity $|\mathbf{q}_0|=0$ up to $|\mathbf{q}|>0$. At the identity one has $\mu_0=0$ and $w_0=1$, hence

$$w(\mu)=\frac{1}{1+\mu} \tag{B5}$$

follows.

We next consider dt'/dt by the same token. We set:

$$\begin{aligned}
\frac{dt_{(1)}}{dt}&=1-2\frac{\mu}{N}\equiv v_{(1)},\\
\frac{dt_{(2)}}{dt}&=\left(1-2\frac{\mu}{N}w_{(1)}\right)v_{(1)}\equiv v_{(2)},\\
\frac{dt_{(3)}}{dt}&=\left(1-2\frac{\mu}{N}w_{(2)}\right)v_{(2)}\equiv v_{(3)},\\
&\cdots\\
\frac{dt_{(N)}}{dt}&=\left(1-2\frac{\mu}{N}w_{(N-1)}\right)v_{(N-1)}\equiv v_{(N)},
\end{aligned} \tag{B6}$$

where (obviously) we have used $dt_{(n)}/dt=(dt_{(n)}/dt_{(n-1)})(dt_{(n-1)}/dt)$. Hence, for the limit

$$\frac{dt'}{dt}=\lim_{N\rightarrow\infty}\frac{dt_{(N)}}{dt}=\lim_{N\rightarrow\infty}v_{(N)}\equiv v_{(\mu)}, \tag{B7}$$

using (B5), one easily obtains

$$v(\mu)=\frac{1}{(1+\mu)^{2}}. \tag{B8}$$

This finishes the proof.

APPENDIX C: THE GROUP OF HOMOFOCAL CONICS IN E_3

In order to get a better understanding on how transformations (22) act freely and transitively on the Kepler manifold W_K, we here present a geometric interpretation of the diffeomorphism

$$\mathbf{x}'=\frac{\mathbf{x}}{1+\mathbf{q}\cdot\mathbf{x}} \tag{C1}$$

of the group $G_T^{(0)}$ acting on E_3 [cf. Eqs. (21)]. With this aim, we need to distinguish two cases.
(I) When $1+\mathbf{q}\cdot\mathbf{x}\geqslant 0$, Eq. (C1) yields

$$r'=\frac{r}{1+qr\cos\alpha}, \tag{C2}$$

where $q=|\mathbf{q}|$, $r=|\mathbf{x}|$, $r'=|\mathbf{x}'|$, and $qr\cos\alpha=\mathbf{q}\cdot\mathbf{x}$. For a fixed value of r, this is the profile $r'=r'(\alpha)$ of a conic with semilatus rectum $L=r$ and eccentricity $\epsilon=qr$. This means that whenever $1+\mathbf{q}\cdot\mathbf{r}\geqslant 0$ (C1) maps *spherical surfaces* of radius r, concentric at the origin, into: (a) *ellipsoids* (if $0<qr<1$), (b) *paraboloids* (if $qr=1$), or (c) *hyperboloids* (if $qr>1$); all with one focus at the origin, having $\mathbf{q}$ as symmetry axis of revolution pointing towards the pericenter. For the hyperboloids, the maximum value of α compatible with $1+\mathbf{q}\cdot\mathbf{x}\geqslant 0$ is given by $\cos\alpha_{\max}=-(qr)^{-1}$, which are precisely the directions of the asymptotes to the branch-sheet of the hyperboloid of revolution in this case. In the same manner, for the paraboloids one has $\cos\alpha_{\max}=-1$. (One gets $r'=\infty$ when $\alpha=\alpha_{\max}$ in these instances, as it must be.) For the ellipsoids, r' remains finite for all α.
(II) When $1+\mathbf{q}\cdot\mathbf{x}<0$, Eq. (C1) yields an inversion through the origin $\hat{\mathbf{r}}'=-\hat{\mathbf{r}}$ (with $\hat{\mathbf{x}}'=r'\hat{\mathbf{r}}'$), and also

$$r'=-\frac{r}{1+qr\cos\alpha}. \tag{C3}$$

This case arises only when $rq>1$ and the object-vector $\mathbf{x}$ in (C1) sweeps a spherical cap with α in the range $\alpha_{\max}<\alpha\leqslant\pi$, with $\cos\alpha_{\max}=-(qr)^{-1}$. It is thus clear that $r'=r'(\alpha)$ in (C3) describes the profile of the second branch-sheet of the hyperboloid (with $\epsilon=qr>1$) whose first branch we have seen in case I.

We note that this interpretation of (C1) has the advantage that nothing really undesirable is happening at the plane $1+\mathbf{q}\cdot\mathbf{x}=0$. It is clear that in the applications of $G_T^{(0)}$ to the complete symmetry problem of Eq. (1), case I is of interest when $K>0$, while case II must be considered if $K<0$ (which requires a special treatment not considered in this paper).

Now, let us consider a given conic in E_3, with a focus at $\mathbf{0}$:

$$\mathbf{x}(\alpha)=\frac{L(\cos\alpha\hat{\mathbf{m}}+\sin\alpha\hat{\mathbf{n}})}{1+\epsilon\cos\alpha}, \tag{C4}$$

which meaning is clear. Using the vector $\mathbf{q}'=-(\epsilon/L)\hat{\mathbf{m}}$ in (C1), we obtain a circumference:

$$\mathbf{x}_0(\alpha)=\frac{\mathbf{x}(\alpha)}{1+\mathbf{q}'\cdot\mathbf{x}(\alpha)}=L(\cos\alpha\hat{\mathbf{m}}+\sin\alpha\hat{\mathbf{n}}), \tag{C5}$$

on the fixed plane $(\hat{\mathbf{m}},\hat{\mathbf{n}})$, with radius L and center at $\mathbf{0}$. Let then

$$\mathbf{x}'(\alpha)=\frac{L(\cos\alpha\hat{\mathbf{m}}+\sin\alpha\hat{\mathbf{n}})}{1+\epsilon'\cos\alpha} \tag{C6}$$

be *another* conic with the specified features ($\epsilon'\neq\epsilon$). One transforms $\mathbf{x}_0(\alpha)$ into $\mathbf{x}'(\alpha)$ using $\mathbf{q}''=(\epsilon'/L)\hat{\mathbf{m}}$ in (C1). Hence, a transformation (C1), with $\mathbf{q}$ given by

$$\mathbf{q}=\frac{\epsilon'-\epsilon}{L}\,\hat{\mathbf{m}}, \tag{C7}$$

changes $\mathbf{x}(\alpha)$ into $\mathbf{x}'(\alpha)$.

Therefore we conclude that the diffeomorphisms

$$x'^j=\frac{\rho R_{jk}x^k}{1+\mathbf{q}\cdot\mathbf{x}} \tag{C8}$$

[cf. Eqs. (22)] constitute a group G_Γ that acts freely and transitively on the manifold Γ of *all* homofocal conics in E_3. The group law is given by Eqs. (4b) and (23). G_Γ is the semidirect product $G_\Gamma=G_\Gamma^{(0)}\times G_L^{(0)}$. It is a seven-dimensional proper subgroups of the 15-fold projective group in E_3.

APPENDIX D: UNIQUENESS OF THE INVERSE-SQUARE FORCE UNDER G_K

Let us here solve Eqs. (9), (10), and (37) with the aim of finding the most general force $\mathbf{F}(\mathbf{x},\dot{\mathbf{x}})$ that is consistent with G_K.

First we note that, since $x^kJ^k=0$, Eq. (37) yields

$$J^kF^k=0. \tag{D1}$$

[Interesting enough, this result does *not* follow from Eqs. (9) and (10).] Hence we write

$$F^j(\mathbf{x},\dot{\mathbf{x}})=x^jA(r,v,w)+\dot{x}^jB(r,v,w) \tag{D2}$$

without loss of generality, where $w=\mathbf{x}\cdot\dot{\mathbf{x}}=rv\cos\theta$, and A and B are scalar functions under rotations. (Recall that $r^2v^2-w^2=J^2$). These are the most general solutions to Eqs. (D1) and (10). Substitution of (D2) into Eqs. (9) and (37) then yields

$$x^j\left(6A+2r\,\frac{\partial A}{\partial r}-v\,\frac{\partial A}{\partial v}\right)+\dot{x}^j\left(2r\,\frac{\partial B}{\partial r}+3B-v\,\frac{\partial B}{\partial v}\right)=0, \tag{D3}$$

$$x^jx^k\left(r\,\frac{\partial A}{\partial r}-v\,\frac{\partial A}{\partial v}+3A\right)+\dot{x}^jx^k\left(r\,\frac{\partial B}{\partial r}-v\,\frac{\partial B}{\partial v}+2B\right)+x^j\dot{x}^kx^l\,\frac{\partial A}{\partial \dot{x}^l}+\dot{x}^j\dot{x}^kx^l\,\frac{\partial B}{\partial \dot{x}^l}=0, \tag{D4}$$

wherefrom, after some manipulations, one gets:

$$\left(r\,\frac{\partial}{\partial r}+3\right)A=0,\quad \frac{\partial A}{\partial v}=0, \tag{D5}$$

$$\left(r\,\frac{\partial}{\partial r}+1\right)B=0,\quad \left(v\,\frac{\partial}{\partial v}-1\right)B=0, \tag{D6}$$

and

$$x^k\,\frac{\partial A}{\partial \dot{x}^k}=0,\quad x^k\,\frac{\partial B}{\partial \dot{x}^k}=0. \tag{D7}$$

Note however that, since $A=A(r,v,w)$, here one has

$$\frac{\partial A}{\partial r}=A_r+\frac{w}{r}\,A_w=-\frac{3}{r}\,A, \tag{D8a}$$

$$\frac{\partial A}{\partial v}=A_v+\frac{w}{v}A_w=0, \tag{D8b}$$

$$x^k\frac{\partial A}{\partial \dot{x}^k}=\frac{w}{v}A_v+r^2A_w=0, \tag{D8c}$$

which one readily integrates, to read:

$$A=C/r^3, \tag{D9}$$

where C is a constant of integration. In the same manner, for $B=B(r,v,w)$, we write

$$\frac{\partial B}{\partial r}=B_r+\frac{w}{r}B_w=-\frac{1}{r}B, \tag{D10a}$$

$$\frac{\partial B}{\partial v}=B_v+\frac{w}{v}B_w=\frac{1}{v}B, \tag{D10b}$$

$$x^k\frac{\partial B}{\partial \dot{x}^k}=\frac{w}{v}B_v+r^2B_w=0; \tag{D10c}$$

i.e.,

$$rB_r+vB_v+2wB_w=0, \tag{D11a}$$

$$wB_v+r^2vB_w=0. \tag{D11b}$$

Using the "method of characteristics" in (D11a), namely,

$$\frac{dr}{r}=\frac{dv}{v}=\frac{dw}{2w}=dc, \tag{D12}$$

one obtains three solutions:

$$B_1(r,v)=\frac{r}{v}=c_1,\quad B_2(r,w)=\frac{r}{\sqrt{w}}=c_2,\quad B_3(v,w)=\frac{v}{\sqrt{w}}=c_3. \tag{D13}$$

(Note that $B_1B_3=B_2$.) However, none of these does satisfy (D11b), neither do they satisfy Eqs. (D10). This means that the only solution to these equations is given by

$$B=0. \tag{D14}$$

[1] See, for instance, P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer Verlag, New York, 1986); this is perhaps the most exhaustive monography on this subject. See also G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer Verlag, New York, 1989).

[2] The geometry of *configuration space–time*, and its role for holonomic rheonomic systems in classical mechanics, has been considered by M. Trümper, Ann. Phys. (NY) **149**, 203 (1983).

[3] Even contact transformations, involving the velocity as an independent variable, have been used to study symmetries of some classical systems [see, i.e., E. L. Hill, Rev. Mod. Phys. **23**, 253 (1951)]; but such transformations fail to produce a "complete symmetry group" in the sense required here. To take a case in point, contact transformations for the classical Kepler problem were studied by J. M. Levy-Leblond, Am. J. Phys. **39**, 502 (1971), but they do not yield a complete group for this particular system.

[4] M. Aguirre and J. Krause, J. Math. Phys. **29**, 9 (1988).

[5] M. Aguirre and J. Krause, J. Math. Phys. **29**, 1746 (1988).

[6] See, for instance, M. Aguirre and J. Krause, J. Phys. A **20**, 3553 (1987), concerning the *finite* point realizations of SL(3,R) yielding the *complete* symmetry group for the harmonic oscillator; cf. also P. G. L. Leach, J. Math. Phys. **21**, 300 (1980), for the realization of the Lie algebra sl(3,R) in the case of the one-dimensional time-dependent harmonic oscillator. Furthermore, it has been shown by G. E. Prince and C. J. Eliezer, J. Phys. A **13**, 815 (1980), that the point symmetry group of the time-dependent n-dimensional harmonic oscillator is SL($n+2$,R); nevertheless, it is not known if this group is complete for that system. All these results on point symmetries, and perhaps some few others, have been found by means of the traditional Lie method. Moreover, much of this subject was known already to Sophus Lie himself [S. Lie, *Vorlesungen über Differential-Gleichungen Mit Bekannten Infinitesimal Transformationen* (B. G. Teubner, Leipzig, 1981; reprinted by Chelsea, New York, 1967)]; however, it seems to be not so well known to most physicists. For instance, the rediscovery eighteen years ago by C. E. Wulfman and B. G. Wyborne, J. Phys. A **9**, 507 (1976), that the equation of motion $\ddot{x}+\omega^2 x=0$ has point symmetry SL(3,R) was a surprise to physicists.

[7] G. E. Prince and C. J. Eliezer, J. Phys. A **14**, 587 (1981). Also: G. E. Prince, **16**, L105 (1983).

[8] One could also enlarge G_L considering its improper pieces in Eq. (3), but we are not concerned with such minor details now.

[9] The energy is given by $E=(\mathbf{M}^2-K^2)/2\mathbf{J}^2$. The Hamilton vector is $\mathbf{W}=(KL)^{-1}\mathbf{J}\times\mathbf{M}=\sqrt{(K/L)}\epsilon\hat{\mathbf{n}}$; cf. C. D. Collinson, Bull. Inst. Math. Applic. **9**, 377 (1973).

[10] P. Havas, Rev. Mod. Phys. **36**, 938 (1964). Newtonian spacetime is today a rather familiar concept; it was also introduced by A. Trautman, in *Perspectives in Geometry and Relativity*, edited by B. Hoffman (Indiana University, Bloomington, 1966), pp. 413–425.

[11] One better solves this problem introducing the *eccentric anomaly* u [i.e., setting $t=t(u)$ and $\mathbf{x}=\mathbf{x}(u)$] used by the astronomers; cf. i.e., A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University, Princeton, NJ, 1947). With this approach, the exercise becomes essentially in a *consistency control* between Eqs. (32a) and (32b).

[12] The existence of the symmetries $O_+(4)$ and SU (3) for all classical central potential problems was established by Fradkin many years ago; D. M. Fradkin, Progr. Theor. Phys. **37**, 798 (1967). The internal symmetry $O_+(4)$ associated with the Kepler problem has been known for a long time; cf. V. Fock, Z. Physik **98**, 145 (1935), also: V. Bargmann, Z. Physik **99**, 576 (1936). During the 1960s much work on the Kepler problem was directed towards imbedding $O_+(4)$ within a larger finite noncompact group, for example the De Sitter group $O(4,1)$; see H. Bacry, Nuovo Cimento **41**, A222 (1966), A. Bohm, Nuovo Cim. **43**, A665 (1966), and references quoted therein. The $O_+(4)$ symmetry of the Kepler system sets an "age" problem that has been much investigated, owing mainly to its important role in the quantum-mechanic theory of the hydrogen atom. Even more recently, the Fock quantization of the hydrogen atom has been reformulated by means of the symmetry SO($2,n+1$) of the n-dimensional Kepler problem; B. Cordani, J. Phys. A **22**, 2695 (1989).

[13] A nonintegrable "quotient of differentials" is also used, for instance, in the Kustaanheimo–Stiefel transformation as a means of transforming the three-dimensional Kepler problem into that for a four-dimensional harmonic oscillator; P. Kustaanheimo and E. Stiefel, J. Reine Angew. Math. **218**, 204 (1965), cf., also, F. H. J. Cornish, J. Phys. A **17**, 2191 (1984), where the same subject is discussed in terms of quaternions' continuous mappings.

[14] J. Krause, J. Phys. A **18**, 1309 (1985).

[15] J. Krause, J. Math. Phys. **32**, 348 (1991); J. Krause, J. Phys. A **26**, 6285 (1993); J. Krause, Int. J. Phys. **32**, 1363 (1993).

[16] J. Krause, J. Math. Phys. **27**, 2922 (1987).

[17] J. Krause, J. Math. Phys. **29**, 1309 (1988).