# A COMMENT ON THE TWIN PARADOX AND THE HAFELE–KEATING EXPERIMENT

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We show that a theoretical and experimental analysis of the twin paradox and the Hafele–Keating experiment recently presented in this journal is non sequitur from the epistemological point of view and wrong according to the mathematical structure of relativity theory.

From time to time, like a phoenix the twin paradox revives in some physical journal with someone trying to show that relativity theory does not imply an unequal age for two twins after one of them takes a trip starting and ending at the location of his brother, or that the canonical calculated ages are wrong. It always happens that a paper of this kind generates a controversy with many fellows presenting their arguments for the unequal age solution whereas others insist on the equal age solution and/or non-"canonical" calculations. Here history repeats itself since we are going to show that the theoretical and experimental analysis of the twin paradox recently put forth by Cornille [1] is non sequitur from the epistemological point of view, being moreover wrong within the mathematical structure of relativity theory.

To begin, let us remember that the most important feature of relativity theory is the hypothesis that the collection of all possible happenings, i.e., all possible events constituting space–time, i.ee., ST = (M, $g$, D) is a connected four-dimensional oriented and time oriented Lorentzian manifold (M, $g$) together with the Levi-Civita connection D of $g$ on M. The events in U $\subset$ M in a particular chart of a given atlas have coordinates ($x^0$, $x^1$, $x^2$, $x^3$), $x^0$ is called the time-like coordinate and the $x^i$, $i$ = 1, 2, 3 are called the space-like coordinates. These labels according to Einstein [2] do not necessary have a metrical meaning, i.e., are not measured by the standard clocks and the standard rulers of the theory. The metrical of the manifold (in a coordinate basis) is

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu , \tag{1}$$

with $g_{\mu\nu} = g(\partial/\partial x^\mu, \partial/\partial x^\nu)$ being calculated, of course, for each $x \in$ M in $T_x$M, the tangent space to M at $x$. (The properties of the vector space $T_x M = R^{1,3}$ (Minkowski vector space), and in particular the so-called anti-Minkowski inequality for time-like vectors in the same class are fundamental for the understanding of the clock problem of relativity theory. We discussed these properties at length in ref. [3] and the reader is adressed to this reference for details of notation and the proofs of the results we are going to use.) Here we quote the

*Anti-Minkowski inequality* (proposition 9 in ref. [3]): Let $v$, $w \in \tau^+ \subset R^{1,3}$ (where $\tau^+$ is the class of future pointing time-like vectors). Then it holds that

$$[g(v+w, v+w)]^{1/2} \geqslant [g(v,v)]^{1/2} + [g(w,w)]^{1/2} . \tag{2}$$

Now, tangent space magnitudes defined by the metric are related to magnitudes on the manifold in the following way:

Let I $\subset$ R be an open interval on the real line and $\Gamma$: I $\to$ M a map. We suppose that $\Gamma$ is a $C^0$, piecewise $C^1$ curve in M. We denote the inclusion function I $\to$ R by $u$, and the distinguished vector field on I by d/d$u$. For each $u \in$ I, $\Gamma_* u$ denotes the tangent vector at $\Gamma u \in$ M; thus

$\Gamma_* u = [\Gamma_*(\mathrm{d}/\mathrm{d}u)](u) \in \mathrm{M}_{\Gamma_u}$.

Finally, the path-length between points $x_1 = \Gamma(a)$, $x_2 = \Gamma(b)$, $a, b \in \mathrm{I}$, $x_1, x_2 \in \mathrm{M}$ along the curve [#1] $\Gamma$: $\mathrm{I} \to \mathrm{M}$ such that $g(\Gamma_* u, \Gamma_* u)$ has the same sign in all points along $\Gamma u$, is the quantity

$$\int_a^b \mathrm{d}u \, |g(\Gamma_* u, \Gamma_* u)|^{1/2} . \tag{3}$$

Observe now that taking the point $\Gamma(a)$ as a reference we can use eq. (3) to define the function $s$: $\Gamma(\mathrm{I}) \to \mathrm{R}$ by

$$s(u) = \int_a^u \mathrm{d}u' \, |g(\Gamma_* u', \Gamma_* u')|^{1/2} . \tag{4}$$

With eq. (3) we can calculate the derivative $\mathrm{d}s/\mathrm{d}u$. We have

$$\frac{\mathrm{d}s}{\mathrm{d}u} = |g(\Gamma_* u, \Gamma_* u)|^{1/2} = \left| g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \frac{\mathrm{d}x^\nu}{\mathrm{d}u} \right|^{1/2} . \tag{5}$$

From eq. (4) old textbooks on differential geometry and general relativity infer the equation

$$(\mathrm{d}s)^2 = g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \tag{6}$$

supposed to represent the square of the length of the "infinitesimal" arc determined by the coordinate displacement

$$x^\mu(a) \to x^\mu(a) + \frac{\mathrm{d}x^\mu}{\mathrm{d}u}(a)\varepsilon ,$$

where $\varepsilon$ is an "infinitesimal" and $a \in \mathrm{I}$.

The abusive and non-careful use of eq. (6) has produced many incorrect interpretations in relativity theory as illustrated, e.g., in the odd paper [4] quoted by Cornille in support of his wrong view. For a critical reply to ref. [4] and also ref. [5] (also examples of the phoenix-like nature of the twin paradox) see ref. [3].

Now, given a time-like curve $\rho$: $\mathrm{R} \supseteq \mathrm{I} \to \mathrm{M}$, any event $c \in \rho(\mathrm{I})$ separates all other events in two disjoint classes, the past and the future [3]. The theory models an observer as

*Definition 1.* An observer in ST is a future-point time curve [3] $\Gamma$: $\mathrm{R} \supseteq \mathrm{I} \to \mathrm{M}$, by $\mathrm{I} \ni u \to \Gamma(\mathrm{I}) \subset \mathrm{M}$, and such that $g(\Gamma_* u, \Gamma_* u) = 1$.

We now introduce

*Definition 2* (Standard clock postulate). Let $\Gamma$ be an observer. Then there exist standard clocks that "can be carried by $\Gamma$" and such that they register (in $\Gamma$) proper time, i.e., the inclusion parameter $u$ of the definition of observer. Standard clocks "tic-tac" with a constant period, which means that in $\Gamma$ there is a sequence of events separated by equal intervals of proper time.

The question regarding the physical objects that realize the standard clocks of relativity theory is of course central to the present issue and will be discussed below. We shall need

*Definition 3.* A reference frame in $\mathrm{U} \subseteq \mathrm{M}$ is a time-like vector field $\mathrm{Q} \in \mathrm{TU}$ such that each one of its integral lines is an observer.

*Definition 4.* A chart in $\mathrm{U} \subseteq \mathrm{M}$ of the maximal oriented atlas of M is said to be a naturally adapted coordinate system to a reference frame Q (nacs/Q) if in the natural coordinate basis of TU associated with the chart the space-like components of Q are null.

Old treatments of the clock problem involve at least two reference frames $\mathrm{Q} \in \mathrm{TU}$ and $\mathrm{Q}' \in \mathrm{TV}$, each one containing a standard clock at (coordinate) rest at the origins of $\langle x^\mu \rangle$ and $\langle x'^\mu \rangle$, respectively the (nacs/Q) and (nacs/Q′). For $\mathrm{U} \cap \mathrm{V} \subseteq \mathrm{M}$ where both Q and Q′ are defined we have the coordinate transformations $\langle x^\mu \rangle \to \langle x'^\mu \rangle$. In particular we have $x'^0 = f(x^0, x^1, x^2, x^3)$ relating the time-like coordinate of an event $e \in \mathrm{U} \cap \mathrm{V}$ in Q′ with the time-like and the space-like coordinates of the same event in Q. In what follows we are not using the coordinate transformation laws to solve the clock problem.

With the above definitions and given the Einstein synchronization procedure we can discover when a given reference frame is synchronizable, i.e., when the time-like coordinate function $x^0$ of the (nacs/Q) has the meaning of proper time registered by the standard clocks at (coordinate) rest in Q. All these

[#1] Curves are classified as time-like, light-like and space-like when (for all $u \in \mathrm{I}$) $g(\Gamma_* u, \Gamma_* u) > 0$, $g(\Gamma_* u, \Gamma_* u) = 0$, $g(\Gamma_* u, \Gamma_* u) < 0$ respectively.

points are discussed at length in ref. [3] and here we quote that the condition for Q to be proper time synchronizable is the existence of the mapping $x^0$: M→R such that $\alpha = \mathrm{d}x^0$ where

$$\alpha = g(Q, \ ) \qquad (7)$$

is the one-form field physically equivalent to Q.

We are now theoretically prepared to analyse the clock (or twin) problem (no paradox, of course), the Hafele–Keating experiment and some other claims done by Cornille.

Let $\Gamma_1$, $\vec{\Gamma}_2$, $\overleftarrow{\Gamma}_2$ be three future pointing time-like and straight lines in $\mathscr{M}$ (the affine Minkowski space). $\Gamma_1$ and $\vec{\Gamma}_2$ have $x_i$ as common point, $\Gamma_1$ and $\overleftarrow{\Gamma}_2$ have $x_f$ as common point and $\vec{\Gamma}_2$ and $\overleftarrow{\Gamma}_2$ have $x_m$ as common point. $\Gamma_1$ represents the path of a standard clock called 1 and $\Gamma_2 = \vec{\Gamma}_2 + \overleftarrow{\Gamma}_2$ represents the path of a standard clock called 2. Now, according to definition 2 and eq. (4) the proper time registered by clock 1 between the events $x_i$ and $x_f$ is given by $T_1 = [g(x_f - x_i, x_f - x_i)]^{1/2}$, i.e., the norm of the vector $x_f - x_i \in \mathrm{R}^{1,3}$. The proper time registered by clock 2 is given by

$$T_2 = [g(x_m - x_i, x_m - x_i)]^{1/2} + [g(x_f - x_m, x_f - x_m)]^{1/2} .$$

According to the anti-Minkowski inequality we have

$$g(x_f - x_i, x_f - x_i)]^{1/2} \geqslant [g(x_m - x_i, x_m - x_i)]^{1/2} + [g(x_f - x_m, x_f - x_m)]^{1/2} \qquad (8)$$

and thus $T_1 \geqslant T_2$.

This result is an intrinsic consequence of the mathematical model of relativity theory. All observers in all reference frames in $\mathscr{M}$ must agree with the validity of the result $T_1 \geqslant T_2$.

We observe that the path $\Gamma_1$ is a geodesic path between $x_f$ and $x_i$ as can be trivially proved, and then it follows that $T_1 > T_2$. We can also prove the following theorem [3] which is valid in a general ST (i.e., D does not need to be flat):

*Theorem.* Among all future pointing time-like curves in ST = (M, $g$, D) passing through the points $x_i = \Gamma(a)$ and $x_f = \Gamma(b)$ the integral in eq. (4) is a maximum when $\Gamma$ is a time-like geodesic.

To find the explicit relation between $T_1$ and $T_2$ we must introduce one reference frame in $\mathscr{M}$ and then give the parametric equations of $\Gamma_1$, $\vec{\Gamma}_2$, $\overleftarrow{\Gamma}_2$ in this frame. In $\mathscr{M}$ there exist infinite inertial reference frames {i}, i.e., frames such that $\mathrm{D}\alpha_i = 0$, $\alpha_i = g(i, \ )$. These frames are proper time synchronizable. Let i be an inertial frame and $\langle x^\mu \rangle$ the (nacs/i), with $x^0$, having the meaning of the proper time registered by standard clocks that are at rest in i, and synchronized à la Einstein.

Let

$$\Gamma_{1*} = \frac{\partial}{\partial x^0} \circ \Gamma_1, \quad 0 \leqslant x^0 \leqslant T_1 ,$$

$$\vec{\Gamma}_{2*} = (1 - v^2)^{-1/2} \frac{\partial}{\partial x^0} \circ \vec{\Gamma}_2 + v(1 - v^2)^{-1/2} \frac{\partial}{\partial x^1} \circ \vec{\Gamma}_2 , \quad 0 \leqslant x^0 \leqslant T_1/2 ,$$

$$\overleftarrow{\Gamma}_{2*} = (1 - v^2)^{-1/2} \frac{\partial}{\partial x^0} \circ \overleftarrow{\Gamma}_2 - v(1 - v^2)^{-1/2} \frac{\partial}{\partial x^1} \circ \overleftarrow{\Gamma}_2 , \quad T_1/2 \leqslant x^0 \leqslant T_1 . \qquad (9)$$

In eq. (9) $0 \leqslant v < 1$ is a positive real constant. If clocks 1 and 2 are put at the same phase at $x^0 = 0$, we get the canonical result

$$T_2 = (1 - v^2)^{1/2} T_1 . \qquad (10)$$

We now must investigate if $T_1 > T_2$ when clock 1 is left at rest in the inertial frame i and clock 2 is at rest in an accelerated frame. We will distinguish two cases:

(i) Clock 2 is at rest in the accelerated frame Q and the tangent vector to its world line $\Gamma_2$ is given by

$$\Gamma_{2*} = [1 - v(x^0)^2]^{-1/2} \frac{\partial}{\partial x^0} \circ \Gamma_2 + v(x^0)[1 - v(x^0)^2]^{-1/2} \frac{\partial}{\partial x^1} \circ \Gamma_2 , \quad 0 \leqslant x^0 \leqslant \tau ,$$

$$= (1-v^2)^{-1/2} \frac{\partial}{\partial x^0} \circ \Gamma_2$$
$$+ v(1-v^2)^{-1/2} \frac{\partial}{\partial x^1} \circ \Gamma_2 , \quad \tau \leqslant x^0 \leqslant 2\tau ,$$
$$= [1 - v'(x^0)^2]^{-1/2} \frac{\partial}{\partial x^0} \circ \Gamma_2$$
$$+ v'(x^0) [1 - v'(x^0)^2]^{-1/2} \frac{\partial}{\partial x^1} \circ \Gamma_2 ,$$
$$2\tau \leqslant x^0 \leqslant 3\tau , \quad (11)$$

where $v(x^0)$ and $v'(x^0)$ are the standard velocity functions [6] of clock 2 with constant accelerations (in the $\partial/\partial x^1$-direction) $a$ and $-a$ respectively. If clocks 1 and 2 are put at the same phase at $x^0=0$, we get from eq. (11) using definition 2 and eq. (4) again

$$T_2 = \tau(1-v^2)^{1/2} + \frac{2}{a} \ln(a\tau + \sqrt{a^2\tau^2+1})$$
$$< T_1 = 3\tau . \quad (12)$$

(ii) Clock 2 is at rest in the frame Q but is rotating with constant angular velocity $\omega$ relative to the inertial frame $\mathrm{i} = \partial/\partial x^0 = \partial/\partial t$. For this problem we use polar coordinates and write the flat metric of $\mathcal{M}$ as

$$g = \mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}r \otimes \mathrm{d}r - r^2\, \mathrm{d}\phi \otimes \mathrm{d}\phi - \mathrm{d}z \otimes \mathrm{d}z$$

and

$$Q = (1-\omega^2 r^2)^{-1/2} \partial/\partial t + \omega(1-\omega^2 r^2)^{-1/2} \partial/\partial\phi , \quad (14)$$

defined in $\mathrm{M} \supseteq \mathrm{U} = \{-\infty < t < \infty;\ 0 < r < 1/\omega;\ 0 \leqslant \phi \leqslant 2\pi;\ -\infty < z < \infty\}$. Then

$$\alpha = g(Q, \ ) = (1-\omega^2 r^2)^{-1/2} \mathrm{d}t$$
$$- \omega r^2 (1-\omega^2 r^2)^{-1/2} \mathrm{d}\phi . \quad (15)$$

A (nacs/Q) is $\langle t, r, \bar{\phi}, z \rangle$, with $\bar{\phi} = \phi + \omega t$. In the canonical non-coordinate basis $(\partial/\partial t, \partial/\partial r, (1/r)\partial/\partial\bar{\phi}, \partial/\partial z)$ associated with this coordinate system we get for the rotation vector [3,7] $\Omega$ associated to $\alpha$,

$$\Omega = \tfrac{1}{2} \hat{g}[*(\mathrm{d}\alpha \wedge \alpha), \ ] = \omega(1-\omega^2 r^2)^{-1} \partial/\partial z ,$$

which shows that Q is indeed rotating with constant angular velocity $\omega$ relative to the $z$-axis of i. (In eq. (16) $\hat{g}$ is the metric of the cotangent bundle.)

Now, the tangent vector field to the world line $\Gamma_2$ of clock 2 is

$$\Gamma_{2*} = (1-\omega^2 R^2)^{-1/2} \frac{\partial}{\partial t} \circ \Gamma_2$$
$$+ \omega(1-\omega^2 R^2)^{-1/2} \frac{\partial}{\partial \phi} \circ \Gamma_2 . \quad (17)$$

If clocks 1 and 2 are put at the same phase at $x^0=0$, we get from eq. (17) using definition 2 and eq. (4) that

$$T_2 = (1-\omega^2 R^2)^{1/2} T_1 . \quad (18)$$

We now come to comments concerning Cornille's paper:

(A) Cornille quotes correctly that several experiments [8–12] done (using the Mössbauer effect) with atomic systems that follow world lines as in eq. (17) are compatible with eq. (18). From this he concludes that eq. (12) is false since it is eq. (18) that is observed experimentally. Well, since both equations are derived for the operationally distinct motions from the same assumptions (definition 2 plus eq. (4)) it is epistemological non sequitur to claim that only one of the equations is valid within relativity theory. Obviously, both equations are theoretically true statements. Whether these statements are realized in the physical world is a question of which pure mathematics cannot say anything. Only experiments can solve the issue.

(B) Cornille says that the experiment [13] shows that eq. (12) is false and is in accord with his own eq. (13). Well, first of all his eq. (13) is non sequitur as a theoretical statement within relativity theory. This point is clear from the theoretical analysis we did above. Also in the experiment [13] the rest mean life-time of muons is determined in a statistical way from muons that are "quickly" stopped after they are produced and the mean life-time of moving muons are compared with the life-time of the muons put to rest in the laboratory. In particular it must be said that each muon produced in an elementary particle collision is born with a fixed velocity $v$. It is not accelerated from zero velocity to the velocity $v$ contrary to Cornille supposition. Of course, the muon suffers accelerations due to its electric charge after they have been produced. The effect of a constant angular acceleration (equivalent to $10^{20}$ times the gravity acceleration) on muons has been measured in experiment [14]. The agreement between experiment and eq. (18) is not so good. Indeed, Apsel [15] found that there is a better agreement if the "proper time" of the moving muons are associated with a Finslerian metric in $R^4$ involving the electromagnetic potentials. This may imply that muons are not standard clocks or that relativity is after all wrong. More experimentation is needed, of course, to have any answer concerning this point.

(C) Cornille said: "Moreover, if there was a time difference after a round trip in the case of a rectilinear motion and if this effect was attributed to a pure velocity effect as most authors think, then we will have an experiment which allows to discriminate a state of rest from a state of rectilinear uniform motion which is contradictory to the Michelson and Morley experiment which fails to measure the rectilinear uniform motion of the earth through space". Well, besides the fact that Cornille did not say which is the experiment he is talking about the fact is that just the opposite is true. More precisely we showed in a rigorous mathematical way [16] that in $\mathscr{M}$ relativity theory forbids the existence of a Lorentz invariant clock, i.e., a clock that when set in motion relative to an inertial reference frame i does not lag behind relative to a series of clocks synchronized à la Einstein in i. Indeed in ref. [16] we showed that the existence of one such clock implies the breakdown of Lorentz invariance.

(D) Eq. (16) is presented only to show that Cornille's comments concerning Davies and Jennison's paper [17] are completely wrong. Eq. (16) has nothing to do with the Thomas precession and even more, the local angular velocity deduced in ref. [17] for the rotation disk is wrong. The mistake is due to the method used in ref. [17] to measure distances which is not the one that follows from the formalism of relativity theory.

Concerning the Hafele–Keating [18,19] experiment it is clear that Cornille's analysis and formulas cannot be applied since they are wrong within relativity theory. Here, we must say that the original Hafele–Keating theoretical analysis is also a little bit misleading. Indeed, to predict correctly the proper times registered by the three sets of clocks in their experiment it is necessary to use the Kerr metric instead of the Schwarzschild metric, write the parametric equations of the world lines of the clocks and finally use definition 2 and eq. (4). However the final equation presented in ref. [19] is a good approximation if we are to believe the precision of the measurements presented by Hafele and Keating. In this respect we unfortunately have to quote that Essen [20], the "builder" of the atomic clocks used in ref. [19], says that the clocks do not have the precision in order to provide a test of relativity theory! For the same reason, of course, the Hafele–Keating data cannot be used to test Cornille's odd formulas. An anonymous referee asked to us if it is possible using clocks to "see" the dragging of inertial frames. According to Cohen, Rosenblum and Clifton [21] this is possible and we agree with them.

We end this paper with the following comment: The question of which real clocks are the standard clocks of relativity theory is not experimentally solved yet in view of the above discussion. (Dirac [22], for example, is of the opinion that atomic clocks do not realize the Lorentzian metric of relativity theory, i.e., they do not satisfy definition 2 and eq. (2).) What is out of the question is the theoretical result (presented above) for the behavior of standard clocks in relativity theory. We hope that the present paper put an end in the "phoenix-like" career of the "twin paradox" at least within the pages of this journal.

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