Article

Lie Group Statistics and Lie Group Machine Learning based on Souriau Lie Groups Thermodynamics & Koszul-Souriau-Fisher Metric: New Entropy Definition as Generalized Casimir Invariant Function in Coadjoint Representation

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Abstract: In 1969, Jean-Marie Souriau introduced a "Lie Groups Thermodynamics" in Statistical Mechanics in the framework of Geometric Mechanics. This Souriau's model considers the statistical mechanics of dynamic systems in their "space of evolution" associated to a homogeneous symplectic manifold by a Lagrange 2-form, and defines in case of non null cohomology (non equivariance of the coadjoint action on the moment map with appearance of an additional cocyle) a Gibbs density (of maximum entropy) that is covariant under the action of dynamic groups of physics (eg, Galileo's group in classical physics). Souriau Lie Group Thermodynamics was also addressed 30 years after Souriau by R. F. Streater in the framework of Quantum Physics by Information Geometry for some Lie algebras, but only in the case of null cohomology. Souriau method could then be applied on Lie Groups to define a covariant maximum entropy density by Kirillov representation theory. We will illustrate this method for homogeneous Siegel domains and more especially for Poincaré unit disk by considering SU(1,1) group coadjoint orbit and by using its Souriau's moment map. For this case, the coadjoint action on moment map is equivariant. For non-null cohomology, we give the case of Lie group SE(2). Finally, we will propose a new geometric definition of Entropy that could be built as a generalized Casimir invariant function in coadjoint representation, and Massieu characteristic function, dual of Entropy by Legendre transform, as a generalized Casimir invariant function in adjoint representation, where Souriau cocycle is a measure of the lack of equivariance of the moment mapping.

Keywords: Lie Groups Thermodynamics, Lie Group Machine Learning, Kirillov Representation Theory, Coadjoint Orbits, Moment Map, Covariant Gibbs Density, Maximum Entropy Density, Souriau-Fisher Metric, Generalized Casimir Invariant Function.

> *La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt…En outre, quantité de notions naturelles concernant les représentations s'interprètent géométriquement en terme d'orbites coadjointes : restriction à un sous-groupe, induction unitaire, produit tensoriel, mesure de Plancherel, la topologie de l'ensemble représentations unitaires irréductibles... Kirillov s'est vite convaincu, et il a convaincu la communauté mathématique que cette « méthode des orbites » devait être applicable à des groupes bien plus généraux que les groupes nilpotents. Il n'a pas hésité à aborder le cas des groupes de Lie connexes quelconques. Evidemment, des difficultés considérables ont surgi immédiatement. Néanmoins, Kirillov a indiqué une voie d'accès, qui ensuite a été largement utilisée.*
> **Jacques Dixmier, Brèves remarques sur l'œuvre de A.A. Kirillov**

On comprend ainsi comment Lagrange a pu développer les lois de la Mécanique des systèmes formés de solides sans s'occuper des variations de la température de ces corps et Fourier traiter des variations de la température de ces mêmes corps solides sans s'occuper de leur mouvement ; comment on peut étudier le mouvement de la Terre, assimilée à un solide rigide, sans se préoccuper de la température de cet astre et étudier le refroidissement du globe terrestre sans se préoccuper de son mouvement. Une telle indépendance entre les problèmes qui ressortissent à la Mécanique et les problèmes qui ressortissent à la Théorie de la chaleur n'existe plus lorsque les systèmes auxquels on a affaire ne sont plus des systèmes classiques ; si, par exemple, au lieu de regarder la Terre comme un solide rigide, d'état invariable, on tient compte des changements de volume, de forme, d'état physique et chimique qui accompagnent son refroidissement, on ne peut plus séparer le problème du mouvement de la Terre et le problème du refroidissement terrestre. ... On sait que cette forme de relations supplémentaires avait été introduite par Newton et les géomètres du XVIIIème siècle dans la théorie du son. Ces considérations montrent que les questions qui ressortissent à la Thermodynamique ont dû solliciter l'attention des physiciens dès qu'on a voulu aborder l'étude des systèmes autres que des systèmes classiques ; et, en fait, c'est la théorie de la propagation du son dans l'air qui a provoqué Laplace à créer la Thermodynamique - **P. Duhem, L'intégrale des forces vives en thermodynamique, JMPA 4:5-19, 1898 [1-4]**

Sous cette aspiration, la physique qui était d'abord une science des "agents" doit devenir une science des "milieux". C'est en s'adressant à des milieux nouveaux que l'on peut espérer pousser la diversification et l'analyse des phénomènes jusqu'à en provoquer la géométrisation fine et complexe, vraiment intrinsèque...Sans doute, la réalité ne nous a pas encore livré tous ses modèles, mais nous savons déjà qu'elle ne peut en posséder un plus grand nombre que celui qui lui est assigné par la théorie mathématique des groupes.

Gaston Bachelard, Etude sur l'Evolution d'un problème de Physique –La propagation thermique dans les solides, 1928

The classical simple gradient descent used in Deep Learning has two drawbacks: the use of the same non-adaptive learning rate for all parameter components, and a non-invariance with respect to parameter re-encoding inducing different learning rates. As the parameter space of multilayer networks forms a Riemannian space equipped with Fisher information metric, instead of the usual gradient descent method, the natural gradient or Riemannian gradient method, which takes account of the geometric structure of the Riemannian space, is more effective for learning. The natural gradient preserves this invariance to be insensitive to the characteristic scale of each parameter direction. The Fisher metric defines a Riemannian metric as the Hessian of two dual potential functions (the Entropy and the Massieu Characteristic Function).

In Souriau's Lie groups thermodynamics, the invariance by re-parameterization in information geometry has been replaced by invariance with respect to the action of the group. In Souriau model, under the action of the group, the entropy and the Fisher metric are invariant. Souriau defined a Gibbs density that is covariant under the action of the group. The study of exponential densities invariant by a group goes back to the work of Muriel Casalis in her 1990 thesis. The general problem was solved for Lie groups by Jean-Marie Souriau in Geometric Mechanics in 1969, by defining a "Lie groups Thermodynamics" in Statistical Mechanics. These new tools are bedrocks for Lie Group Statistics and Lie Group Machine Learning. Souriau introduced a Riemannian metric, linked to a generalization of the Fisher metric for homogeneous Symplectic manifolds. This model considers the KKS 2-form (Kostant-Kirillov-Souriau) defined on the coadjoint orbits of the Lie group in the case of non-null cohomology, with the introduction of a Symplectic cocycle, called "Souriau's cocycle", characterizing the non-equivariance of the coadjoint action (action of the Lie group on the moment map).

We can observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group is invariant $S\left(Ad_g^{\#}(Q)\right)=S(Q)$ with Souriau affine definition of coadjoint action $Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta(g)$ where $\theta(g)$ is called Souriau cocyle. Based on Souriau Lie groups Thermodynamics, we will propose a new geometric definition of ***Entropy that could be built as a generalized Casimir invariant function in coadjoint representation***, ***and Massieu characteristic function, dual of Entropy by Legendre transform, as a generalized Casimir function in adjoint representation***. This geometric structure of Entropy is a foundation for a new geometric theory of information, where the ***Entropy is no longer defined axiomatically as Shannon***

or von Neumann Entropies, but built as Casimir Invariant function in coadjoint representation and solution to the Casimir equation extended $\left(ad^*_{\frac{\partial S}{\partial Q}}Q\right)_j+\Theta\left(\frac{\partial S}{\partial Q}\right)_j=C_{ij}^k ad^*_{\left(\frac{\partial S}{\partial Q}\right)^i}Q_k+\Theta_j=0$ ***, with Souriau cocycle*** $\theta(g)$ ***where*** $\Theta(X)=T_e\theta(X(e))$ with $\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$, $X,Y\in\mathfrak{g}$ ***in case of non null cohomology.***

The dual Lie algebra foliates into coadjoint orbits that are also the level sets on the entropy. The KKS (Kostant-Kirillov Souriau) 2-form, and the Souriau-Koszul-Fisher metric make each orbit into homogeneous Symplectic manifold. The information manifold foliates into level sets of the entropy that could be interpreted in Thermodynamics: motion remaining on this complex surfaces is non-dissipative, whereas motion transversal to these surfaces is dissipative, where the dynamic is given by: $\frac{dQ}{dt}=\{Q,H\}_{\tilde{\Theta}}=ad^*_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)$ with stable equilibrium given when $H=S\Rightarrow\frac{dQ}{dt}=\{Q,S\}_{\tilde{\Theta}}=ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$.

We will introduce the link between Koszul geometry of homogeneous bounded domains, Souriau "Lie Groups Thermodynamics", Information Geometry and Kirillov representation theory [12] to define probability densities as Souriau covariant Gibbs densities (density of Maximum of Entropy). We will illustrate this case for the matrix Lie group SU(1,1) (case with null cohomology) through the computation of Souriau's moment map, and Kirillov's orbit method. We will also indicate application for SE(2) Lie group (case with non-null cohomology) where a Souriau cocycle should be take into account for the default of equivariance of the coadjoint action on moment map.

0. Historical Preamble

Lie groups Thermodynamics is a "knot of high denisity" between different disciplines and make links between different branches of statistical sciences from statistical physics, to Information Geometry and theory of inference. We give in the following the main contributions and tools that are covered by this topic:

- **Statistical Physics, Massieu & Poincaré Characteristic Functions**
  - 1724, **Alexis Claude Clairaut** introduced **Clairaut Equation.** In 1787, **Adrien-Marie Legendre** introduced **Legendre transform** to solve a minimal surface problem given by **Gaspard Monge**
  - In 1869, **François Massieu** introduced in **Thermodynamics** dual (Massieu) **Characteristic Functions**, related by **Legendre transform**.
  - In 1912, **Henri Poincaré**, inspired by Massieu, introduced a **Characteristic Function** in **Probability** by **Laplace Transform** (logarithm of Poincaré characteristic function if Massieu characteristic function).
- **Fisher Metric and Information Geometry**
  - In 1939, **Maurice Fréchet** gave a Lecture at IHP introducing (Cramer-Rao) Bound and Clairaut-Legendre equation of Information Geometry (for « densités distinguées », distributions where the variance of estimators reached the Cramer-Rao-Fréchet-Darmois bound). Fréchet also observed that we have to consider hessian of a real function that was link to characteristic function.
  - In 1945, **C. R. Rao** has rediscovered Cramer-Rao Bound and used Fisher Information Matrix to define a metric in space of densities of probabilities.
  - In 1982, **N. Chentsov** has axiomatized Information Geometry and characterized the Fisher information metric as the only Riemannian metric that is invariant under sufficient statistics, based on category theory.
- **Learning « Natural Gradient » of Information Geometry**
  - In 1998, **Sun-Ishi Amari** in the framework of Information geometry, introduced Natural Gradient in percepton parameter space, proved to be Fisher efficient. The parameter space of multilayer networks forms a Riemannian space equipped with Fisher information metric. Instead of the usual gradient descent method, the natural gradient or Riemannian gradient

method, which takes account of the geometric structure of the Riemmanian space, is more effective for learning.
- In 2015, **Yann Ollivier** and in 2016 with **Gaëtan Marceau-Caron**, mitigating difficulty of Fisher information matrix inversion, introduced a Practical Natural Gradient for Deep Learning on very deep neural networks with many layers and parameters.

- **Thermodynamics & statistical Physics**
  - With **J.C. Maxwell** and **Ludwig Boltzmann**, **J.W. Gibbs** founded statistical mechanics.
  - Influenced by **J. W. Gibbs**, in 1891, **Pierre Duhem** extended Thermodynamics Potentials and Heat Capacities, and introduced Clausius-Duhem Inequality.
  - In 1909, **C. Carathéodory** formulated the Laws of Thermodynamics axiomatically.
  - Influenced by **V. Bargmann** [5], in 1969, **Jean-Marie Souriau** introduced Symplectic Geometry in (Statistical) Mechanics, and extension of Fisher Metric in Lie Groups Thermodynamics by studying non-equivariance of coadjoint operator on moment map (KKS 2 form in non-null cohomology and Souriau Symplectic Cocycle). Souriau gave a new definition of Entropy with a new geometric thermodynamics that is fully covariant under the action of the group acting on the system.
- **Geometry of Homogeneous Convex Cones**
  - **Jean-Louis Koszul**, introduced the **Koszul-Vinberg** Characteristic Function, and a Koszul 2 form on sharp convex cones**, that constitutes the algebraic foundation of Fisher Metric.**
  - Based on **J. Diximier** work, in 1972, **Alexandre A. Kirrilov** introduced Representation Theory and Coadjoint Orbit Method for Harmonic Analysis on Lie Groups

Figure 1. Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning.

The study of exponential densities invariant by a group goes back to the work of Muriel Casalis in her 1990 thesis supervised by Gérard Letac, and more recently by Ishi and Tojo [30-32]. This problem was previously studied in a geometric framework by Jean-Louis Koszul in the 1960s, in parallel with the work of Ernest Vinberg in Russia, to define Riemannian metric on sharp convex cones, invariant by the automorphisms of these cones.

The general problem was solved for Lie groups by Jean-Marie Souriau in Geometric Mechanics in 1969, by defining a "Lie groups Thermodynamics" in Statistical Mechanics (http://souriau2019.fr/) [6-11, 13, 42-45]. This Souriau's model considers the statistical mechanics of dynamic systems in their "space of evolution" associated with a symplectic manifold, and defines in case of non null cohomology (non equivariance of the coadjoint operator on the moment map with appearance of a cocyle) a density (of Gibbs) that is covariant under the action of dynamic groups of physics (eg, Galileo's group in classical physics) [14]. The family of exponential densities invariant by a group is a special case associated with the affine group. Koszul and

Souriau's approach uses the affine representation of Lie groups and Lie algebra (https://fgsi2019.sciencesconf.org/). V. Arnorld [47-48] studied Geometric approach of Thermodynamics with Contact and Symplectic geometries in Fluid Mechanics and R. Balian [49-58] in Quantum Physics.

Jean-Louis Koszul returns to this model of Jean-Marie Souriau in 1987 in Book "Introduction to Symplectic Geometry" (work that has just been translated into English). It is the tool by excellence for dealing with symmetries in Symplectic Geometry. These geometric structures associated with the exponential families make it possible to define generalizations of the Fisher metric in Information geometry (Koszul-Fisher metric related to the Koszul's 2-form on the sharp convex cones and Souriau-Fisher metric tied to coadjoint orbits and the moment map). Maurice Fréchet concomitantly with his discovery of the Fréchet-Darmois bound (called Cramer-Rao Bound) discovered the equation of Clairaut-Legendre at the foundation of these geometric structures in 1943. These new tools are bedrocks for Lie Group Machine Learning, due to Souriau Theorem relating to each Coadjoint Orbits of Lie Group an homogeneous Symplectic Manifold structure.

In the domain of Calculus of Variations, Jean-Marie Souriau studied the manifold of system movment in his book "Structure des systèmes dynamiques" published in 1969. When the system is Hamiltonian, the manifold of movements has a natural symplectic structure discovered in 1809 by Lagrange by introducing the Lagrange brackets. Jean-Marie Souriau gave a global geometric formulation of this Lagrange model. When a Lie group G acts by symplectomorphisms on a symplectic manifold under certain cohomological conditions, there is a natural map defined on the manifold and with values in the dual of the Lie algebra of G. Souriau called this map the "moment map". If we chose a base of the Lie algebra, the components of this map in the dual base are functions having for associated Hamiltonian vectors field the infinitesimal generators of the action of the group. The moment map is a geometrization of Emmy Noether's first theorem. This theorem asserts that every differentiable symmetries of the action of a physical system has a corresponding conservation law. If we consider the associated Hamiltonian system, the notion of moment map gives to this theorem a more geometric form: if the Hamiltonian of the system has a Lie algebra of infinitesimal symmetries, the corresponding moment map is constant on each integral curve of the system. The usual form of the theorem is obtained by considering each component of the moment application separately. In chapter IV of his book, Jean-Marie Souriau has applied this model for statistical mechanic and build a geometric model of thermodynamics that he called "Lie Groups Thermodynamics. This is the main achievement of Souriau in the domain of Calculus of Variations for a geometrization of statistical physics.

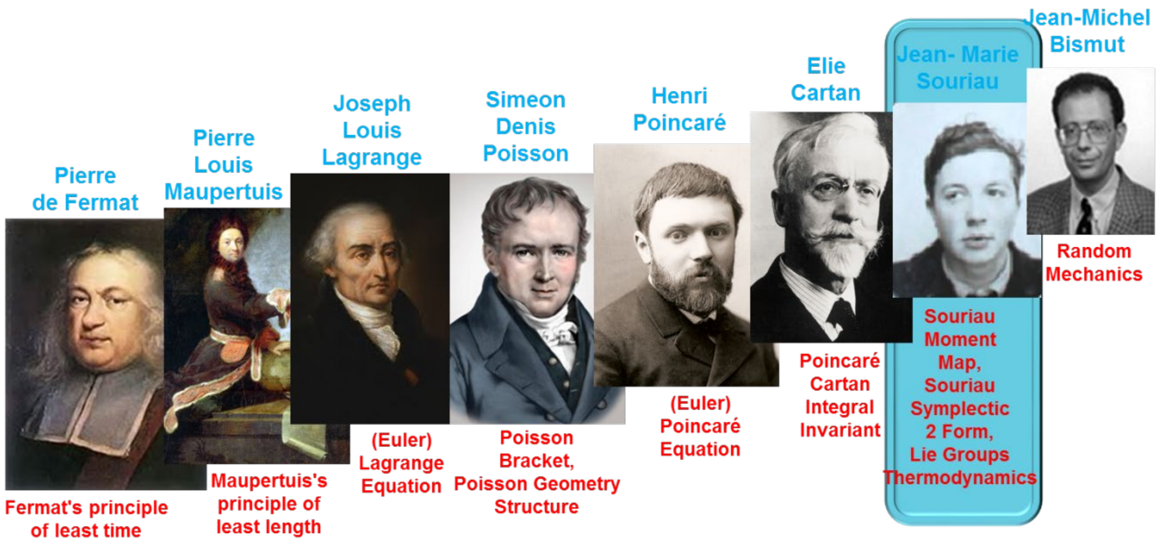

Figure 2. Jean-Marie Souriau contribution in the domain of Calculus of Variations.

We will develop Souriau Lie Groups Thermodynamics model and expression of covariant Gibbs density and Souriau-Koszul-Fisher Metric. After mathematical details about Souriau Moment map, we will illustrate the model for Poincaré Unit Disk considered as a symplectic manifold where the Lie Group SU(1,1)/K acts transitively (case of null cohomology). We will give expression of SU(1,1)/K Moment map to compute by mean of Kirillov representation theory and kirillov character, Souriau covariant Gibbs density. By this way, we can define this Gibbs density as a generalization of "Gauss density" in Poincaré Unit disk through representation theory. This "Gaussian density" and its statistical moments will be invariant under the action of the group. We will indicate also method to extend this approach to non-null cohomology case for particular use-case of SE(2) Lie Group.

1. Lie Groups Thermodynamics and Covariant Gibbs Density

We identify the Riemanian metric introduced by Souriau based on cohomology, in the framework of "Lie groups thermodynamics" as an extension of classical Fisher metric introduced in information geometry. We have observed that Souriau metric preserves Fisher metric structure as the Hessian of the minus logarithm of a partition function, where the partition function is defined as a generalized Laplace transform on a sharp convex cone. Souriau's definition of Fisher metric extends the classical one in case of Lie groups or homogeneous manifolds. Souriau has developed this "Lie groups thermodynamics" theory in the framework of homogeneous symplectic manifolds in geometric statistical mechanics for dynamical systems, but as observed by Souriau, these model equations are no longer linked to the symplectic manifold but equations only depend on the Lie group and the associated cocycle [77-78]. This analogy with Fisher metric opens potential applications in machine learning, where the Fisher metric is used in the framework of information geometry, to define the "natural gradient" tool for improving ordinary stochastic gradient descent sensitivity to rescaling or changes of variable in parameter space. In machine learning revised by natural gradient of information geometry, the ordinary gradient is designed to integrate the Fisher matrix. Amari has theoretically proved the asymptotic optimality of the natural gradient compared to classical gradient. With the Souriau approach, the Fisher metric could be extended, by Souriau-Fisher metric, to design natural gradients for data on homogeneous manifolds. Information geometry has been derived from invariant geometrical structure involved in statistical inference. The Fisher metric defines a Riemannian metric as the Hessian of two dual potential functions, linked to dually coupled affine connections in a manifold of probability distributions. With the Souriau model, this structure is extended preserving the Legendre transform between two dual potential function parametrized in Lie algebra of the group acting transentively on the homogeneous manifold.

1.1. Inference by natutal gradient and Legendre structure

Classically, to optimize the parameter θ of a probabilistic model, based on a sequence of observations y_t, is an online gradient descent:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t \frac{\partial l_t(y_t)^T}{\partial \theta} \tag{1}$$

with learning rate η_t, and the loss function $l_t = -\log p(y_t / \hat{y}_t)$. This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. Amari has introduced the natural gradient to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the Fisher information matrix with respect to parameter θ, given by:

$$I(\theta) = \left[g_{ij} \right] \text{ with } g_{ij} = \left[-E_{y \sim p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} \tag{2}$$

with natural gradient:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta)^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta} \tag{3}$$

Amari has proved that the Riemannian metric in an exponential family is the Fisher information matrix defined by:

$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j} \right]_{ij} \text{ with } \Phi(\theta) = -\log \int e^{-\langle \theta, y \rangle} dy \tag{4}$$

and the dual potential, the Shannon entropy, is given by the Legendre transform:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \text{ with } \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \text{ and } \theta_i = \frac{\partial S(\eta)}{\partial \eta_i} \tag{5}$$

We can observe that $\Phi(\theta) = -\log \int e^{-\langle \theta, y \rangle} dy = -\log \psi(\theta)$ is linked with the cumulant generating function.

J.L. Koszul and E. Vinberg have introduced an affinely invariant Hessian metric on a sharp convex cone through its characteristic function:

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy = -\log \psi_\Omega(\theta) \text{ with } \theta \in \Omega \text{ sharp convex cone}$$
$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy \text{ with Koszul-Vinberg Characteristic function} \quad (6)$$

Jean-Louis Koszul has introduced the following forms

1st Koszul form α : $\alpha = d\Phi_\Omega(\theta) = -d \log \psi_\Omega(\theta)$ (7)

2nd Koszul form γ : $\gamma = D\alpha = Dd \log \psi_\Omega(\theta)$ (8)

with the following property of positive definitiveness:

$$\left(Dd \log \psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \quad (9)$$
$$\text{with } F(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle} \text{ and } G(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle}\langle u, \xi \rangle$$

Koszul has defined the following Diffeomorphism:

$$\eta = \alpha = -d \log \psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi \text{ with } p_\theta(\xi) = \frac{e^{-\langle \xi, \theta \rangle}}{\int_{\Omega^*} e^{-\langle \xi, \theta \rangle} d\xi} \quad (10)$$

with preservation of Legendre transform:

$$S_\Omega(\eta) = \langle \theta, \eta \rangle - \Phi_\Omega(\theta) \text{ with } \eta = d\Phi_\Omega(\theta) \text{ and } \theta = dS_\Omega(\eta) \quad (11)$$

1.2. Souriau Lie Groups Thermodynamique and Souriau-Koszul-Fisher metric

This relations have been extended by Jean-Marie Souriau in geometric statistical mechanics, where he developed a "Lie groups thermodynamics" of dynamical systems where the (maximum entropy) Gibbs density is covariant with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda_\omega \text{ and } U : M \to \mathfrak{g}^* \quad (12)$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g} \quad (13)$$

In the Souriau Lie groups thermodynamics model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group. Souriau has proposed a Riemannian metric that we have identified as a generalization of the Fisher metric:

$$I(\beta) = \left[g_\beta\right] \text{ with } g_\beta\left([\beta, Z_1], [\beta, Z_2]\right) = \tilde{\Theta}_\beta\left(Z_1, [\beta, Z_2]\right) \quad (14)$$

$$\text{with } \tilde{\Theta}_\beta(Z_1, Z_2) = \tilde{\Theta}(Z_1, Z_2) + \langle Q, ad_{Z_1}(Z_2) \rangle \text{ where } ad_{Z_1}(Z_2) = [Z_1, Z_2] \quad (15)$$

Souriau has proved that all co-adjoint orbit of a Lie Group given by $O_F = \left\{ Ad_g^* F = g^{-1} F g, g \in G \right\}$ subset of $\mathfrak{g}^*$, $F \in \mathfrak{g}^*$ carries a natural homogeneous symplectic structure by a closed G-invariant 2-form. If we define $K = Ad_g^* = \left(Ad_{g^{-1}}\right)^*$ and $K_*(X) = -\left(ad_X\right)^*$ with $\left\langle Ad_g^* F, Y\right\rangle = \left\langle F, Ad_{g^{-1}} Y\right\rangle, \forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$ where if $X \in \mathfrak{g}$, $Ad_g(X) = gXg^{-1} \in \mathfrak{g}$, the G-invariant 2-form is given by the following expression $\sigma_\Omega\left(ad_X F, ad_Y F\right) = B_F\left(X,Y\right) = \left\langle F, \left[X,Y\right]\right\rangle, X,Y \in \mathfrak{g}$. Souriau Foundamental Theorem is that « every symplectic manifold is a coadjoint orbit ». We can observe that for Souriau model, Fisher metric is an extension of this 2-form in non-equivariant case $g_\beta\left(\left[\beta,Z_1\right],\left[\beta,Z_2\right]\right) = \tilde{\Theta}\left(Z_1,\left[\beta,Z_2\right]\right) + \left\langle Q,\left[Z_1,\left[\beta,Z_2\right]\right]\right\rangle$.

The Souriau additional term $\tilde{\Theta}\left(Z_1,\left[\beta,Z_2\right]\right)$ is generated by non-equivariance through Symplectic cocycle. The tensor $\tilde{\Theta}$ used to define this extended Fisher metric is defined by the moment map $J(x)$, application from M (homogeneous symplectic manifold) to the dual Lie algebra $\mathfrak{g}^*$, given by:

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \left\{J_X, J_Y\right\} \tag{16}$$

with $J(x): M \to \mathfrak{g}^*$ such that $J_X(x) = \left\langle J(x), X\right\rangle, X \in \mathfrak{g}$ (17)

This tensor $\tilde{\Theta}$ is also defined in tangent space of the cocycle $\theta\left(g\right) \in \mathfrak{g}^*$ (this cocycle appears due to the non-equivariance of the coadjoint operator Ad_g^*, action of the group on the dual lie algebra; the action of the group on dual Lie algebra is modified with a cocycle so that the momentu map becomes equivariant relative to this new affine action):

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right) \tag{18}$$

$\theta\left(g\right) \in \mathfrak{g}^*$ is called nonequivariance one-cocycle, and it is a measure of the lack of equivariance of the moment map.

$$\begin{aligned} &\tilde{\Theta}\left(X,Y\right): \mathfrak{g} \times \mathfrak{g} \to \Re \qquad \text{with } \Theta(X) = T_e\theta\left(X(e)\right) \\ &\qquad X,Y \mapsto \left\langle \Theta(X), Y\right\rangle \end{aligned} \tag{19}$$

The cocycle should verify:

$$\begin{aligned} &\theta(st) = J\left(\left(st\right).x\right) - Ad_{st}^* J(x) \\ &\theta(st) = \left[J(s.\left(t.x\right)) - Ad_s^* J\left(t.x\right)\right] + \left[Ad_s^* J\left(t.x\right) - Ad_s^* Ad_t^* J(x)\right] \\ &\theta(st) = \theta(s) + Ad_s^*\left[J\left(t.x\right) - Ad_t^* J(x)\right] \\ &\theta(st) = \theta(s) + Ad_s^*\theta(t) \end{aligned} \tag{20}$$

We can also compute tangent of one-cocycle θ at neutral element, to compute 2-cocycle Θ:

$$\begin{aligned} &\zeta \in \mathfrak{g} \ , \ \theta_\zeta\left(s\right) = \left\langle \theta\left(s\right), \zeta\right\rangle = \left\langle J\left(s.x\right), \zeta\right\rangle - \left\langle Ad_s^* J\left(x\right), \zeta\right\rangle = \left\langle J\left(s.x\right), \zeta\right\rangle - \left\langle J\left(x\right), Ad_{s^{-1}}\zeta\right\rangle \\ &T_e\theta_\zeta\left(\xi\right) = \left\langle T_x J.\xi_p(x), \zeta\right\rangle + \left\langle J\left(x\right), ad_\xi\zeta\right\rangle \ \text{ with } \ \xi_p = X_{\langle J,\xi\rangle} \\ &T_e\theta_\zeta\left(\xi\right) = X_{\langle J(x),\xi\rangle}\left[\left\langle J(x),\zeta\right\rangle\right] + \left\langle J\left(x\right),\left[\xi,\zeta\right]\right\rangle \\ &T_e\theta_\zeta\left(\xi\right) = -\left\{\left\langle J,\xi\right\rangle,\left\langle J,\zeta\right\rangle\right\} + \left\langle J(x),\left[\xi,\zeta\right]\right\rangle = \Theta\left(\xi\right) \end{aligned} \tag{21}$$

We can also write: $T_x J\left(\xi_p(x)\right) = -ad_\xi^* J(x) + \Theta\left(\xi,.\right)$

By differentiating the equation on affine action, we have:

$$dJ(Xx) = ad_X J(x) + d\theta(X) \ , \ x \in M, X \in \mathfrak{g} \tag{22}$$

$$\begin{aligned} &\left\langle dJ(Xx), Y\right\rangle = \left\langle ad_X J(x), Y\right\rangle + \left\langle \mathrm{d}\theta(X), Y\right\rangle, \ x \in M, X, Y \in \mathfrak{g} \\ &\left\langle dJ(Xx), Y\right\rangle = \left\langle J(x),\left[X,Y\right]\right\rangle + \left\langle \mathrm{d}\theta(X), Y\right\rangle \ = \left\{\left\langle J,X\right\rangle,\left\langle J,Y\right\rangle\right\}(x) \\ &\left\langle J(x),\left[X,Y\right]\right\rangle - \left\{\left\langle J,X\right\rangle,\left\langle J,Y\right\rangle\right\}(x) = -\left\langle \mathrm{d}\theta(X), Y\right\rangle \end{aligned} \tag{23}$$

It can be then deduced that the tensor could be also written:

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \left\{J_X, J_Y\right\} = -\left\langle d\theta(X), Y\right\rangle \ , \ X,Y \in \mathfrak{g} \tag{24}$$

with the cocycle property:

$$\tilde{\Theta}\left(\left[X,Y\right],Z\right) + \tilde{\Theta}\left(\left[X,Y\right],Z\right) + \tilde{\Theta}\left(\left[X,Y\right],Z\right) = 0 \ , \ X,Y,Z \in \mathfrak{g} \tag{25}$$

By noting the action of the group on the dual Lie algebra:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (s,\xi) \mapsto s\xi = Ad_s^*\xi + \theta(s) \tag{26}$$

Associativity is also derived:

$$(s_1s_2)\xi = Ad^*_{s_1s_2}\xi + \theta(s_1s_2) = Ad^*_{s_1}Ad^*_{s_2}\xi + \theta(s_1) + Ad^*_{s_1}\theta(s_2)$$
$$(s_1s_2)\xi = Ad^*_{s_1}\left(Ad^*_{s_2}\xi + \theta(s_2)\right) + \theta(s_1) = s_1\left(s_2\xi\right) \quad , \ \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^* \tag{27}$$

This study of the moment map J equivariance, and the existence of an affine action of G on $\mathfrak{g}^*$, whose linear part is the coadjoint action, for which the moment J is equivariant, is at the cornerstone of Souriau theory of geometric mechanics and Lie groups thermodynamics.

1.3. Souriau Entropy and Souriau-Fisher-Koszul metric Invariance under the action of the group and Covariant Souriau Gibbs density

In Souriau's Lie groups thermodynamics, the invariance by re-parameterization in information geometry has been replaced by invariance with respect to the action of the group. When an element of the group g acts on the element $\beta \in \mathfrak{g}$ of the Lie algebra, given by adjoint operator Ad_g. Under the action of the group $Ad_g(\beta)$, the entropy $S(Q)$ and the Fisher metric $I(\beta)$ are invariant:

$$\beta \in \mathfrak{g} \rightarrow Ad_g(\beta) \Rightarrow \begin{cases} S\left[Q\left(Ad_g(\beta)\right)\right] = S(Q) \\ I\left[Ad_g(\beta)\right] = I(\beta) \end{cases} \tag{28}$$

In the framework of Lie group action on a symplectic manifold, equivariance of moment map could be studied to prove that there is a unique action $a(.,.)$ of the Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra for which the moment map J is equivariant, that means for each $x \in M$:

$$J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad^*_g\left(J(x)\right) + \theta(g) \tag{29}$$

When coadjoint action is not equivariant, the symmetry is broken, and new "cohomological" relations should be verified in Lie algebra of the group. A natural equilibrium state will thus be characterized by an element of the Lie algebra of the Lie group, determining the equilibrium temperature β. The entropy $s(Q)$, parametrized by Q the geometric heat (mean of energy U, element of the dual Lie algebra) is defined by the Legendre transform of the Massieu potential $\Phi(\beta)$ parametrized by β ($\Phi(\beta)$ is the minus logarithm of the partition function $\psi_\Omega(\beta)$).

Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle \beta, U(\xi)\rangle} = \frac{e^{-\langle \beta, U(\xi)\rangle}}{\int\limits_M e^{-\langle \beta, U(\xi)\rangle} d\lambda_\omega} \quad , \text{ with } \Phi(\beta) = -\log \int\limits_M e^{-\langle \beta, U(\xi)\rangle} d\lambda_\omega$$
$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int\limits_M U(\xi) e^{-\langle \beta, U(\xi)\rangle} d\lambda_\omega}{\int\limits_M e^{-\langle \beta, U(\xi)\rangle} d\lambda_\omega} = \int\limits_M U(\xi) p(\xi) d\lambda_\omega \tag{30}$$

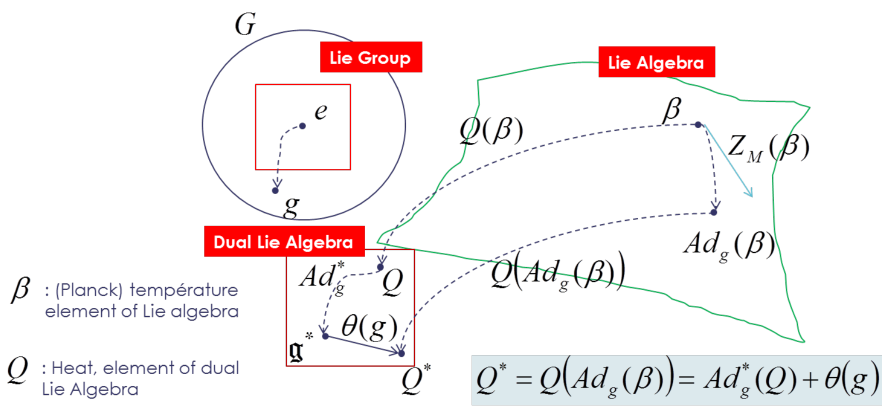

Figure 1. Fondamental Equation of Souriau Lie Groups Thermodynamics.

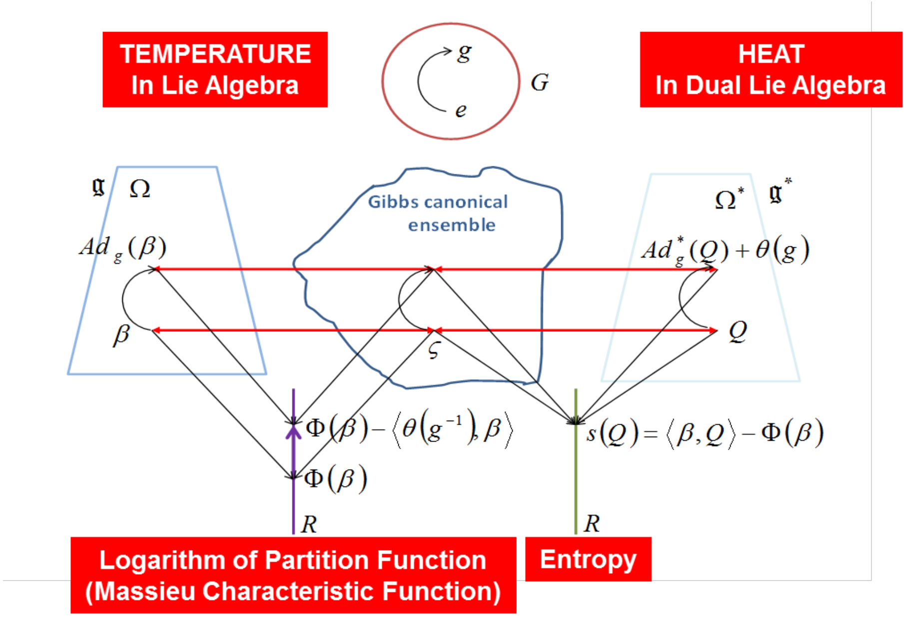

Figure 2. Souriau Model of Lie Groups Thermodynamics.

Souriau completed his "geometric heat theory" by introducing a 2-form in the Lie algebra, that is a Riemannian metric tensor in the values of adjoint orbit of β, $[\beta, Z]$ with Z an element of the Lie algebra. This metric is given for (β, Q):

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \langle \Theta(Z_1),[\beta,Z_2]\rangle + \langle Q,[Z_1,[\beta,Z_2]]\rangle \tag{31}$$

where Θ is a cocycle of the Lie algebra, defined by $\Theta = T_e\theta$ with θ a cocycle of the Lie group defined by $\theta(M) = Q(Ad_M(\beta)) - Ad_M^* Q$.

We observe that Souriau Riemannian metric, introduced with symplectic cocycle, is a generalization of the Fisher metric, that we call the Souriau-Fisher metric, that preserves the property to be defined as a Hessian of the partition function logarithm $g_\beta = -\dfrac{\partial^2 \Phi}{\partial \beta^2} = \dfrac{\partial^2 \log \psi_\Omega}{\partial \beta^2}$ as in classical information geometry. We will establish the equality of two terms, between Souriau definition based on Lie group cocycle Θ and parameterized by "geometric heat" Q (element of dual Lie algebra) and "geometric temperature" β (element of Lie algebra) and hessian of characteristic function $\Phi(\beta) = -\log \psi_\Omega(\beta)$ with respect to the variable β:

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \langle \Theta(Z_1),[\beta,Z_2]\rangle + \langle Q,[Z_1,[\beta,Z_2]]\rangle = \frac{\partial^2 \log \psi_\Omega}{\partial \beta^2} \tag{32}$$

If we differentiate this relation of Souriau theorem $Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$, this relation occurs:

$$\frac{\partial Q}{\partial \beta}(-[Z_1,\beta],.) = \tilde{\Theta}(Z_1,[\beta,.]) + \langle Q, Ad_{.Z_1}([\beta,.])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,.]) \tag{33}$$

$$-\frac{\partial Q}{\partial \beta}([Z_1,\beta],Z_2.) = \tilde{\Theta}(Z_1,[\beta,Z_2]) + \langle Q, Ad_{.Z_1}([\beta,Z_2])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \tag{34}$$

$$\Rightarrow -\frac{\partial Q}{\partial \beta} = g_\beta([\beta,Z_1],[\beta,Z_2]) \tag{35}$$

Souriau-Fisher Metric $I(\beta)=\left[g_\beta\right]$ with $g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$

with $\tilde{\Theta}_\beta\left(Z_1,Z_2\right)=\tilde{\Theta}\left(Z_1,Z_2\right)+\left\langle Q,[Z_1,Z_2]\right\rangle$

Non-null cohomology: aditional term from Souriau Cocycle **Equivariant KKS 2 form**

$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\}$ with $J(x):M\to\mathfrak{g}^*$ such that $J_X(x)=\left\langle J(x),X\right\rangle, X\in\mathfrak{g}$

$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re$ with $\Theta(X)=T_e\theta\left(X(e)\right)$ $\tilde{\Theta}(\beta,Z)+\left\langle Q,[\beta,Z]\right\rangle=0$

$X,Y\mapsto\left\langle\Theta(X),Y\right\rangle$ $\beta\in Ker\,\tilde{\Theta}_\beta$

Souriau Fundamental Equation of Lie Group Thermodynamics $Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$

Figure 3. Souriau-Fisher metric as extension of KKS 2-form in case of non-null Cohomogy.

As the entropy is defined by the Legendre transform of the characteristic function, a dual metric of the Fisher metric is also given by the hessian of "geometric entropy" $S(Q)$ with respect to the dual variable given by Q: $\frac{\partial^2 S(Q)}{\partial Q^2}$.

For the maximum entropy density (Gibbs density), the following three terms coincide: $\frac{\partial^2 \log\psi_\Omega}{\partial\beta^2}$ that describes the convexity of the log-likelihood function, $I(\beta)=-E\left[\frac{\partial^2\log p_\beta(\xi)}{\partial\beta^2}\right]$ the Fisher metric that describes the covariance of the log-likelihood gradient, whereas $I(\beta)=E\left[(\xi-Q)(\xi-Q)^T\right]=Var(\xi)$ that describes the covariance of the observables. We can also observe that the Fisher metric $I(\beta)=-\frac{\partial Q}{\partial\beta}$ is exactly the Souriau metric defined through symplectic cocycle:

$$I(\beta)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)=g_\beta\left([\beta,Z_1],[\beta,Z_2]\right) \tag{36}$$

The Fisher metric $I(\beta)=-\frac{\partial^2\Phi(\beta)}{\partial\beta^2}=-\frac{\partial Q}{\partial\beta}$ has been considered by Souriau as a *generalization of* "*heat capacity*". Souriau called it K the "*geometric capacity*".

1.4. Covariant Souriau Gibbs density and Information Manifold Foliation

R.F. Streater has studied in 1999, Information Geometry for some Lie algebra where for certain unitary representation of a Lie algebra, he has defined the statistical manifold of states as convex cone for which the partition function is finite, making reference to Bogoliubov-Kubo-Mori metric. But Streater has only developed the case with null cohomology for so(3) and sl(2,R) Lie alebras. Nevertheless, as observed by R.F. Streater in his paper "**Information Geometry for some Lie algebras**" [83], referring to Kirillov work and Roger Balian paper, "*We can expect further natural structures to arise in this case. Indeed, it is known (*) that* ***the dual to the Lie algebra****, which parametrizes the state-space in this case,* ***foliates into coadjoint orbits; there are also the level sets on the entropy; Kirillov form, and the BKM (Bogoliubov-Kubo-Mori) metric, together make each orbit into kähler space****, along the lines proposed by Kostant.* ***Motion along these holomorphic directions is nondissipative. The transversal to the orbits is a real half-line, which represents the dissipative direction****…We study the case of sl(2,R) in the discrete series of representations. We show* ***the information manifold foliates into level sets of the entropy****, each being isometric to H, the Poincaré upper half-plane… The states of constant entropy are the hyperboloids and* β *is the dissipative coordinate…For an integrable system described by a Lie algebra in a traceable representation, we find that* ***the information manifold foliates into***

***complex spaces; the level sets of entropy can be given a complex structure by the method of Kostant**. Motion remaining on the complex surfaces is nondissipative, whereas motion transversal to these surfaces is dissipative. In information geometry, the state is parametrized by the canonical coordinates. Which function of them is measured by a thermometer? In our models, it is reasonable to designate $1/\beta$ to be the temperature; it is a dissipative coordinate, and it increases with time, showing that the system is thermalizing".*

2. Mathematical definition of Souriau Moment map

In previous chapter, we have introduced Souriau moment map. In this chapter, we provide mathematical definition of Souriau moment map, as defined in Souriau's book "Structure of Dynamical System: A Symplectic View of Physics" [6] with modern notations [105-106,108]. More details could be also find in Jean-Louis Koszul Book "Introduction to Symplectic Geometry", that we have recently supervised translation [109].

2.1. Operations on Vector Fields

Consider a map $F: X \subset R^M \to Y \subset R^N$, $y = F(x)$, the derivative of F at $x \in X$, $DF: X \to R^{N \times M}$ is given by:

$$\begin{pmatrix} \delta y^1 \\ \vdots \\ \delta y^N \end{pmatrix} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^M} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^N}{\partial x^1} & \cdots & \frac{\partial y^N}{\partial x^M} \end{pmatrix} \begin{pmatrix} \delta x^1 \\ \vdots \\ \delta x^M \end{pmatrix} = DF(x)(\delta x) = \underset{t \to 0}{Lim} \frac{F(x + t\delta x) - F(x)}{t} \quad (37)$$

Second derivative is given by the linear map $D^2F: X \to R^{N \times M \times M}: \delta \left[\frac{\partial y}{\partial x} \right] = \frac{\partial^2 y}{\partial x^2}(\delta x) = D^2F(x)(\delta x)$ (38)

Consider a vector Field V on $X \subset R^M$ defined by : $V: X \subset R^M \to R^M$, operations on vector fields are given by adjoint action and Lie bracket:

$$Ad_F V(y) = \frac{d}{dt}\left[F \circ e^{tV} \circ F^{-1} \right](y)\Big|_{t=0} = DF(x)\big(V(x)\big) \text{ with } x = F^{-1}(y) \quad (39)$$

$$[U,V](x) = \frac{d}{ds} Ad_{e^{sU}} V(x)\Big|_{s=0} = DU(x)\big(V(x)\big) - DV(x)\big(U(x)\big) \quad (40)$$

0-form is a scalar, 1-form are row $\omega = (\omega_1 \cdots \omega_M)$ in dual space. 2-forms can be regarded as antisymmetric matrices (ω_{ij}) with $\omega(u,v) = u^t \begin{pmatrix} \omega_{11} & \cdots & \omega_{1M} \\ \vdots & \ddots & \vdots \\ \omega_{M1} & \cdots & \omega_{MM} \end{pmatrix} v$. m-forms are all scalar multiples of the standard volume form vol, defined by $Vol(v_1, \cdots, v_m) = \det(\text{matrix with columns } v_1, ..., v_m)$.

2.2. Derivative rules by Sophus Lie, Elie Cartan and Henri Cartan

With the following classical definitions:

- ***Pull back***: $F^*\omega$ is a p-form on X

$$F^*\omega(v_1, \cdots, v_p) = \omega_{F(x)}\big(DF(x)(v_1), \cdots, DF(x)(v_p)\big) \quad (41)$$

- ***Interior product***: $i_V\omega$ is the (p-1)form on M obtained by inserting $V(x)$ as the first argument of ω

$$i_V\omega\left(v_2,\cdots v_p\right)=\omega\left(V(x),v_2,\cdots,v_p\right) \tag{42}$$

- ***Exterior product:*** $\theta\wedge\omega$ is the (p + 1)-form on X where ω is a p-form and θ is a 1-form on M (where the hat indicates a term to be omitted).:

$$\theta\wedge\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\,\theta(v_i)\omega\left(v_0,\cdots,\hat{v}_i,\cdots,v_p\right) \tag{43}$$

- ***Lie derivative*** : $L_V\omega$ is a p-form on M, and $L_V\omega=0$ if the flow of V consists of symmetries of ω:

$$L_V\omega\left(v_1,\cdots,v_p\right)=\frac{d}{dt}e^{tV*}\omega\left(v_1,\cdots,v_p\right)\Big|_{t=0} \tag{44}$$

$d\omega$ is the (p+1)-form on M defined by taking the ordinary derivative of ω and then antisymmetrizing:

- ***Exterior derivative:***

$$d\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\frac{\partial\omega}{\partial x}(v_i)(v_0,\cdots,\hat{v}_i,\cdots,v_p) \tag{45}$$

$$p=0,[d\omega]_i=\partial_i\omega\ ;\ p=1,[d\omega]_{ij}=\partial_i\omega_j-\partial_j\omega_i\ ;\ p=2,[d\omega]_{ijk}=\partial_i\omega_{jk}+\partial_j\omega_{ki}+\partial_k\omega_{ij} \tag{46}$$

From these definitions, the properties of the exterior and Lie Derivative were established by Sophus Lie, Elie Cartan, and Henri Cartan:

- $L_V\omega=di_V\omega+i_Vd\omega$ ***(lie. Cartan)*** (47)
- $i_{[U,V]}\omega=i_VL_U\omega-L_Ui_V\omega$ ***(Henri Cartan)*** (48)
- $L_{[U,V]}\omega=L_VL_U\omega-L_UL_V\omega$ ***(Sophus Lie)*** (49)

2.3. Souriau Moment Map

Considering Manifolds and Lie groups, We define the tangent bundle TX of X as the disjoint union of the T_xX, or the set of all pairs $\begin{pmatrix}\delta x\\ x\end{pmatrix}$ with $x\in X$ and $\delta x\in T_xX$. If $F:X\to Y$ is a smooth map between manifolds, its tangent map is the map: $F_*\begin{pmatrix}\delta x\\ x\end{pmatrix}=\begin{pmatrix}DF(x)(\delta x)\\ F(x)\end{pmatrix}$ (50)

A Lie group is a group G with a manifold structure such that the product $(g,h)\mapsto gh$ and the inversion $g\mapsto g^{-1}$ are smooth maps from $G\times G$ (resp. G) to G. Its Lie algebra is the tangent space $g=T_eG$ at the identity element. A smooth action of G on a manifold X is a group morphism:

$$\begin{aligned}&\Phi:G\times X\to Diff(X)\\&(g,x)\mapsto g.x\end{aligned} \tag{51}$$

The orbit of $x\in X$ is $G(x)=\{g.x:g\in G\}$.

The tangent space to an orbit at x:

$$T_xG(x)=\{Z(x):Z\in\mathfrak{g}\}=\mathfrak{g}/\mathfrak{g}_x \text{ with } Z(x)=\frac{d}{dt}e^{tZ}(x)\Big|_{t=0} \text{ and where } \mathfrak{g}_x=\{Z\in\mathfrak{g}:Z(x)=0\} \tag{52}$$

Let (M,σ) be a connected symplectic manifold. A vector field η on M is called symplectic if its flow preserves the 2-form : $L_\eta\sigma=0$. If we use Elie Cartan's formula, we can deduce that $L_\eta\sigma=di_\eta\sigma+i_\eta d\sigma=0$ but as $d\sigma=0$ then $di_\eta\sigma=0$. We observe that the 1-form $i_\eta\sigma$ is closed. When this 1-form is exact, there is a smooth function $x\mapsto H$ on M with:

$$i_\eta\sigma=-dH \tag{53}$$

This vector field η is called Hamiltonian and could be defined as symplectic gradient $\eta=\nabla_{Symp}H$.

Let a Lie group G that acts on M and that also preserve σ. A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map $J:M\to\mathfrak{g}^*$ exists with:

$$i_{Z_X}\sigma=-dH_Z \quad\text{where}\quad H_Z=\langle J(x),Z\rangle \tag{54}$$

The Poisson bracket of two functions H, H' is defined by :

$$\{H,H'\}=\sigma(\eta,\eta')=\sigma\left(\nabla_{Symp}H',\nabla_{Symp}H\right) \text{ with } i_\eta\sigma=-dH \text{ and } i_{\eta'}\sigma=-dH' \tag{55}$$

If G is connected, then the moment map is G-equivariant if and only if it satisfies $\{H_Z, H_{Z'}\} = H_{[Z,Z']}$. Souriau has proved thet every coadjoint orbit of a Lie group is a homogeneous symplectic manifold when endowed with the KKS 2-form $\sigma\left(Z(x), Z'(x)\right) = \left\langle x, [Z', Z] \right\rangle$, and conversely, every homogeneous symplectic manifold of a connected Lie group G is, up to a possible covering, a coadjoint orbit of some central extension of G. σ is G-invariant.

3. Poincaré Unit Disk, SU(1,1) Lie Group and Souriau Moment map

We will introduce Souriau moment map for SU(1,1)/K group that acts transitively on Poincaré Unit Disk.

3.1. Poincaré Unit Disk and SU(1,1) Lie Group

The group of complex unimodular pseudo-unitary matrices $SU(1,1)$, is the set of elements u such that [20-22,27-29, 79-82]:

$$uMu^+ = M \quad \text{with} \quad M = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{56}$$

We can show that the most general matrix u belongs to the Lie group given by:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a,b \in C \right\} \tag{57}$$

Its Cartan decomposition is given by:

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} = |a| \begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix} \begin{pmatrix} a/|a| & 0 \\ 0 & a^*/|a| \end{pmatrix} \text{ with } z = b\left(a^*\right)^{-1}, |a| = \left(1 - |z|^2\right)^{-1/2} \tag{58}$$

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix} = |a'| \begin{pmatrix} 1 & z' \\ z'^* & 1 \end{pmatrix} \begin{pmatrix} a'/|a'| & 0 \\ 0 & a'^*/|a'| \end{pmatrix} \text{ with } \begin{cases} a' = bz^* + a \\ z' = \dfrac{az+b}{b^* z + a^*} \end{cases} \tag{59}$$

$SU(1,1)$ is associated to group of holomorphic automorphisms of the Poincaré unit disk $D = \{ z = x + iy \in C / |z| < 1 \}$ in the complex plane, by considering its action on the disk as $g(z) = (az+b)/(b^* z + a^*)$. The following measure on Unit disk:

$$d\mu_0\left(z, z^*\right) = \frac{1}{2\pi i} \frac{dz \wedge dz^*}{\left(1 - |z|^2\right)^2} \tag{60}$$

is invariant under the action of $SU(1,1)$ captured by the fractional holomorphic transformation:

$$\frac{dz' \wedge dz'^*}{\left(1 - |z'|^2\right)^2} = \frac{dz \wedge dz^*}{\left(1 - |z|^2\right)^2} \tag{61}$$

The complex unit disk admits a Kähler structure determined by potential function:

$$\Phi\left(z', z^*\right) = -\log\left(1 - z' z^*\right) \tag{62}$$

The invariant 2-form is: $$\Omega = \frac{1}{i} \frac{\partial^2 \Phi\left(z, z^*\right)}{\partial z \partial z^*} dz \wedge dz^* = \frac{1}{i} \frac{dz \wedge dz^*}{\left(1 - |z|^2\right)^2} \tag{63}$$

which is closed $d\Omega = 0$. This group $SU(1,1)$ is isomorphic to the group $SL(2,\mathbb{R})$ as a real Lie group, and the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$g=\left\{\begin{pmatrix} -ir & \eta \\ \eta^{*} & ir \end{pmatrix} / r\in R, \eta\in C\right\} \tag{64}$$

with the bases $(u_1,u_2,u_3)\in \mathfrak{g}$: $u_1=\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $u_2=\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $u_3=\frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

with the commutation relation: $[u_3,u_2]=u_1,[u_3,u_1]=-u_2,[u_2,u_1]=-u_3$ (65)

Dual base on dual Lie algebra is named $(u_1^*,u_2^*,u_3^*)\in\mathfrak{g}^*$. The dual vector space $\mathfrak{g}^*=\mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,C)$ of the form:

$$g^{*}=\left\{\begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix}=x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}+y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}+z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x,y,z\in R\right\} \tag{66}$$

Coadjoint action of $g\in G$ on dual Lie algebra $\xi\in\mathfrak{g}^*$ is written $g.\xi$.

3.2. Coadjoint Orbit of SU(1,1) and Souriau Moment Map

We will use results of C. Cishahayo and S. de Bièvre [15] and B. Cahen [16,17] for computation of moment map of $SU(1,1)$. Let $r\in R^{*+}$, orbit $O(ru_3^*)$ of ru_3^* for the coadjoint action of $g\in G$ could be identified with the upper half sheet $x_3>0$ of $\left\{\xi=x_1u_1^*+x_2u_2^*+x_3u_3^* / -x_1^2-x_2^2+x_3^2=r^2\right\}$, the two-sheet hyperboloid. The stabilizer of ru_3^* for the coadjoint action of G is torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},\theta\in\square\right\}$. K induces rotations of the unit disk, and leaves 0 invariant. The stabilizer for the origin 0 of unit disk is maximal compact subgroup K of *SU(1,1)*. We can observe [16] that $O(ru_3^*)=G/K$. On the other hand $O(ru_3^*)=G/K$ is diffeomorphic to the unit disk $D=\{z\in C/|z|<1\}$, then by composition, the Souriau moment map is given by:

$$J:D\rightarrow O(ru_3^*)$$
$$z\mapsto J(z)=r\left(\frac{z+z^*}{(1-|z|^2)}u_1^*+\frac{z-z^*}{i(1-|z|^2)}u_2^*+\frac{1+|z|^2}{(1-|z|^2)}u_3^*\right) \tag{67}$$

J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $O(ru_3^*)=G/K$. J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $C\cup\infty$ [46]. In case $r=\frac{n}{2}$ where $n\in N^{+}, n\geq 2$ then the coadjoint orbit is given by $O_n=O(\zeta_n)$ with $\xi_n=\frac{n}{2}u_3^*\in\mathfrak{g}^*$, with stabilizer of ξ_n for coadjoint action the torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},\theta\in R\right\}$ with Lie algebra Ru_3. $O_n=O(\zeta_n)$ is associated with a holomorphic discrete series representation π_n of G by the KKS (Kirillov-Kostant-Souriau) method of orbits.

$$J:D\rightarrow O_n$$
$$z\mapsto J(z)=\frac{n}{2}\left(\frac{z+z^*}{(1-|z|^2)}u_1^*+\frac{z-z^*}{i(1-|z|^2)}u_2^*+\frac{1+|z|^2}{(1-|z|^2)}u_3^*\right) \tag{68}$$

Group G act on D by homography $g.z=\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}.z=\frac{az+b}{b^*z+a^*}$. ***This action corresponds with coadjoint action of*** G ***on*** O_n. The Kirillov-Kostant-Souriau 2-form of O_n is given by:

$$\Omega_n(\zeta)(X(\zeta),Y(\zeta))=\langle\zeta,[X,Y]\rangle \ , X,Y\in\mathfrak{g} \text{ and } \zeta\in O_n \tag{69}$$

and is associated in the frame by J with: $\omega_n=\frac{in}{\left(1-|z|^2\right)^2}dz\wedge dz^*$ (70)

with the corresponding Poisson Bracket: $\{f,g\}=i\left(1-|z|^2\right)^2\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial g}{\partial z}\right)$ (71)

It has been also observed that there are 3 basic observables generating the $SU(1,1)$ symmetry on classical level:

$$\begin{cases} D\to R \\ z\mapsto k_3(z)=\frac{1+|z|^2}{1-|z|^2} \end{cases},\begin{cases} D\to R \\ z\mapsto k_1(z)=\frac{1}{i}\frac{z-z^*}{1-|z|^2} \end{cases},\begin{cases} D\to R \\ z\mapsto k_2(z)=\frac{z+z^*}{1-|z|^2} \end{cases} \tag{72}$$

With the Poisson commutation rule: $\{k_3,k_1\}=k_2,\{k_3,k_2\}=-k_1,\{k_1,k_2\}=-k_3$ (73)

(k_1,k_2,k_3) vector points to the upper sheet of the two-sheeted hyperboloid in R^3 given by $k_3^2-k_1^2-k_2^2=1$, whose the stereographic projection onto the open unit disk is:

$$\begin{cases} (k_1,k_2,k_3)\in H^+\to D \\ z=\frac{k_2+ik_1}{1+k_3}=\sqrt{\frac{k_3-1}{k_3+1}}e^{i\arg z} \end{cases} \tag{74}$$

Under the action of $g\in G=SU(1,1)=\left\{\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}/|a|^2-|b|^2=1,\ a,b\in C\right\}$:

$\begin{pmatrix} k_- & k_3 \\ k_3 & k_+ \end{pmatrix}=\begin{pmatrix} k_2+ik_1 & k_3 \\ k_3 & k_2-ik_1 \end{pmatrix}=\frac{1}{1-|z|^2}\begin{pmatrix} 2z & 1+|z|^2 \\ 1+|z|^2 & 2z^* \end{pmatrix}$ is transform in:

$$\begin{pmatrix} k_-^{'} & k_3^{'} \\ k_3^{'} & k_+^{'} \end{pmatrix}=\begin{pmatrix} k_-(g^{-1}.z) & k_3(g^{-1}.z) \\ k_3(g^{-1}.z) & k_+(g^{-1}.z) \end{pmatrix}=g^{-1}\begin{pmatrix} k_- & k_3 \\ k_3 & k_+ \end{pmatrix}(g^{-1})^t \tag{75}$$

This transform can be viewed as the co-adjoint action of $SU(1,1)$ ***on the coadjoint orbit identified with*** $k_3^2-k_1^2-k_2^2=1$.

4. Covariant Gibbs Density by Souriau Thermodynamics for Poincaré Unit Disk

4.1. Covariant Gibbs densité for Poincaré Unit Disk

Representation theory studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, and algebraic objects (Lie groups, Lie algebras) by describing its elements by matrices and the algebraic operations in terms of matrix addition and matrix multiplication, reducing problems of abstract algebra to problems in linear algebra. Representation theory generalizes Fourier analysis via harmonic analysis. The modern development of Fourier analysis during XXth century has explored the generalization of Fourier and Fourier-Plancherel formula for non-commutative harmonic analysis, applied to locally compact non-Abelian groups. This has been solved by geometric approaches based on "orbits methods" (Fourier-Plancherel formula for G is given by coadjoint representation of G in dual vector space of its Lie algebra) with many contributors (Dixmier, Kirillov, Bernat, Arnold, Berezin, Kostant, Souriau, Duflo, Guichardet, Torasso, Vergne, Paradan, etc.) [23-26, 33-38, 40-41].

For classical commutative harmonic analysis, we consider the following groups:

$G = \mathrm{T}^n = R^n / Z^n$ for Fourier series, $G = R^n$ for Fourier Transform

G group character (linked to e^{ikx}): $\chi : G \to U$ with $U = \{ z \in C / |z| = 1 \}$

$\hat{G} = \{ \chi / \chi_1 . \chi_2 (g) = \chi_1 (g) \chi_2 (g) \}$ and Fourier transform is given by:

$$\begin{array}{l} \varphi : G \to C \\ g \mapsto \varphi(g) = \int\limits_{\hat{G}} \hat{\varphi}(\chi) \chi(g)^{-1} d\chi \end{array} \quad \text{and} \quad \begin{array}{l} \hat{\varphi} : \hat{G} \to C \\ \chi \mapsto \hat{\varphi}(\chi) = \int\limits_{G} \varphi(g) \chi(g) dg \end{array} \tag{76}$$

For non-commutative harmonic analysis, Group unitary irreductible representation is $\mathrm{U} : G \to U(\mathrm{H})$ with H Hilbert space and character by $\chi_{\mathrm{U}}(g) = tr \mathrm{U}_g$. Fourier transform for non-commutative group is $\mathrm{U}_\varphi = \int\limits_G \varphi(g) \mathrm{U}_g dg$ with character $\chi_{\mathrm{U}}(g) = tr \mathrm{U}_\varphi$. If we describe group element with exponential map $\mathrm{U}_\psi = \int\limits_{\mathfrak{g}} \psi(X) \mathrm{U}_{\exp(X)} dX$, we have:

$$\begin{array}{l} \mathrm{trU}_\psi = \dim \tau . \mu_{G.f} \left(\hat{\psi . j^{-1}} \right) \\ \hat{\psi . j^{-1}} : \mathfrak{g} \to \mathfrak{g}^*, \text{Four. Transf.} \end{array} \quad \text{with} \quad \begin{cases} \mu_{G.f} : \text{Liouville meas. on } \mathrm{O} = G.f , f \in \mathfrak{g}^* \\ \mu_{G.f} \left(\hat{\psi . j^{-1}} \right) : \text{Integral of } \hat{\psi . j^{-1}} \text{ wrt } \mu_{G.f} \end{cases} \tag{77}$$

$$\text{where} \quad j(X) = \left(\det s(ad_X) \right)^{1/2} \quad \text{with} \quad s(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{x}{2} \right)^{2n} = sh\left(\frac{x}{2} \right) \Big/ \left(\frac{x}{2} \right) \tag{78}$$

Kirillov Character formula is: $\chi_U(\exp(X)) = \mathrm{trU}_{\exp(X)} = j(X)^{-1} \int\limits_{\mathrm{O}} e^{i\langle f, X \rangle} d\mu_{\mathrm{O}}(f)$ (79)

$$\int\limits_{\mathrm{O}} e^{i\langle f, X \rangle} d\mu_{\mathrm{O}}(f) = j(X) \mathrm{trU}_{\exp(X)} \quad \text{with} \quad j(X) = \left(\det \left(\frac{e^{ad_X/2} - e^{-ad_X/2}}{ad_X/2} \right) \right)^{1/2} \tag{80}$$

We will use Kirillov representation theory and his character formula to compute Souriau covariant Gibbs density in the unit Poincaré disk. For any Lie group G, a coadjoint orbit $\mathrm{O} \subset \mathfrak{g}^*$ has a canonical symplectic form ω_0 given by KKS 2-form. As seen, if G is finite dimensional, the corresponding volume element defines a G-invariant measure supported on O, which can be interpreted as a tempered distribution. The Fourier transform (where d is the half of the dimension of the orbit O):

$$\Im(x) = \int\limits_{\mathrm{O} \subset \mathfrak{g}^*} e^{-i\langle x, \lambda \rangle} \frac{1}{d!} d\omega_{O^d} \quad \text{with } \lambda \in \mathfrak{g}^* \text{ and } x \in \mathfrak{g} \tag{81}$$

is Ad G-invariant. When $\mathrm{O} \subset \mathfrak{g}^*$ is an integral coadjoint orbit, Kirillov formula, given previously, expresses Fourier transform $\Im(x)$ by Kirillov character χ_{O}:

$$\Im(x) = j(x)\chi_O\left(e^x\right) \text{ where } j(x) = \det{}^{1/2}\left(\frac{\sinh\left(ad\left(x/2\right)\right)}{ad\left(x/2\right)}\right) \tag{82}$$

χ_O is, as defined previously, the "*Kirillov character*" of a unitary representation associated to the orbit. We will consider the universal covering of $PSU(1,1)$, the Lie algebra is:

$$\mathfrak{g}^* = \mathfrak{su}(1,1)^* = \left\{\begin{pmatrix} iE & p^* \\ p & -iE \end{pmatrix} / E \in R, p \in C\right\} \tag{83}$$

As observed in [18-19], the Ad-invariant form $m^2 = E^2 - |p|^2$ allows to identify the following operator Ad and Ad^*, m could be considered analogously as rest mass, E as energy, and $p = p_1 + ip_2$ as the momentum vector. The coadjoint orbits are the rest mass shells. Let $D = \left\{w \in C / |w| < 1\right\}$ Poincaré unit disk, for any $m > 0$, there is a corresponding action of the universal covering of $PSU(1,1)$ on $\kappa^{m/2}$ (with κ the holomorphic cotangent bundle of unit disk), with the invariant symplectic form:

$$\omega = curv(\kappa) = -i\partial\partial^* \log|dw|^2 = 2i\frac{dw \wedge dw^*}{\left(1-|w|^2\right)^2} \tag{84}$$

The moment map is an equivariant isomorphism (O_m^+ coadjoint orbit for $m^2 > 0$ and $E > 0$):

$$J : w \in \left(D, curv\left(\kappa^{m/2}\right)\right) \mapsto (p, E) = \frac{m}{\left(1-|w|^2\right)}\left(2iw, 1+|w|^2\right) \in O_m^+ \tag{85}$$

In case $m > 1$, the Kirillov character formula is given by:

$$\chi_m\left(\exp\left(\begin{pmatrix} x & \cdot \\ \cdot & -x \end{pmatrix}\right)\right) = j(x)^{-1} \int_{O_{m-1}^+} e^{-i\left\langle\begin{pmatrix} x & \cdot \\ \cdot & -x \end{pmatrix}, \begin{pmatrix} iE & p^* \\ p & -iE \end{pmatrix}\right\rangle} \omega_{O_{m-1}^+} \tag{86}$$

$$\text{where } j(x) = \det{}^{1/2}\left[\sinh\left(ad\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}\right) / ad\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}\right] = \frac{\sinh(x)}{x} \tag{87}$$

$$\text{which reduces to}: \frac{e^{mx}}{1-e^{2x}} j(x) = \int_D e^{(m-1)x\frac{1+|w|^2}{1-|w|^2}} \frac{1}{\left(1-|w|^2\right)^2} dw \wedge dw^* \tag{88}$$

Finally, the Souriau-Gibbs density is given by:

$$p_{Gibbs}(w)=\frac{e^{-i\left\langle \begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix},\begin{pmatrix} iE & p^* \\ p & -iE \end{pmatrix}\right\rangle}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)} \text{ with } (p,E)=\frac{m}{\left(1-|w|^2\right)}\left(2iw,1+|w|^2\right)\in O_m^+$$

$$p_{Gibbs}(w)=\frac{e^{-\left\langle \begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix},\begin{pmatrix} im\frac{1+|w|^2}{1-|w|^2} & 2m\frac{w^*}{1-|w|^2} \\ -2m\frac{w}{1-|w|^2} & -im\frac{1+|w|^2}{1-|w|^2} \end{pmatrix}\right\rangle}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)}=\frac{e^{-2m\left(x\frac{1+|w|^2}{1-|w|^2}-\frac{\left(\eta w+\eta^* w^*\right)}{1-|w|^2}\right)}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)}=\frac{e^{-2m\left(x\frac{x-\left(\eta w+\eta^* w^*\right)+x|w|^2}{1-|w|^2}\right)}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)} \quad (89)$$

with $\beta=\begin{pmatrix} ix & -\eta \\ -\eta^* & -ix \end{pmatrix}=\Lambda^{-1}(Q)\in\mathfrak{g}$ where $Q=\dfrac{\partial\Phi(\beta)}{\partial\beta}=\Lambda(\beta)\in\mathfrak{g}^*$ and

$$\text{Mean moment map}: Q=E\left[\begin{bmatrix} m\frac{1+|w|^2}{1-|w|^2} & -i2m\frac{w}{1-|w|^2} \\ i2m\frac{w}{1-|w|^2} & -m\frac{1+|w|^2}{1-|w|^2} \end{bmatrix}\right] \text{ where } w\in D$$

<u>Nota 1:</u> Localization for Fourier-Laplace transform

We can observe that there is another way to compute the Laplace or Fourier transform:

$$\Im(x)=\int_M e^{i\langle\lambda,x\rangle}\frac{1}{(2\pi)^n n!}d\omega^n \text{ with } \lambda:M\to\mathfrak{g}^* \text{ and } x\in\mathfrak{g} \quad (90)$$

Let $\lambda:M\to\mathfrak{g}^*$ and $x\in\mathfrak{g}$ be the moment map, Duistermaat and Heckman have used the method of exact stationary phase to prove a formula that expresses this integral explicitely in term of local invariant. When x is purely imaginary, the integral is the partition function of a statistical system with phase space M. This approach is also valid for $\mathrm{Im}(x)>0$ belonging to a special cone in $\mathfrak{g}$. The non-abelian measure was first evaluated by Duflo and Vergne. For each regular $x\in\mathfrak{g}^C,\mathrm{Im}(x)\in Int(C)$, the integral exist and:

$$\Im(x)=\int_M e^{i\langle\lambda,x\rangle}\frac{1}{(2\pi)^n n!}d\omega^n=i^n\sum_p\frac{e^{i\langle\lambda(p),x\rangle}}{\prod_{i=}^n\alpha_i^p(x)} \quad (91)$$

<u>Nota 2:</u> Extension for *SU(p,q)* Unitary Group for Siegel Unit Disk

To address computation of covariant Gibbs density for Siegel Unit Disk, we can consider $SU(p,q)$ Unitary Group:

$$G=SU(p,q) \text{ and } K=S\left(U(p)\times U(q)\right)=\left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}/A\in U(p),D\in U(q),\det(A)\det(D)=1\right\} \quad (92)$$

We can use the following decomposition for $g\in G^C$:

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in G^C, g=\begin{pmatrix} I_p & BD^{-1} \\ 0 & I_q \end{pmatrix}\begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I_p & 0 \\ D^{-1}C & I_q \end{pmatrix} \quad (93)$$

and consider the action of $g \in G^C$ on Siegel Unit Disk $SD = \{ Z \in M_{pq}(C) / I_p - ZZ^+ > 0 \}$ given by:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^C, g = \begin{pmatrix} I_p & BD^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_p & 0 \\ D^{-1}C & I_q \end{pmatrix} \tag{94}$$

Benjamin Cahen has study this case and introduced the moment map by identifing G-equivariantly $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the Killing form β on $\mathfrak{g}^C$:

$\mathfrak{g}^*$ G-equivariant with $\mathfrak{g}$ by Killing form $\beta(X,Y) = 2(p+q)Tr(XY)$

The set of all elements of $\mathfrak{g}$ fixed by K is $\mathfrak{h}$:

$$\mathfrak{h} = \{ \text{element of } G \text{ fixed by } K \} \quad , \xi_0 \in \mathfrak{h}, \xi_0 = i\lambda \begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$$
$$\Rightarrow \langle \xi_0, [Z, Z^+] \rangle = -2i\lambda (p+q)^2 Tr(ZZ^+), \forall Z \in D \tag{95}$$

Then, we the equivatiant moment map is given by:

$$\forall X \in g^C \ , Z \in D, \ \psi(Z) = Ad^*\left(\exp(-Z^+)\zeta(\exp Z^+ \exp Z)\right)\xi_0$$
$$\forall g \in G, Z \in D \ \text{ then } \psi(g.Z) = Ad_g^* \psi(Z) \tag{96}$$
$$\psi \text{ is a diffeomorphism from SD onto orbit } O(\xi_0)$$

with:

$$\psi(Z) = i\lambda \begin{pmatrix} (I_p - ZZ^+)^{-1}(-pZZ^+ - qI_p) & (p+q)Z(I_q - Z^+Z)^{-1} \\ -(p+q)(I_q - Z^+Z)^{-1} Z^+ & (pI_q + qZ^+Z)(I_q - Z^+Z)^{-1} \end{pmatrix} \tag{97}$$

and $$\zeta(\exp Z^+ \exp Z) = \begin{pmatrix} I_p & Z(I_q - Z^+Z)^{-1} \\ 0 & I_q \end{pmatrix} \tag{98}$$

5. New Entropy Definition as Generalized Casimir Invariant Functions for Coadjoint and Adjoint Representation

5.1. Casimir Invariant and Generalized Casimir Invariant

Hendrik Brugt Gerhard Casimir, a Dutch physicist, studied what is called Casimir operators and Casimir invariants (H. Casimir and Van der Waerden studied the SU(2) group, the group of isospin/angular momentum, as the model of the algebraic approach to the study of the unitary representations of semi-simple compact Lie groups). Kirillov has explained that Casimir operators are in one-to-one correspondence with polynomial invariants characterizing orbits of the coadjoint representation. Solutions are not necessarily polynomials and the nonpolynomial solutions are called ***generalized Casimir invariants***. For certain classes of Lie algebras, all invariants of the coadjoint representation are functions of polynomial ones. In physics, Hamiltonians and integrals of motion of classical integrable Hamiltonian systems are not polynomials in the momenta [84-100,103-104].

5.2. Souriau Entropy as Generalized Casimir Invariant in Coadjoint representation

In Souriau Lie groups Thermodynamics, we will see that coadjoint orbits lie on level sets of the Entropy that could be considered as a Casimir invariant function:

$$\begin{aligned} S &: \mathfrak{g}^* \to R \\ Q &\mapsto S(Q) \end{aligned} \tag{99}$$

We will consider first the case of null-cohomology, Entropy as Casimir invariant function is a conserved quantity, because Casimir function has null Lie Poisson brackets functions [100,101]:

$$\begin{gathered} \{S,H\}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle = 0 \ , \ \forall H : \mathfrak{g}^* \to R, \ Q \in \mathfrak{g}^* \ , \ \langle A, B \rangle = B(A,B) \ \text{ Cartan-Killing form} \\ \text{with } \partial S(Q) = \frac{d}{d\varepsilon} S(Q + \delta Q) \Big|_{\varepsilon=0} = \left\langle \delta Q, \frac{\partial S}{\partial Q} \right\rangle \end{gathered} \tag{100}$$

We can observe that $\beta = \frac{\partial S}{\partial Q}$, then:

$$\left\langle Q, \left[\beta, \frac{\partial H}{\partial Q} \right] \right\rangle = \left\langle Q, ad_\beta \frac{\partial H}{\partial Q} \right\rangle = 0 \ , \ \forall H : \mathfrak{g}^* \to R, \ Q \in \mathfrak{g}^* \ , \ ad_a b = [a,b] \tag{101}$$

We can also write:

$$\left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle = \left\langle Q, ad_{\frac{\partial S}{\partial Q}} \frac{\partial H}{\partial Q} \right\rangle = \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q, \frac{\partial H}{\partial Q} \right\rangle = 0 \ , \forall H : \mathfrak{g}^* \to R \tag{102}$$

It means that $ad^*_{\frac{\partial S}{\partial Q}} Q = ad^*_\beta Q = 0 \ , \ \beta = \frac{\partial S}{\partial Q}$. We can remark that if we note $\left(ad^*_{\frac{\partial S}{\partial Q}} Q \right)_j = C^k_{ij} ad^*_{\left(\frac{\partial S}{\partial Q}\right)^i} Q_k = 0$ with C^k_{ij} the structure tensor, we observe that this equation is in fact the Casimir condition for invariant function in coadjoint representation as we will see hereafter. The restriction of the Lie-Poisson bracket to an orbit generates a symplectic structure on the orbit, called the KKS (Kirillov-Kostant-Souriau) structure, or the canonical symplectic structure. Casimir function is characterized as a quantity which commutes with each linear functional on the Poisson manifold, and then it is conserved by dynamics of any Hamiltonian.

Given a Hamiltonian $H : \mathfrak{g}^* \to R$, the equation of motion for $Q \in \mathfrak{g}^*$ is:

$$\frac{dQ}{dt} = \{Q, H\} = ad^*_{\frac{\partial H}{\partial Q}} Q \ \text{ with } H = S \Rightarrow \frac{dQ}{dt} = \{Q, S\} = ad^*_{\frac{\partial S}{\partial Q}} Q = 0 \tag{103}$$

In case of non-null cohomology, the Lie Poisson brackets functions are given by:

$$\begin{gathered} \{S,H\}_{\tilde{\Theta}}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right] \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q} \right) = 0 \ , \ \forall H : \mathfrak{g}^* \to \mathbb{R}, \ Q \in \mathfrak{g}^* \\ \text{with } \tilde{\Theta}(X,Y) = J_{[X,Y]} - \{J_X, J_Y\} \ \text{ where } J_X(x) = \langle J(x), X \rangle \\ \tilde{\Theta}(X,Y) : \mathfrak{g} \times \mathfrak{g} \to \Re \qquad \text{with } \Theta(X) = T_e\theta(X(e)) \\ X,Y \mapsto \langle \Theta(X), Y \rangle \end{gathered} \tag{104}$$

That we can develop in the following:

$$\begin{aligned}
&\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\\
&\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,ad_{\frac{\partial S}{\partial Q}}\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\\
&\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\\
&\forall H,\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)+,\frac{\partial H}{\partial Q}\right\rangle=0\Rightarrow ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0
\end{aligned} \tag{105}$$

We have found the generalized Casimir equation for Entropy in the non-null cohomology case:

$$\{S,H\}_{\tilde{\Theta}}(Q)=0 \tag{106}$$

That coulbe also written:

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \tag{107}$$

This equation was observed by Souriau in his paper of 1974, where he has written that geometric temperature β is a kernel of $\tilde{\Theta}_\beta$, that is written:

$$\beta\in Ker\tilde{\Theta}_\beta\Rightarrow\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0 \tag{108}$$

That we can develop to recover the Casimir equation:

$$\begin{aligned}
&\Rightarrow\langle Q,ad_\beta Z\rangle+\tilde{\Theta}(\beta,Z)=0\Rightarrow\langle ad^*_\beta Q,Z\rangle+\tilde{\Theta}(\beta,Z)=0\\
&\beta=\frac{\partial S}{\partial Q}\Rightarrow\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q,Z\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),Z\right\rangle=0,\forall Z\\
&\Rightarrow ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0
\end{aligned} \tag{109}$$

Then the generalized Casimir Equation in non-null cohomogy is given by:

$$\left(ad^*_{\frac{\partial S}{\partial Q}}Q\right)_j+\Theta\left(\frac{\partial S}{\partial Q}\right)_j=C^k_{ij}ad^*_{\left(\frac{\partial S}{\partial Q}\right)^i}Q_k+\Theta_j=0 \tag{110}$$

Given a Hamiltonian $H:\mathfrak{g}^*\to R$, the equation of motion for $Q\in\mathfrak{g}^*$ is:

$$\frac{dQ}{dt}=\{Q,H\}_{\tilde{\Theta}}=ad^*_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)\ \text{with}\ H=S\Rightarrow\frac{dQ}{dt}=\{Q,S\}_{\tilde{\Theta}}=ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \tag{111}$$

Level sets of the Casimir Entropy function, on which the coadjoint orbits lie, are symplectic manifolds.

5.3. Souriau Entropy invariance in coadjoint representation

If we note $\mathrm{An}(\mathfrak{g}^*)$ the space of analytic function on the dual Lie agebra $\mathfrak{g}^*$, a function $F^*\in\mathrm{An}(\mathfrak{g}^*)$ is a Casimir invariant if for any $g\in G, X\in\mathfrak{g}^*$, we have $F^*\left(Ad^*_g X\right)=F^*(X)$. We have observed previously that Souriau's Entropy analytic function $S(Q)$ defined on dual Lie algebra $\mathfrak{g}^*$ by Legendre transform of Massieu Characteric analytic function $\Phi(\beta)$ (minus logarithm of Laplace transform) defined on Lie algebra $\mathfrak{g}$ was an invariant function under the affine coadjoint action $S\left[Q\left(Ad_g(\beta)\right)\right]=S\left[Ad_g^*(Q)+\theta\left(g\right)\right]=S\left(Q\right)$. In case of null-cohomology, Souriau cocycle cancels $\theta\left(g\right)=0$, and we recover Casimir invariant function in coadjoint representation $S\left[Ad_g^*(Q)\right]=S\left(Q\right)$.

We can then claim that Souriau Entropy is an extended Casimir invariant function in case of non-null cohomogy. This characteristic of Souriau Entropy could be a new general definition of Entropy. In Souriau Lie groups Thermodynamics, Entropy $S(Q)$ is a generalized Casimir invariant function for coadjoint representation in case of non-null cohomology, and Massieu Characteristic function by Legendre duality is a generalized Casimir function for adjoint representation.

We will explain how to prove that Souriau Entropy is invariant under the action of the group, starting from its definition:

$$S(Q)=\langle\beta,Q\rangle-\Phi(\beta) \text{ with } Q=\frac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^* \text{ and } \beta=\frac{\partial S(Q)}{\partial Q}\in\mathfrak{g} \tag{112}$$

$$\text{with } \Phi(\beta)=-\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda_\omega \text{ and } U:M\rightarrow\mathfrak{g}^* \tag{113}$$

Considering Souriau Entropy $S\left(Q\right)$ where the heat $Q=\dfrac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^*$ an element of the dual Lie algebra is parameterized by $\beta\in\mathfrak{g}$ an element of the Lie algebra, the Lie group G acts through $g\in G$ by adjoint operator Ad_g, the entropy is given by $S\left[Q\left(Ad_g(\beta)\right)\right]$ with $Q\left(Ad_g(\beta)\right)$ given by fundamental Souriau equation:

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta\left(g\right) \tag{114}$$

The invariance of Souriau Entropy is deduced from the following developments:

$$\begin{aligned}
&\beta\in\mathfrak{g}\rightarrow Ad_g(\beta)\Rightarrow\Psi(Ad_g(\beta))=\int_M e^{-\langle Ad_g(\beta),U\rangle}d\lambda_\omega\\
&\Psi(Ad_g(\beta))=\int_M e^{-\left\langle\beta,Ad_{g^{-1}}^*U\right\rangle}d\lambda_\omega=\int_M e^{-\left\langle\beta,U\left(Ad_{g^{-1}}\beta\right)-\theta\left(g^{-1}\right)\right\rangle}d\lambda_\omega\\
&\Psi(Ad_g(\beta))=e^{\left\langle\beta,\theta\left(g^{-1}\right)\right\rangle}\Psi\left(\beta\right)\\
&\theta\left(g^{-1}\right)=-Ad_{g^{-1}}^*\theta(g)\Rightarrow\Psi(Ad_g(\beta))=e^{-\left\langle\beta,Ad_{g^{-1}}^*\theta(g)\right\rangle}\Psi\left(\beta\right)\\
&\Phi\left(\beta\right)=-\log\Psi(\beta)\Rightarrow\Phi\left(Ad_g(\beta)\right)=\Phi\left(\beta\right)-\left\langle\beta,\theta\left(g^{-1}\right)\right\rangle=\Phi\left(\beta\right)+\left\langle\beta,Ad_{g^{-1}}^*\theta(g)\right\rangle
\end{aligned} \tag{115}$$

Based on this expression of Massieu Characteristic function transform by action of the group, we can use Legendre transform to study how Souriau Entropy is changed:

$$
\begin{aligned}
&S(Q)=\langle\beta,Q\rangle-\Phi(\beta)\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g\beta,Q\left(Ad_g\beta\right)\right\rangle-\Phi\left(Ad_g\beta\right)\\
&\begin{cases}Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)\\ \Phi\left(Ad_g(\beta)\right)=-\log\Psi(Ad_g(\beta))=-\left\langle\beta,\theta\left(g^{-1}\right)\right\rangle+\Phi(\beta)\end{cases}\\
&\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g\beta,Ad_g^*(Q)+\theta(g)\right\rangle+\left\langle\beta,\theta\left(g^{-1}\right)\right\rangle-\Phi(\beta)\\
&\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g\beta,Ad_g^*(Q)+\theta(g)\right\rangle-\left\langle\beta,Ad_{g^{-1}}^*\theta(g)\right\rangle-\Phi(\beta)\\
&\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle\beta,Ad_{g^{-1}}^*Ad_g^*(Q)+Ad_{g^{-1}}^*\theta(g)\right\rangle-\left\langle\beta,Ad_{g^{-1}}^*\theta(g)\right\rangle-\Phi(\beta)\\
&Ad_{g^{-1}}^*Ad_g^*(Q)=Q\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\langle\beta,Q\rangle-\Phi(\beta)=S(\beta)
\end{aligned}
\tag{116}
$$

We finally prove that Souriau Entropy is invariant in coadjoint representation $S\left(Ad_g^*(Q)+\theta(g)\right)=S(\beta)$ in general case of non-null cohomology, that we could write $S\left(Ad_g^\#(Q)\right)=S(\beta)$, if we note affine coadjoint action $Ad_g^\#(Q)=Ad_g^*(Q)+\theta(g)$. This is also true in case of null-cohomology when the Souriau cocycle cancels $\theta(g)=0$, and we recover classical generalized Casimir invariant function definition on coadjoint representation for Entropy $S\left(Ad_g^*(Q)\right)=S(\beta)$ generalized Casimir invariant function definition on adjoint representation for Massieu Characteristic function $\Phi\left(Ad_g(\beta)\right)=\Phi(\beta)$.

5.4. Souriau Entropy given by Casimir Invariant Functions Equations

Based on development given in the following we can state that:

As the Entropy S is a generalized Casimir invariant function in the coadjoint representation, $S\left(Ad_{e^{t\xi}}^* h\right)=S(h)$, then S should be solution of the following differential equation:

$$
C_{ij}^k Q_k\frac{\partial S(Q)}{\partial Q_j}=0\quad,\quad i,j,k=\dim\mathfrak{g},\ \text{with}\ \begin{cases}C_{ij}^k Q_k=C_{ij}(Q)=B_{ij}\\ B_Q(x,y)=B_{ij}x^i y^j=\left\langle Q,[x,y]\right\rangle\end{cases}
\tag{117}
$$

Where C_{ij}^k is the structure tensor of the Lie algebra $\mathfrak{g}$ in the basis $(e_1,e_2,...,e_n)$, while X_k are the coordinates in $\mathfrak{g}^*$ in the basis $(e^1,e^2,...,e^n)$ defined by $\left\langle e^j,e_i\right\rangle=\delta_{ij}$. The structure tensor s given by $\left[\phi(e_i),\phi(e_i)\right]=C_{ij}^k\phi(e_k)$ with $\phi(e_i)=C_{ij}^k X_k\dfrac{\partial}{\partial X_j}$, $i=1,...,n$.

5.5. Characterization of Generalized Casimir Invariant Functions in Coadjoint Representation

We will describe recent characterization of generalized Casimir invariant functions by Oleg L. Kurnyavko and Igor V. Shirokov [85,89,96] who have proposed Algebraic method for construction of Casimir invariants of Lie groups coadjoint representations. Modern invariant theory based on geometric methods, which was credited classically as non-constructive, has some exception admitting a constructive solution related to the constructing invariants of Lie Groups representations.

Let T be a connected Lie group, $T(G)$ a representation of the group G in the linear space V, T_g the operators associated to the representation of the group G on the linear space V, then the invariants are given by:

$$F\left(T_g x\right) = F(x) \ , \ x \in V, g \in G, T_g \in T(G), F(x) \in C^{\infty}(V) \tag{118}$$

With the properties that:

$$T_e = I \ , \ T_{g_a g_b} = T_{g_a} T_{g_b} \ , \ T_{g^{-1}} = \left(T_g\right)^{-1} \tag{119}$$

Solution is given by the following differential equation:

$$-\sum_{i,j}^{\dim V} t_{kj}^{i} x^{j} \frac{\partial F(x)}{\partial x^{i}} = 0 \ \text{ with } \ t_{kj}^{i} = \left.\frac{\partial \left(T_g\right)_j^i}{\partial g^k}\right|_{g=e} \ \text{ and } \ k = 1,\ldots,\dim G \tag{120}$$

t_{kj}^{i} are elements of the matrices of the Lie algebra representation basis of G.

That we can write $t_k = -t_{kj}^{i} x^{j} \dfrac{\partial}{\partial x^{i}}$ and $t_k F(x) = 0$.

If we consider the dual space V^*, the co-tangent representation is given by:

$$\left\langle T^*(g)X, T(g)x \right\rangle = \left\langle X, x \right\rangle \tag{121}$$

And co-represnetation invariants are given by:

$$t_k^* F^*(X) = 0 \ \text{ with } \ t_k^* = t_{kj}^{i} X_i \frac{\partial}{\partial X_j} \tag{122}$$

They have underlined the relationship between invariants of representations and conjugate representations, where the algebraic construction of Lie groups representations invariants are given by invariants of the conjugate representation with respect to the invariants of the original representation.

Shirokov Theorem 1:

Let $F(x)$ be a non-degenerate invariant of the representation $T(G)$, then conjugate representation invariant can be found by Legrendre tranform:

$$F^*(X) = x^i X_i - F(x) = \left\langle x, X \right\rangle - F(x) \ \text{ with } X = \frac{\partial F(x)}{\partial x} \ \text{ such that } X_i = \frac{\partial F(x)}{\partial x^i} \tag{123}$$

and also the converse problem:

$$F(x) = x^i X_i - F^*(X) = \left\langle x, X \right\rangle - F^*(X) \ \text{ with } x = \frac{\partial F^*(X)}{\partial X} \ \text{ such that } x^i = \frac{\partial F^*(X)}{\partial X_i} \tag{124}$$

Shirokov has considered $F(x)$ the representation invariant $T(G)$, and $F^*(X)$ the representation invariant $T^*(G)$ conjugate to $T(G)$, with the conditions :

$$-t_{kj}^{i} x^{j} \frac{\partial F(x)}{\partial x^{i}} = 0 \ \text{ and } \ t_{lj}^{i} X_i \frac{\partial F^*(X)}{\partial X_j} = 0 \tag{125}$$

$$t_{lj}^i X_i \frac{\partial F^*(X)}{\partial X_j} = t_{lj}^i X_i \frac{\partial}{\partial X_j}\left[x^k(X)X_k - F\left(x(X)\right)\right] = t_{lj}^i X_i \frac{\partial x^k}{\partial X_j} X_k + t_{lj}^i X_i x^k \frac{\partial X_k}{\partial X_j} - t_{lj}^i X_i \frac{\partial F(x)}{\partial x^k}\frac{\partial x^k}{\partial X_j}$$
$$t_{lj}^i X_i \frac{\partial F^*(X)}{\partial X_j} = t_{lj}^i X_i \frac{\partial x^k}{\partial X_j}\frac{\partial F(x)}{\partial x^k} + t_{lj}^i \frac{\partial F(x)}{\partial x^i} x^k \delta_k^j - t_{lj}^i \frac{\partial F(x)}{\partial x^i}\frac{\partial F(x)}{\partial x^k}\frac{\partial x^k}{\partial X_j} \quad (126)$$
$$t_{lj}^i X_i \frac{\partial F^*(X)}{\partial X_j} = t_{lj}^i x_j \frac{\partial F(x)}{\partial x^i} = 0$$

Invariant Casimir Functions of the coadjoint representation has been studied for completely integrable Hamiltonian systems, as classical systems on the orbits of the coadjoint representation. Oleg L. Kurnyavko and Igor V. Shirokov have considered the relationship between invariants of representations of Lie groups and their conjugate dual representations.

Considering the coadjoint action given by:

$$\left\langle Ad_g^* X, x\right\rangle = \left\langle X, Ad_{g^{-1}} x\right\rangle \ , \ g \in G, X \in \mathfrak{g}^*, x \in \mathfrak{g} \quad (127)$$

Invariants of a coadjoint representation are called Casimir functions, with the property:

$$F^*\left(Ad_g^* X\right) = F^*\left(X\right) \quad (128)$$

the infinitesimal invariance is given by the equations:

$$C_{ij}(X)\frac{\partial F^*\left(X\right)}{\partial X_j} = 0 \quad \text{with} \quad C_{ij}(X) = C_{ij}^k X_k, \quad i,j,k = \dim \mathfrak{g} \quad (129)$$

The number of functionally independent invariants is given by the rank of the matrix $C_{ij}(X)$, called the index of the Lie algebra $\mathfrak{g}$: $ind\mathfrak{g} = \dim \mathfrak{g}^* - \sup_{X \in \mathfrak{g}^*} rank C_{ij}(X)$

From these adjoint and coadjoint representation, Shirokov has introduced the following theorem:

Shirokov Theorem 2:

Let $F\left(Ad_g x\right) = F(x)$ be a non-degenerate invariant of the adjoint representation Ad_G, then conjugate representation invariant, invariant of coadjoint representation Ad_G^* can be found by formula:

$$F^*(X) = x^i X_i - F(x) = \left\langle x, X\right\rangle - F(x) \quad \text{with } X = \frac{\partial F(x)}{\partial x} \text{ such that } X_i = \frac{\partial F(x)}{\partial x^i} \quad (130)$$

and also the converse problem , let $F^*\left(Ad_g^* X\right) = F^*\left(X\right)$, invariant of coadjoint representation Ad_G is given by:

$$F(x) = x^i X_i - F^*(X) = \left\langle x, X\right\rangle - F^*(X) \quad \text{with } x = \frac{\partial F^*(X)}{\partial X} \text{ such that } x^i = \frac{\partial F^*(X)}{\partial X_i} \quad (131)$$

Nota: $C_{ij}^k X_k \frac{\partial F^*\left(X\right)}{\partial X_j} = 0 \ , \ i,j,k = \dim \mathfrak{g}, \text{ with } \begin{cases} C_{ij}^k X_k = C_{ij}(X) = B_{ij} \\ B_X\left(x,y\right) = B_{ij} x^i y^j = \left\langle X, [x,y]\right\rangle \end{cases} \quad (132)$

5.6. Constructing Generalized Casimir Invariant Functions in Coadjoint Representation

I. V. Shirokov has proposed a method for constructing invariants of the coadjoint representation of Lie groups with an arbitrary dimension and structure based on local symplectic coordinates on the coadjoint orbits. Oleg L. Kurnyavko and Igor V. Shirokov have also proposed a general method for constructing Casimir invariants.

We will give some other developments of Casimir Invariant Functions by A.T. Fomenko and V.V. Trofimov, related to Orbits of the coadjoint representation and the associated canonical symplectic structure

The coadjoint orbit O_h passing through the point $h \in \mathfrak{g}^*$ is given by

$$O_h = \left\{ Ad_g^* h \,/\, g \in G \right\} \text{ where } h \in \mathfrak{g}^* \tag{133}$$

$$T_h O_h = \left\{ ad_\rho^* h \,/\, \rho \in \mathfrak{g} \right\} \subset \mathfrak{g}^*, h \in \mathfrak{g}^*$$

$$v = \left. \frac{d\left[Ad^*_{\exp^{t\rho}} h \right]}{dt} \right|_{t=0} \in T_h O_h \ , \ \rho \in \mathfrak{g} \tag{134}$$

Let $(e_1,...,e_n)$ basis of $\mathfrak{g}$, $(e^1,...,e^n)$ basis of $\mathfrak{g}^*$, with $\left\langle e^i, e_j \right\rangle = \delta_{i,j}$

$$\begin{aligned} & h = h_i e^i \Rightarrow v^i = \left. \frac{d\left[\left\langle h_i, Ad^*_{\exp^{t\rho}} h \right\rangle \right]}{dt} \right|_{t=0} = \left. \frac{d\left[\left\langle Ad^*_{\exp^{t\rho}} h, e_i \right\rangle \right]}{dt} \right|_{t=0} \\ & v^i = \left. \frac{d\left[\left\langle h, Ad_{\exp^{-t\rho}} e_i \right\rangle \right]}{dt} \right|_{t=0} = \left\langle h, \left. \frac{d\left[Ad_{\exp^{-t\rho}} e_i \right]}{dt} \right|_{t=0} \right\rangle \\ & v^i = \left\langle h, -[\rho, e_i] \right\rangle = -\left\langle ad_\rho^* h, e_i \right\rangle = \left\langle v, e_i \right\rangle \Rightarrow v = -ad_\rho^* h \end{aligned} \tag{135}$$

Kirillov, Kostant and Souriau have introduced a KKS 2-form on co-adjoint co-orbits that then inherit a structure of homogeneous symplectic manifold:

$$\begin{aligned} & \xi, \eta \in T_h O_h = \left\{ ad_\chi^* h \,/\, \rho \in \mathfrak{g} \right\} \subset \mathfrak{g}^*, h \in \mathfrak{g}^* \\ & \omega_h(\xi,\eta) = \omega\left(ad_{\xi_1}^* h, ad_{\eta_1}^* h \right) = \left\langle h, [\xi_1, \eta_1] \right\rangle \text{ with } \xi = ad_{\xi_1}^* h \text{ and } \eta = ad_{\eta_1}^* h \end{aligned} \tag{136}$$

This KKS 2-form ω is invariant with respect to the coadjoint action $\omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \omega_h(\xi,\eta)$:

$$\begin{aligned} & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \omega_g\left(Ad_f^* ad_{\xi_1}^* h, Ad_f^* ad_{\eta_1}^* h \right) \\ & \text{with } g = Ad_f^* h \ , \ \xi = ad_{\xi_1}^* h \ , \ \eta = ad_{\eta_1}^* h \text{ and } f \in G, g, h \in \mathfrak{g}^* \\ & Ad_f^* ad_{\xi_1}^* h = ad^*_{Ad_f \xi_1}\left(Ad_f^* h \right) \text{ and } Ad_f^* ad_{\eta_1}^* h = ad^*_{Ad_f \eta_1}\left(Ad_f^* h \right) \\ & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \omega_g\left(ad^*_{Ad_f \xi_1}\left(Ad_f^* h \right), ad^*_{Ad_f \eta_1}\left(Ad_f^* h \right) \right) \\ & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \omega_g\left(ad^*_{Ad_f \xi_1} g, ad^*_{Ad_f \eta_1} g \right) = \left\langle g, \left[Ad_f \xi_1, Ad_f \eta_1 \right] \right\rangle \\ & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \left\langle g, Ad_f [\xi_1, \eta_1] \right\rangle = \left\langle Ad^*_{f^{-1}} g, [\xi_1, \eta_1] \right\rangle \text{ with } h = Ad^*_{f^{-1}} g \\ & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \left\langle h, [\xi_1, \eta_1] \right\rangle \\ & \omega_g\left(Ad_f^* \xi, Ad_f^* \eta \right) = \omega_h(\xi,\eta) \end{aligned} \tag{137}$$

The symplectic structure is given due to the property that $d\omega = 0$, that could be proved making link with Jacobi identity.

Let $grad_{skew}m$ such that $\omega\left(v, grad_{skew}m\right) = v(m) = \sum_i v^i \frac{\partial m}{\partial x^i}$ smooth vector field on M

$$\{m,n\} = \omega\left(grad_{skew}m, grad_{skew}n\right) = \sum_{i<j} \omega_{ij}\left(grad_{skew}m\right)^i \left(grad_{skew}n\right)^j \tag{138}$$

with $\left(grad_{skew}m\right)^i = \sum_j \omega^{ij}\frac{\partial m}{\partial x^j} \Rightarrow \{m,n\} = \sum_{i<j}\omega^{ij}\frac{\partial m}{\partial x^i}\frac{\partial n}{\partial x^j}$

We can compute terms of Jacobi identity:

$\{m,\{n,p\}\} = -\left(grad_{Skew}m\right)\{n,p\} = -L_{grad_{Skew}m}\{n,p\}$ with L_ξ: Lie derivative

If $\xi = grad_{Skew}m$

$$L_\xi\{n,p\} = L_\xi\left(\omega^{ij}\frac{\partial n}{\partial x^i}\frac{\partial p}{\partial x^j}\right) = L_\xi(\omega)^{ij}\frac{\partial n}{\partial x^i}\frac{\partial p}{\partial x^j} + \omega^{ij}\frac{\partial(\xi n)}{\partial x^i}\frac{\partial p}{\partial x^j} + \omega^{ij}\frac{\partial n}{\partial x^i}\frac{\partial(\xi p)}{\partial x^j} \tag{139}$$

$$L_\xi\{n,p\} = L_\xi(\omega)^{ij}\frac{\partial n}{\partial x^i}\frac{\partial p}{\partial x^j} + \{\xi n, p\} + \{n, \xi p\} = L_\xi(\omega)^{ij}\frac{\partial n}{\partial x^i}\frac{\partial p}{\partial x^j} - \{\{m,n\},p\} - \{n,\{m,p\}\}$$

$$\Rightarrow \{m,\{n,p\}\} + \{\{m,n\},p\} + \{n,\{m,p\}\} = L_\xi(\omega)^{ij}\frac{\partial n}{\partial x^i}\frac{\partial p}{\partial x^j}$$

Using Elie Cartan formula $L_\xi\omega = i(\xi)d\omega + di(\xi)\omega$. If ξ is a Hamiltonian vector field, $di(\xi)\omega = 0$ and then $L_\xi\omega = i(\xi)d\omega$ .We can the observe that if $d\omega = 0$, then the Jacobi identity is satisfied $\{m,\{n,p\}\} + \{\{m,n\},p\} + \{n,\{m,p\}\} = 0$ and conversely.

Let consider the Berezin Bracket:

$$\{m,n\} = -C_{ij}^k x_k \frac{\partial m}{\partial x^i}\frac{\partial n}{\partial x^j} \text{ with } \left[e_i, e_j\right] = C_{ij}^k e_k \tag{140}$$

where $\left(e_1, e_2, ..., e_n\right)$ basis of Lie algebra $\mathfrak{g}$, $\left(e^1, e^2, ..., e^n\right)$ basis of dual Lie algebra $\mathfrak{g}^*$ of corresponding coordinates $x^1, ..., x^n$ for $\mathfrak{g}$, $x_1, ..., x_n$ for $\mathfrak{g}^*$

This Berezin Bracket is given by:

$$\{m,n\}_x = dm_x\left(ad^*_{dn(x)}(x)\right) = \left(ad^*_{dn(x)}(x)\right)\left(dm_x\right)$$

$$\{m,n\}_x = \left\langle x, \left[dn_x, dm_x\right]\right\rangle = C_{ij}^k x_k \frac{\partial n}{\partial x^i}\frac{\partial m}{\partial x^j} \text{ with } dn_x = \frac{\partial n}{\partial x_i}e_i \ ,\ dm_x = \frac{\partial m}{\partial x_j}e_j \tag{141}$$

By developping Berezin Bracket $\{m,n\} = -C_{ij}^k x_k \frac{\partial m}{\partial x^i}\frac{\partial n}{\partial x^j}$ with $\left[e_i, e_j\right] = C_{ij}^k e_k$, we can prove that the bracket verify jacoby identy $\{m,\{n,p\}\} + \{\{m,n\},p\} + \{n,\{m,p\}\} = 0$ and then $d\omega = 0$.

We will see that differential equation for (semi-)invariants of the coadjoint representations could be established. We will note $\mathrm{An}\left(\mathfrak{g}^*\right)$ the space of analytic function on the dual Lie agebra $\mathfrak{g}^*$. A function $F^* \in \mathrm{An}\left(\mathfrak{g}^*\right)$ is an invariant if for any $g \in G, X \in \mathfrak{g}^*$, we have $F^*\left(Ad_g^* X\right) = F^*(X)$, and is semi-invariant if $F^*\left(Ad_g^* X\right) = \chi(g)F^*(X)$ where $\chi(g)$ is a character of the Lie group G.

We have a representation of Lie algebras $\phi : \mathfrak{g} \to Vec(\Gamma)$ defined on basis $(e_1, e_2, ..., e_n)$ in $\mathfrak{g}$,where $Vec(\Gamma)$ is the space of vector fields on Γ an open subset in $\mathfrak{g}^*$, given by:

$$\phi(e_i) = C_{ij}^k X_k \frac{\partial}{\partial X_j} \quad , \; i = 1,...,n \tag{142}$$

Where C_{ij}^k is the structure tensor of the Lie algebra $\mathfrak{g}$ in the basis $(e_1, e_2, ..., e_n)$, while X_k are the coordinates in $\mathfrak{g}^*$ in the basis $(e^1, e^2, ..., e^n)$ defined by $\langle e^j, e_i \rangle = \delta_{ij}$. The representation is not dependent of the choice of the basis, with the property: $[\phi(e_i), \phi(e_i)] = C_{ij}^k \phi(e_k)$.

We have the property, that:

$$\left. \frac{d^n F^*\left(Ad_{e^{t\xi}}^* h\right)}{dt^n} \right|_{t=0} = \left[\left(-\phi(\xi)\right)^n F^* \right](h) \tag{143}$$

This result is obtained by the following development:

$$\begin{gathered}
\left. \frac{dF^*\left(Ad_{e^{t\xi}}^* h\right)}{dt} \right|_{t=0} = \frac{\partial F^*}{\partial X_i}(h) . \left. \frac{d\left\langle Ad_{e^{t\xi}}^* h, X_i \right\rangle}{dt} \right|_{t=0} = \frac{\partial F^*}{\partial X_i}(h) . \left. \frac{d\left\langle h, Ad_{e^{-t\xi}} e_i \right\rangle}{dt} \right|_{t=0} \\
\left. \frac{d\left\langle h, Ad_{e^{-t\xi}} e_i \right\rangle}{dt} \right|_{t=0} = \left\langle h, -[\xi, e_i] \right\rangle = -\left\langle h, C_{ij}^k \xi^j e_k \right\rangle \text{ with } [\xi, e_i] = \left[\xi^j e_j, e_i\right] = C_{ji}^k \xi^j e_k \\
\left. \frac{d\left\langle h, Ad_{e^{-t\xi}} e_i \right\rangle}{dt} \right|_{t=0} = \left\langle h_k e^k, -C_{ji}^k \xi^j e_k \right\rangle = -C_{ji}^k \xi^j h_k \\
\left. \frac{dF^*\left(Ad_{e^{t\xi}}^* h\right)}{dt} \right|_{t=0} = -C_{ji}^k \xi^j h_k \frac{\partial F^*}{\partial X_i}(\xi) = \left(-\phi(\xi) F^*\right)(h)
\end{gathered} \tag{144}$$

We use then Taylor expansion of $F^*\left(Ad_{e^{t\xi}}^* h\right)$ given by:

$$F^*\left(Ad_{e^{t\xi}}^* h\right) = F^*(h) + \sum_{n=1}^{\infty} \frac{\left(-\phi(\xi)\right)^n F^*}{n!}(h).t^n \tag{145}$$

We can observe that F^* is invariant if $F^*\left(Ad_{e^{t\xi}}^* h\right) = F^*(h)$ and then $(-\phi(\xi))^n F^* = 0$ or $\phi(\xi) F^* = 0$ that could be written $C_{ji}^k \xi^j h_k \frac{\partial F^*}{\partial X_i}(\xi) = 0$.

If F^* is semi-invariant of the coadjoint representation of group if and only if:

$\phi(e_i) F^* = -\lambda_i F^*$ with $\lambda_i = d\chi(e_i)$ ($d\chi$: derivative of χ at the group G identity element

$$\begin{gathered}
F^*\left(Ad_{e^{t\xi}}^* h\right) = \chi\left(e^{t\xi}\right) F^*(h) \text{ with } \chi\left(e^{t\xi}\right) = e^{t\chi_*(\xi)} \\
\left[\left(-\phi(\xi)\right)^n F^* \right](h) = \chi_*(\xi) F^*(h) \\
\Rightarrow F^*\left(Ad_{e^{t\xi}}^* h\right) = \left[1 + \sum_{n=1}^{\infty} \frac{\left[\chi_*(\xi)\right]^n}{n!} t^n \right] . F^*(h)
\end{gathered} \tag{146}$$

6. Lie Groups Thermodynamics for SE(2) Lie group

After $SU(1,1)$ Lie group with null cohomology and then without Souriau one-cocycle, we will consider Souriau model for $SE(2)$ Lie group with non-null cohomology and then with introduction of Souriau one-cocycle [107].

We will consider first $SO(2)$ Lie group:

$$SO(2)=\left\{R_\varphi=\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}/\varphi\in R\right\} \tag{147}$$

A vector at the identity to $SO(2)$ is given by:

$$\left.\frac{dR_{t\eta}}{dt}\right|_{t=0}=-\eta\Im \text{ with } \Im=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \Im^T=\Im^{-1}=-\Im \tag{148}$$

We consider the special Euclidean group $SE(2)=SO(2)\times R^2$.

$$SE(2)=\left\{\begin{bmatrix}R_\varphi & \tau\\ 0 & 1\end{bmatrix}/R_\varphi\in SO(2),\tau\in R^2\right\} \tag{149}$$

the group operation is given by:

$$\begin{bmatrix}R_{\varphi_1} & \tau_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}R_{\varphi_2} & \tau_2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}R_{\varphi_1}R_{\varphi_2} & R_{\varphi_1}\tau_2+\tau_1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}R_{\varphi_1+\varphi_2} & R_{\varphi_1}\tau_2+\tau_1\\ 0 & 1\end{bmatrix}$$
$$\Rightarrow\left(R_{_1},\tau_1\right).\left(R_{\varphi_2},\tau_2\right)=\left(R_{\varphi_1+\varphi_2},R_{\varphi_1}\tau_2+\tau_1\right) \tag{150}$$

$$\begin{bmatrix}R_{\varphi_1} & \tau_1\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}R_{-\varphi_1} & -R_{-\varphi_1}\tau_1\\ 0 & 1\end{bmatrix}\Rightarrow\left(R_{\varphi_1},\tau_1\right)^{-1}=\left(R_{-\varphi_1},-R_{-\varphi_1}\tau_1\right) \tag{151}$$

The Lie algebra $se(2)$ of $SE(2)$ has underlying vector space R^3 and Lie bracket:

$$(\xi,u)\in se(2)=R\times R^2\Rightarrow\begin{bmatrix}-\xi\Im & u\\ 0 & 0\end{bmatrix}\in se(2) \tag{152}$$

Lie bracket is given by:

$$\left[(\xi,u),(\eta,v)\right]=\left(0,\xi\Im v+\eta\Im u\right) \tag{153}$$

Adjoint action of $SE(2)$ is given by:

$$Ad_{(R_\varphi,\tau)}(\xi,u)=\begin{bmatrix}R_\varphi & \tau\\ 0 & 1\end{bmatrix}\begin{bmatrix}-\xi\Im & u\\ 0 & 0\end{bmatrix}\begin{bmatrix}R_{-\varphi} & -R_{-\varphi}\tau\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\xi\Im & \xi\Im\tau+R_\varphi u\\ 0 & 0\end{bmatrix}$$
$$Ad_{(R_\varphi,\tau)}(\xi,u)=\left(\xi,R_\varphi u+\xi\Im\tau\right) \tag{154}$$

Coadjoint action of $SE(2)$ is given by:

$$Ad^{*}_{(R_\varphi,\tau)}(m,\rho)=\left(m+\Im R_\varphi \rho.\tau , R_\varphi \rho\right) \tag{155}$$

The moment map $J:R^2\to se^{*}(2)$ of $SE(2)$ is defined by:

$$J_{(\xi,u)}(x)=J(x).(\xi,u) \tag{156}$$

with the right action of $SE(2)$ on R^2:

$$x.(R\varphi,\tau)=R_{-\varphi}(x-\tau) \tag{157}$$

the infinitesimal generator of $(\xi,u)\in se(2)$ has the expression:

$$(\xi,u)_{R^2}(x)=\left.\frac{d\left[x.\left(R_{t\xi},tu\right)\right]}{dt}\right|_{t=0}=\left.\frac{d\left[R_{-t\xi}\left(x-tu\right)\right]}{dt}\right|_{t=0}=\xi\Im x-u \tag{158}$$

Let $J_{(\xi,u)}(x):R^2\to se^{*}(2)$ be the moment map of this action relative to the symplectic form, we can compute it from its definition:

$$\begin{aligned}
&dJ_{(\xi,u)}(x).y=-2\omega\left((\xi,u)_{R^2},y\right)\\
&\text{with } \ \omega\left((\xi,u)_{R^2},y\right)=\omega(\xi\Im x-u,y)=(\xi\Im x-u).\Im y=(\xi x+\Im u).y\\
&\Rightarrow dJ_{(\xi,u)}(x).y=-2(\xi x+\Im u).y\\
&\Rightarrow J_{(\xi,u)}(x)=-2\left(\frac{1}{2}\xi\|x\|^2+\Im u.x\right)=-2\left(\frac{1}{2}\|x\|^2,-\Im x\right).(\xi,u)\\
&J_{(\xi,u)}(x)=J(x).(\xi,u)\Rightarrow J(x)=-2\left(\frac{1}{2}\|x\|^2,-\Im x\right)\ ,\ x\in R^2
\end{aligned} \tag{159}$$

We then compute the one-cocycle of $SE(2)$ from the moment map:

$$\begin{aligned}
&\theta\left(\left(R_{\varphi,\tau}\right)\right)=J\left(0.\left(R_\varphi,\tau\right)\right)-Ad^{*}_{(R_\varphi,\tau)}J(0)=J\left(-R_{-\varphi}\tau\right)\\
&\theta\left(\left(R_{\varphi,\tau}\right)\right)=-2\left(\frac{1}{2}\|\tau\|^2,\Im R_{-\varphi}\tau\right)=-2\left(\frac{1}{2}\|\tau\|^2,R_{-\varphi-\frac{\pi}{2}}\tau\right)
\end{aligned} \tag{160}$$

Coadjoint orbit of $SE(2)$ are generated by:

$$\begin{aligned}
&O_{(m,\rho)}=\left\{A^{*}_{(R_{\varphi,\tau})}(m,\rho)+\theta\left(\left(R_\varphi,\tau\right)\right)/\left(R_\varphi,\tau\right)\in SE(2)\right\}\\
&O_{(m,\rho)}=\left\{\left(x-R_{-\frac{\pi}{2}}\rho.\tau-\|\tau\|^2,R_{-\varphi}\rho-2R_{-\varphi-\frac{\pi}{2}}\tau\right)/\left(R_\varphi,\tau\right)\in SE(2)\right\}
\end{aligned} \tag{161}$$

The Souriau Symplectic form in this case of non-null cohomology is given by:

$$\begin{aligned}
&\omega_{(m,\rho)(m',\rho')}\left(ad^{*}_{(\xi,u)}(m',\rho')-(0,2\Im u),ad^{*}_{(\eta,v)}(m',\rho')-(0,2\Im v)\right)=\rho'.(-\xi\Im v+\eta\Im u)+2u.\Im v\\
&\text{with } \ (m',\rho')=\left(x-R_{-\frac{\pi}{2}}\rho.\tau-\|\tau\|^2,R_{-\varphi}\rho-2R_{-\varphi-\frac{\pi}{2}}\tau\right)\in O_{(m,\rho)}\subset R^3
\end{aligned} \tag{162}$$

With the expression of moment map and Fourier Transform for $SE(2)$ by Kirillov Orbit method and Kirillov character, we can compute Souriau Gibbs density, in the same way that we have developed the case for $SU(1,1)$.

7. Conclusion: Lie Groups Thermodynamics for Machine Learning

Lie Group tools based on Representation Theory and Orbits Methods could be used with Souriau-Fisher Metric on Coadjoint Orbits that is an extension of Fisher Metric for Lie Group through homogeneous Symplectic Manifolds on Lie Group Co-Adjoint Orbits. In appendix C, we provide an algorithm invented by Jean-Marie Souriau to compute Exponential Map of matrices, that we can use for “geodesic shooting”.

Different tools developed based on Souriau Lie Groups Thermodynamics and Kirillov Representation Theory for :

- **Supervised Machine Learning**

  o Geodesic Natural Gradient on Lie Algebra: Extension of Neural Network Natural Gradient from Information Geometry on Lie Algebra for Lie Groups Machine Learning.

  o Souriau Maximum Entropy Density on Co-Adjoint Orbits: Covariant Maximum Entropy Probability Density for Lie Groups defined with Souriau Moment Map, Co-Adjoint Orbits Method & Kirillov Representation Theory

  o Symplectic Integrator preserving Moment Map: Extension of Neural Network Natural Gradient to Geometric Integrators as Symplectic integrators that preserve moment map

- **Non-Supervised Machine Learning**

  o Souriau Exponential Map on Lie Algebra: Exponential Map for Geodesic Natural Gradient on Lie Algebra based on Souriau Algorithm for Matrix Characteristic Polynomial

  o Fréchet Geodesic Barycenter by Hermann Karcher Flow: Extension of Mean/Median on Lie Group by Fréchet Definition of Geodesic Barycenter on Souriau-Fisher Metric Space, solved by Karcher Flow.

  o Mean-Shift on Lie Groups with Souriau-Fisher Distance: Extension of Mean-Shift for Homogeneous Symplectic Manifold and Souriau-Fisher Metric Space

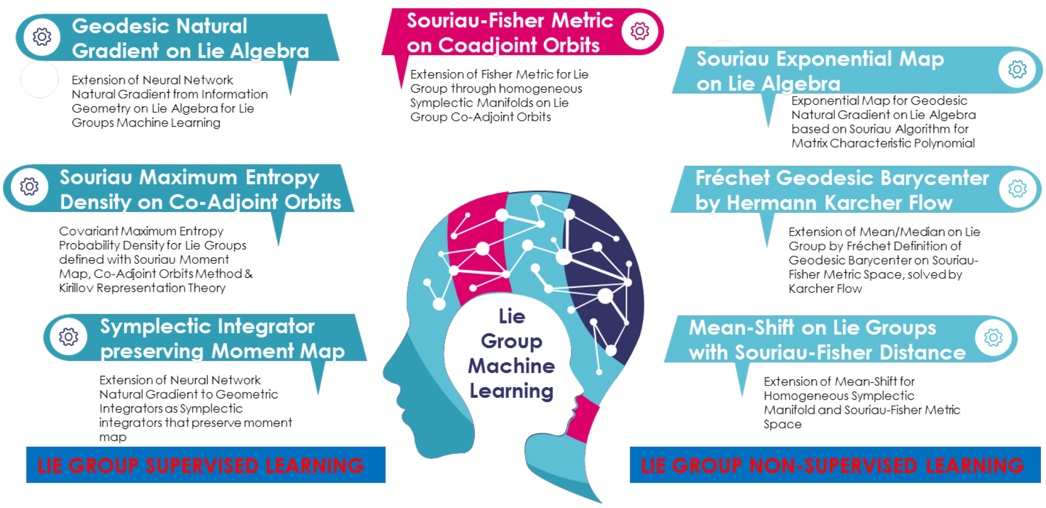

Figure 4. Supervised/Non-Supervised Machine Learning based on Lie Groups Thermodynamics

La notion classique d'ensemble canonique de Gibbs est étendue au cas d'une variété symplectique sur laquelle un groupe de Lie possède une action symplectique ("groupe dynamique"). La définition rigoureuse donnée ici permet d'étendre un certains nombre de propriétés thermodynamiques classiques (la température est ici un élément de l'algèbre de Lie du groupe, la chaleur un élément de son dual), notamment des inégalités de convexité. Dans le cas de groupes non commutatifs, des propriétés particulières apparaissent : la symétrie est spontanément brisée, certaines relations de type cohomologique sont vérifiées dans l'algèbre de Lie du groupe – **Jean-Marie Souriau, Mécanique Statistique, Groupes de Lie et Cosmologie, colloque CNRS n°237 – Géométrie Symplectique et physique mathématique**

Appendix A: Coadjoint orbits and Moment Map for SU(1,1)

We give more details developpements to obtain SU(1,1)/K coadjoint orbit and moment map [39].

If we consider hyperbolic Group $SL(2,R)$

$$SL(2,R)=\left\{\begin{pmatrix} m & p \\ q & n \end{pmatrix} \in GL(2,R) / mn-pq=1\right\} \tag{A1}$$

Elements of $SL(2,R)$ could be written with Iwasawa decomposition:

$$g=k_\theta.a_t.n_b \quad \text{with } k_\theta \in K, a_t \in A, n_b \in N \tag{A2}$$

$$\begin{aligned}
K &= \left\{k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} / 0 \le \theta < 2\pi \right\} \text{ with } e^{-i\theta} = \frac{m-iq}{\sqrt{m^2+q^2}} \\
A &= \left\{a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} / t \in R \right\} \text{ with } e^{2t} = m^2+q^2 \\
N &= \left\{n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} / b \in R \right\} \text{ with } b = \frac{mp+qn}{\sqrt{m^2+q^2}}
\end{aligned} \tag{A3}$$

K is a maximal compact sub-group of G_{SL}.

Group of Unit Disk automorphisms

Consider $SU(1,1)$ sub-group of $SL(2,R)$ given by

$$SU(1,1)=\left\{A=\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / \det(A)=|a|^2-|b|^2=1,\ a,b \in R\right\} \tag{A4}$$

The following interior automorphism that transforms $SL(2,R)$ to $SU(1,1)$, inducing an isomorphism between them:

$$\begin{aligned}
&SL(2,R) \to SU(1,1) \\
&\qquad g \mapsto CgC^{-1} \quad \text{with } C=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}
\end{aligned} \tag{A5}$$

$SU(1,1)$ acts on the Poincaré unit disk $D=\{z \in C / |z|<1\}$:

$$g^{-1}.z=\frac{az+b}{b^*+a^*} \quad \text{with } g^{-1}=\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \in SU(1,1) \text{ and } z \in D \tag{A6}$$

Valentine Bargmann has parameterized $SU(1,1)$:

$$\begin{cases} \gamma = \dfrac{b}{a} \\ \omega = \arg(\alpha) \bmod 2\pi \end{cases} \Rightarrow \begin{cases} |\gamma|<1 \\ a=e^{i\omega}\left(1-|\gamma|^2\right)^{-1/2} \\ b=e^{i\omega}\gamma\left(1-|\gamma|^2\right)^{-1/2} \end{cases} \tag{A7}$$

Then $SU(1,1)=\{(\gamma,\omega)/|\gamma|<1,\omega\in]-\pi,+\pi]\}$ with Group composition law given by:

$$(\gamma,\omega).(\gamma',\omega')=(\gamma'',\omega'')$$
$$\begin{cases}\gamma''=\left[\gamma'+\gamma e^{-2i\omega'}\right]\left[1+\gamma\gamma'^{*}e^{2i\omega'}\right]^{-1}\\ \omega''=\omega+\omega'+\frac{1}{2i}\log\left(\left[1+\gamma'^{*}\gamma e^{-2i\omega'}\right]\left[1+\gamma'\gamma^{*}e^{-2i\omega'}\right]^{-1}\right)\ \bmod\ 2\pi\end{cases} \tag{A8}$$
$$g=(\gamma,\omega)\Rightarrow g^{-1}=\left(-\gamma e^{2i\omega},-\omega\right)$$

$SU(1,1)$ is topological product of unit disk and circle.

Universal covering of $SL(2,R)$

If we consider $G=\{(\gamma,\omega)/|\gamma|<1,\omega\in R\}$, the following mapping:

$$\begin{array}{l}\Theta:G\to SU(1,1)\\ \Theta(\gamma,\omega)=(\gamma,\omega \bmod 2\pi)\end{array}\quad\text{with}\quad Ker\Theta=\{(0,2k\pi)/k\in Z\} \tag{A9}$$

Topological product of unit Disk D dans C and real straight line R is the universal covering of $SU(1,1)$.

A maximal compact subgroup of $SU(1,1)$ is $CKC^{-1}=\left\{\begin{pmatrix}e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{pmatrix}/\theta\in R\right\}$ and the subgroup for G is $\Theta^{-1}\left(CKC^{-1}\right)=\{(0,\theta)/\theta\in R\}$.

Pukanszky and Sally have defined irreductible unitary representation of $S\tilde{L}(2,R)$, classified in principal serie, discrete serie and complemantary serie.

The Lie algebra $\mathfrak{g}$ of G and $SU(1,1)$ is given by:

$$l_1=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},l_2=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},l_0=\frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{A10}$$

with the commutation relation: $[l_0,l_1]=l_2,[l_1,l_2]=-l_0,[l_2,l_0]=l_1$.

Dual Lie algebra $\mathfrak{g}^*$ of $\mathfrak{g}$ to $\mathfrak{g}$ thanks to Killing form:

$$B\left(\sum_{i=1,2,0}x_i l_i,\sum_{i=1,2,0}y_i l_i\right)=2\left(x_1y_1+x_2y_2-x_0y_0\right) \tag{A11}$$

Coadjointes orbits of $SU(1,1)$

Considering adjoint representation $Ad_g:\mathfrak{g}\to\mathfrak{g}$, and coadjointe representation as transpose linear mapping of $Ad_{g^{-1}}$, written by $Ad_g^*=\left(Ad_{g^{-1}}\right)^*$, $Ad_g^*:\mathfrak{g}^*\to\mathfrak{g}^*$.

Let $f=(\eta,0,0)$ in base $\{l_1,l_2,l_0\}$ with $\eta>0$ coadjoint orbit of f is:

$$O_\eta = \left\{ gfg^{-1} / g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,R) \right\} \tag{A12}$$
$$O_\eta = \left\{ \eta(1+2bc), \eta(bd-ac), \eta(bd+ac) / a,b,c,d \in R \text{ with } ad-bc=1 \right\}$$

Stabilizer of f is $G_1(f) = \{ g \in G / g.f = f \} = A \cup \{-A\}$ with $A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} / t \in R \right\}$ (A13)

$SL(2,R) = K'NA'$ with $A' = A \cup \{-A\}$ where $N = \left\{ n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} / b \in R \right\}$ and $K' = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} / 0 \le \theta < \pi \right\}$ and as $SL(2,R)/G_1(f)$ is a bijection with O_η, then O_η is in bijection with $K'N$, diffeomorph to $K/\{I_d\} \times N$. Then all element $x \in O_\eta$ can be written through this bijection $x = k_\theta n_b, k_\theta \in K', n_b \in N$. O_η is set of points $l = (x_1, x_2, x_0) \in \mathfrak{g}^*$ such that:

$x_1^2 + x_2^2 - x_0^2 = \eta^2 > 0$ a one sheet hyperboloid in $\mathfrak{g}^*$.

We will study discrete sequence.

Quantization of Kähler Manifold

Let (M,ω) a Kalherian manifold of dimension 2n (complex manifold with complex structure J) with 2-form $g(X,Y) = \omega(X,JY)$, a Riemannian structure on M.

We have seen that $SU(1,1)$ is conjugate of $SL_2(R)$ in $SL(2,R)$

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \in SU(1,1), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R) \tag{A14}$$
$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \frac{-i}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} \alpha = \frac{1}{2}\left[(a+d) + i(b-c)\right] \\ \beta = \frac{1}{2}\left[(a-d) - i(b+c)\right] \end{cases} \Rightarrow \begin{cases} \alpha + \alpha^* = a+d \\ \alpha - \alpha^* = i(b-c) \\ \beta + \beta^* = a-d \\ \beta - \beta^* = -i(b+c) \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2}\left[(\alpha+\alpha^*) + (\beta+\beta^*)\right] \\ b = \frac{1}{2i}\left[(\alpha-\alpha^*) - (\beta-\beta^*)\right] \\ c = \frac{-1}{2i}\left[(\alpha-\alpha^*) + (\beta-\beta^*)\right] \\ d = \frac{1}{2}\left[(\alpha+\alpha^*) - (\beta+\beta^*)\right] \end{cases} \tag{A15}$$

Let $f = \left(0, 0, h - \frac{1}{2}\right)$ in base $\{l_1, l_2, l_0\}$ where $l_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, l_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, l_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with $h > \frac{1}{2}$ (respectively $f = \left(0, 0, h + \frac{1}{2}\right)$ with $h < -\frac{1}{2}$) , the coadjoint orbit of f is:

$$O_h = \left\{ gfg^{-1} / g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,R) \right\}$$
$$O_h = \left\{ \left(|h| - \frac{1}{2} \right) \mathrm{sign} h \left(ac+bd \right); \frac{\left(|h| - \frac{1}{2} \right) \mathrm{sign} h}{2} \left(d^2 + c^2 - \left(a^2 + b^2 \right) \right); \frac{\left(|h| - \frac{1}{2} \right) \mathrm{sign} h}{2} \left(d^2 + c^2 + a^2 + b^2 \right) \right\} \quad \text{(A16)}$$

O_h appears as points $l = (x_1, x_2, x_0) \in \mathfrak{g}^*$ such that:

$$x_1^2 + x_2^2 - x_0^2 = -\left(|h|^2 - \frac{1}{2} \right)^2 < 0 \text{ with } \begin{cases} x_0 > 0 \text{ if } h > \frac{1}{2} \\ x_0 < 0 \text{ if } h < -\frac{1}{2} \end{cases} \quad \text{(A17)}$$

O_h is one of hyperboloid sheets of $\mathfrak{g}^*$, associated to representation π_h of discrete serie of G.

$$\begin{cases} x_1 = h'(ac+bd) \\ x_2 = \frac{h'}{2}\left(d^2 - a^2 + c^2 - b^2 \right) \text{ with } h' = \left(|h| - \frac{1}{2} \right) \mathrm{sign} h \\ x_0 = \frac{h'}{2}\left(d^2 + a^2 + c^2 + b^2 \right) \end{cases} \quad \text{(A18)}$$

Then, we obtain:

$$\begin{cases} x_1 = ih'\left(\alpha\beta - \alpha^* \beta^* \right) \\ x_2 = -h'\left(\alpha\beta + \alpha^* \beta^* \right) \\ x_0 = h'\left(\alpha\alpha^* + \beta\beta^* \right) \end{cases} \quad \text{(A19)}$$

If we set:

$$\begin{cases} \alpha = \frac{e^{i\theta}}{\sqrt{1-|z|^2}} \\ \beta = \frac{ze^{-i\theta}}{\sqrt{1-|z|^2}} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{ih'\left(z - z^* \right)}{1-|z|^2} \\ x_1 = \frac{-h'\left(z + z^* \right)}{1-|z|^2} \\ x_0 = \frac{h'\left(1 + |z|^2 \right)}{1-|z|^2} \end{cases} \quad \text{(A20)}$$

We use the parametrization of O_h by unit disk D:

$$\begin{aligned} D = \left\{ z \in C / |z| < 1 \right\} &\rightarrow O_h \\ z &\mapsto \left(x_1, x_2, x_0 \right) \end{aligned} \quad \text{(A21)}$$

$$\begin{cases} x_1 = -2\left(|h| - \frac{1}{2}\right)\operatorname{sign}h \frac{\operatorname{Im}(z)}{1-|z|^2} \\ x_1 = -2\left(|h| - \frac{1}{2}\right)\operatorname{sign}h \frac{\operatorname{Re}(z)}{1-|z|^2} \\ x_0 = \left(|h| - \frac{1}{2}\right)\operatorname{sign}h \frac{1+|z|^2}{1-|z|^2} \end{cases} \tag{A22}$$

This parametrisation provides a kahlerian structure for O_h, inherit from D.

We have $O_h = G/\tilde{K} = SL(2,R)/K$, and as $SL(2,R)$ is isomorph to $SU(1,1)$, O_h is identified with $SU(1,1)/K_0 = D$. Stabilizer of the origin is $K_0 = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} / |\alpha| = 1\right\}$. Then O_h is globally diffeomorph to D for all h.

We can compute Liouville measure on O_h. This measure is $\omega_h = \frac{\omega_h^{'}}{2\pi}$ where $\omega_h^{'}$ is the canonical 2-form on orbit. Parameterization on O_h gives:

$$\omega_h = \frac{\omega_h^{'}}{2\pi} = \frac{-i\left(2|h|-1\right)}{2\pi\left(1-|z|^2\right)}\operatorname{sign}h|dz|^2 \tag{A23}$$

We can observe that:

$$\begin{cases} dx_1 = \frac{i}{\left(1-|z|^2\right)}\left[\left(z-z^*\right)\left(1-zz^*\right)dh' + h'\left(1-z^{*2}\right)dz - h'\left(1-z^2\right)dz^*\right] \\ dx_2 = \frac{-1}{\left(1-|z|^2\right)}\left[\left(z+z^*\right)\left(1-zz^*\right)dh' + h'\left(1+z^{*2}\right)dz - h'\left(1+z^2\right)dz^*\right] \\ dx_0 = \frac{1}{\left(1-|z|^2\right)}\left[\left(1+zz^*\right)\left(1-zz^*\right)dh' + 2z^*h'dz - 2zh'dz^*\right] \end{cases} \tag{A24}$$

$$dx_1dx_2dx_0 = \frac{-ih'^2\left(1-|z|^2\right)}{\left(1-|z|^2\right)^6}\det\begin{pmatrix} z-z^* & 1-|z|^2 & -\left(1-z^2\right) \\ z+z^* & 1+z^{*2} & 1+z^2 \\ 1+|z|^2 & 2z^* & 2z \end{pmatrix} \tag{A25}$$

$$dx_1dx_2dx_0 = \frac{-2ih'^2}{\left(1-|z|^2\right)}dh'|dz|^2 = \frac{2}{\pi}h'dh'\omega_h \tag{A26}$$

This the measure defined on open set of $\mathfrak{g}^*$ given by $l_1^2 + l_2^2 - l_0^2 < 0$.

Appendix B: Bargman Parameterization of SU(1,1)

$SU(1,1)$ is isomorphic to $SL(2,R)=Sp(2,R)$ through the complex unitary matrix W:

$$SL(2,R)=\left\{g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} / \det g = ad-bc=1\right\} \tag{A27}$$

$$Sp(2,R)=\left\{g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} / gJg^T=J, J=\begin{pmatrix} 0 & +1 \\ -1 & 0\end{pmatrix}\right\} \tag{A28}$$

$$W=\frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{-1} & \omega^{-1} \\ -\omega & \omega\end{pmatrix}=\left(W^{+}\right)^{-1} \text{ with } \omega=e^{i\pi/4}=\frac{1}{\sqrt{2}}(1+i) \tag{A29}$$

If we observe that $W^{-1}JW=-iM$, the isomorphism is given explicitely by:

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}=g(u)=WuW^{-1}=\begin{pmatrix} \operatorname{Re}(\alpha+\beta) & -\operatorname{Im}(\alpha-\beta) \\ \operatorname{Im}(\alpha+\beta) & \operatorname{Re}(\alpha-\beta)\end{pmatrix} \tag{A30}$$

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^*\end{pmatrix}=u(g)=W^{-1}gW=\frac{1}{2}\begin{pmatrix} (a+d)-i(b-c) & (a-d)+i(b+c) \\ (a-d)-i(b+c) & (a+d)+i(b-c)\end{pmatrix} \tag{A31}$$

We can also make also a link with $SO(2,1)$ of "1+2" pseudo-orthogonal matrices:

$$SO(2,1)=\left\{\Gamma\in GL(3,3)/\det(\Gamma)=1, \Gamma K\Gamma^T=\Gamma, K=\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}\right\} \tag{A32}$$

$$\Gamma(g)=\begin{pmatrix} \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) & \frac{1}{2}\left(a^2-b^2+c^2-d^2\right) & -cd-ab \\ \frac{1}{2}\left(a^2+b^2-c^2-d^2\right) & \frac{1}{2}\left(a^2-b^2-c^2+d^2\right) & cd-ab \\ -bd-ac & bd-ac & ad+bc\end{pmatrix} \tag{A33}$$

with $\Gamma(g_1)\Gamma(g_2)=\Gamma(g_1g_2), \Gamma(I)=I, \Gamma\left(g^{-1}\right)=\Gamma(g)^{-1}$

The $SO(2,1)$ matrix corresponds to any $SU(1,1)$:

$$\Gamma(u)=\begin{pmatrix} |\alpha|^2+|\beta|^2 & 2\operatorname{Re}\alpha\beta^* & 2\operatorname{Im}\alpha\beta^* \\ 2\operatorname{Re}\alpha\beta & \operatorname{Re}\left(\alpha^2+\beta^2\right) & \operatorname{Im}\left(\alpha^2-\beta^2\right) \\ -2\operatorname{Im}\alpha\beta & -\operatorname{Im}\left(\alpha^2+\beta^2\right) & \operatorname{Re}\left(\alpha^2-\beta^2\right)\end{pmatrix} \tag{A34}$$

and $\alpha=\pm\sqrt{\frac{1}{2}(\Gamma_{11}+\Gamma_{12})+i(\Gamma_{12}-\Gamma_{21})}, \ \beta=\frac{1}{2\alpha}(\Gamma_{10}-i\Gamma_{20})$

The properties of connectivity of $Sp(2,R)$ is described by its isomorphy with $SU(1,1)$. Using unimodular condition:

$|\alpha|^2-|\beta|^2=1\Rightarrow \alpha_R^2+\alpha_I^2-\beta_R^2=1+\beta_I^2\geq 1$ with $\alpha=\alpha_R+i\alpha_I$ and $\beta=\beta_R+i\beta_I$

If β_I is fixed, $(\alpha_R,\alpha_I,\beta_R)$ are constrained to define a one-sheeted revolution hyperboloid, with its circular waist in the α plane.

To $SU(1,1)$, we can associate the simply-connected universal covering group, using the maximal compact subgroup $U(1)$ and corresponding to the Iwasawa decomposition (factorization of a noncompact semisimple group into its maximal compact subgroup times a solvable subgroup).

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^*\end{pmatrix}=\begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{i\omega}\end{pmatrix}\begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda\end{pmatrix} \text{ with } \begin{cases} \omega=\arg\alpha=\frac{1}{2}i\ln\left(\alpha^*\alpha^{-1}\right) \\ \lambda=|\alpha|>0 \\ \mu=e^{-i\omega}\beta=\sqrt{\frac{\alpha^*}{\alpha}}\beta \end{cases} \tag{A35}$$

$$\beta = e^{i\omega}\mu, |\alpha|^2 - |\beta|^2 = \lambda^2 - |\mu|^2 = 1 \text{ so } |\mu| < \lambda \tag{A36}$$

Bargmann has generalized this parameterization for $Sp(2N,R)$, more convenient but difficult to generalize to N dimensions. For $SU(1,1)$:, Bargmann has used (ω,γ):

$$\gamma = \frac{\mu}{\lambda} = \frac{\beta}{\alpha} \ (|\gamma| < 1),\ \lambda = \frac{1}{\sqrt{1-|\gamma|^2}}, \mu = \frac{\gamma}{\sqrt{1-|\gamma|^2}} \tag{A37}$$

For $SL(2,R) = Sp(2,R)$, the Bargman, parameterization is given by this decomposition of a non-singular matrix into the product of an orthogonal and a positive definite symmetric matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \lambda + \operatorname{Re}\mu & \operatorname{Im}\mu \\ \operatorname{Im}\mu & \lambda - \operatorname{Re}\mu \end{pmatrix} \tag{A38}$$

Conversely: $\omega = \arg\left[(a+d) - i(b-c)\right], \mu = e^{-i\omega}\left[(a-d) + i(b+c)\right]$

ω is counted modulo 2π, $\omega \equiv \omega(\operatorname{mod} 2\pi)$.

$SU(1,1)$ and $SL(2,R) = Sp(2,R)$ are described when ω is counted modulo 2π, $\omega \equiv \omega(\operatorname{mod} 2\pi)$.

Valentine Bargmann has proposed the covering of the general symplectic group $Sp(2N,R)$:

$$Sp(2N,R) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} / gJ_{2N}g^T = J_{2N}, J_{2N}^T = -J_{2N}, J_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \right\} \tag{A39}$$

with relations:

$$AB^T = BA^T, AC^T = CA^T, BD^T = DB^T, CD^T = DC^T, AD^T - BC^T = I_N \tag{A40}$$

$$g \in Sp(2N,R) \Rightarrow g^{-1} = M_{2N} g^T M_{2N} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \tag{A41}$$

Bargmann has observed that although $Sp(2N,R)$ is not isomorphic to any pseudo-unitary group, its inclusion in $U(N,N)$ will display the connectivity properties through its unitary $U(N)$ maximal compact subgroup, generalizing the role of $U(1) = SO(2)$ in $Sp(2,R)$.

$W_N = W \otimes I_N$ a $2N \times 2N$ matrix where $W = W_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_{\pi/4}^{-1} & \omega_{\pi/4}^{-1} \\ -\omega_{\pi/4} & \omega_{\pi/4} \end{pmatrix}$ with $\omega = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$, which gives the $N \times N$ block coefficients.

$$u(g) = W_N^{-1} g W_N = \frac{1}{2}\begin{pmatrix} [A+D] - i[B-C] & [A-D] + i[B+C] \\ [A-D] - i[B+C] & [A+D] + i[B-C] \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{A42}$$

With $\begin{cases} \alpha\alpha^+ - \beta\beta^+ = I_N, \alpha^+\alpha - \beta^T\beta^* = I_N \\ \alpha\beta^T - \beta\alpha^T = 0, \alpha^T\beta^* - \beta^+\alpha = 0 \end{cases}$ (A43)

and $u^{-1} = M_{2N} u^+ M_{2N}^{-1} = \begin{pmatrix} \alpha^+ & -\beta^T \\ -\beta^+ & \alpha^T \end{pmatrix}$ (A44)

The symplecticity property of g becomes:

$$uM_{2N}u^+ = M_{2N}, M_{2N} = iW_N^{-1}J_{2N}W_N = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} \tag{A45}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = g(u) = W_N u W_N^{-1} = \begin{pmatrix} \operatorname{Re}(\alpha+\beta) & -\operatorname{Im}(\alpha-\beta) \\ \operatorname{Im}(\alpha+\beta) & \operatorname{Re}(\alpha-\beta) \end{pmatrix} \tag{A46}$$

Valentine Bargmann has extended the well-know theorem that any real matrix R may be decomposed into the product of an orthogonal Q and a symmetric positive definite matrix S, uniquely as $R = QS$. Bargmann has shown that if $R \in Sp(2N,R)$, then $R = QS$ with $Q, S \in Sp(2N,R)$ where Q maps onto unitary matrix and S maps onto Hermitian positive definite matrix:

$$u(Q) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}, \alpha\alpha^+ = I_N, \alpha \in U(N) \text{ and } u(S) = \exp\begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix}, \xi = \xi^T \tag{A47}$$

We can generalize Bargmann parameterization of $SU(1,1)$ to $Sp(2N,R)$:

$$u\{\omega,\lambda,\mu\} = \begin{pmatrix} e^{i\omega}I_N & 0 \\ 0 & e^{-i\omega}I_N \end{pmatrix}\begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix} \oplus, \det\lambda > 0 \tag{A48}$$

Then the Bargmann parameters are:

$$\omega = \frac{1}{N}\arg\det\alpha, \lambda = e^{-i\omega}\alpha, \mu = e^{-i\omega}\beta, e^{iN\omega} = \frac{\det\alpha}{|\det\alpha|}, \det\lambda = |\det\alpha| > 0 \tag{A49}$$

The $Sp(2N,R)$ matrices in terms of the Bargmann parameters are:

$$g\{\omega,\lambda,\mu\} = \begin{pmatrix} \cos\omega I_N & -\sin\omega I_N \\ \sin\omega I_N & \cos\omega I_N \end{pmatrix}\begin{pmatrix} \mathrm{Re}(\lambda+\mu) & -\mathrm{Im}(\lambda-\mu) \\ \mathrm{Im}(\lambda+\mu) & \mathrm{Re}(\lambda-\mu) \end{pmatrix} \tag{A50}$$

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$$AB^T = BA^T, AC^T = CA^T, BD^T = DB^T, CD^T = DC^T, AD^T - BC^T = I_N \tag{A52}$$

Bargmann has observed that although $Sp(2N,R)$ is not isomorphic to any pseudo-unitary group, its inclusion in $U(N,N)$ will display the connectivity properties through its unitary $U(N)$ maximal compact subgroup, generalizing the role of $U(1) = SO(2)$ in $Sp(2,R)$:

$W_N = W \otimes I_N$, $2N \times 2N$ matrix where $W = W_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_{\pi/4}^{-1} & \omega_{\pi/4}^{-1} \\ -\omega_{\pi/4} & \omega_{\pi/4} \end{pmatrix}$ with $\omega = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$.

$$u(g) = W_N^{-1} g W_N = \frac{1}{2}\begin{pmatrix} [A+D]-i[B-C] & [A-D]+i[B+C] \\ [A-D]-i[B+C] & [A+D]+i[B-C] \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{A53}$$

with $\alpha\alpha^+ - \beta\beta^+ = I_N, \alpha^+\alpha - \beta^T\beta^* = I_N$ and $\alpha\beta^T - \beta\alpha^T = 0, \alpha^T\beta^* - \beta^+\alpha = 0$ (A54)

The symplecticity property of g becomes:

$$uM_{2N}u^+ = M_{2N}, M_{2N} = iW_N^{-1}J_{2N}W_N = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} \tag{A55}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = g(u) = W_N u W_N^{-1} = \begin{pmatrix} \mathrm{Re}(\alpha+\beta) & -\mathrm{Im}(\alpha-\beta) \\ \mathrm{Im}(\alpha+\beta) & \mathrm{Re}(\alpha-\beta) \end{pmatrix} \tag{A56}$$

Appendix C: Souriau Algorithm for Exponential Map

The algorithm to compute characteristic polynomial of a matrix was discovered by Urbain Jean Joseph Leverrier in 1840, and was rediscovered in 1948 by Jean-Marie Souriau and modified to its present form, but published only in French. Other authors, P. Horst, D. K. Faddejew and Sominski, J.S. Frame, U. Wegner and L. Csanky, were credited with rediscovering the technique. As soon as 1955, Souriau algorithm was tested and benchmarked by the National Bureau of Standards, Los Angeles, under the sponsorship of the Wright Air Development Center, U. S. Air Force, and the Office of Naval Research, and was concluded at the University of California, by the Office of Naval Research. As observed and illustrated by Souriau, for $n=10$, his algorithm uses only 8 thousands of additions and multiplications, compared to 37 million of additions and 62 million of multiplications for classical approach (Gaussian elimination). Main drawback of most efficient classical algorithm based on Krylov iterates cannot be parallelized. Souriau algorithm has a complexity $O(n^4)$ or $O(n^{\omega+1})$ in sequential computation, and so cannot compete with Krylov-based algorithm, but Souriau algorithm has been parallelized by L. Csanky, proving that characteristic polynomial computation could be solved in parallel time $log^2 n$ with a polynomial number of processors. Souriau algorithm parallelization by Czansky has been improved more recently by Preparata & Sarwate using fast matrix product, and by Keller-Gehrig using matrix reduction. Reduction to complexity $O(n^{\omega})$ is given for generic matrices, but for non-generic ones, only $O(n^{\omega} logn)$ complexity could be achieved. A disadvantage of both algorithms (Le verrier and Gaussian elimination) is the presence of divisions.
The computation of the matrix exponential is a classical problem in numerical mathematics as explained in 1978 paper of Moler & van Loan and many efficient algorithms described in 1998 paper Hochbruck, Lubich & Selhofer 1998. But this problem is very far from being fully solved , especially to approximate an exponential of a matrix which resides in a Lie algebra, a central problem in geometric integration as studied by Iserles, Munthe-Kaas, Nørsett, Zanna & Celledoni.

From Le Verrier to Souriau Algorithm

The algorithm to compute characteristic polynomial of a matrix was discovered by Urbain Jean Joseph Leverrier in 1840, and was rediscovered in 1948 by Jean-Marie Souriau and modified to its present form, but published only in French. Other authors, P. Horst, D. K. Faddejew and Sominski, J.S. Frame, U. Wegner and L. Csanky, were credited with rediscovering the technique. As soon as 1955, Souriau algorithm was tested and benchmarked by the National Bureau of Standards, Los Angeles, under the sponsorship of the Wright Air Development Center, U. S. Air Force, and the Office of Naval Research, and was concluded at the University of California, by the Office of Naval Research. As observed and illustrated by Souriau, for $n=10$, his algorithm uses only 8 thousands of additions and multiplications, compared to 37 million of additions and 62 million of multiplications for classical approach (Gaussian elimination). Main drawback of most efficient classical algorithm based on Krylov iterates cannot be parallelized. Souriau algorithm has a complexity $O(n^4)$ or $O(n^{\omega+1})$ in sequential computation, and so cannot compete with Krylov-based algorithm, but Souriau algorithm has been parallelized by L. Csanky, proving that characteristic polynomial computation could be solved in parallel time $log^2 n$ with a polynomial number of processors. Souriau algorithm parallelization by Czansky has been improved more recently by Preparata & Sarwate using fast matrix product, and by Keller-Gehrig using matrix reduction. Reduction to complexity $O(n^{\omega})$ is given for generic matrices, but for non-generic ones, only $O(n^{\omega} logn)$ complexity could be achieved. A disadvantage of both algorithms (Le verrier and Gaussian elimination) is the presence of divisions.

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Souriau matrix characteristic polynomial computation

Jean-Marie Souriau introduced his algorithm in the framework of his lecture on Multilinear Algebra by consideration on *volume form* [59-62]. In a vector space E of dimension *n*, we can prove that vector space of n-forms (*n-form* as an anti-symmetric n-linear operator with scalar value). After Selecting a frame $(e_1, e_2, ..., e_n)$ of E, we can define an *n-form* called "*volume form*" with: $vol(e_1)(e_2)...(e_n) = 1$ (A57)

Volume of parallelepiped generated by vectors $(x_1, x_2, ..., x_n)$ is given by: $|vol(x_1)(x_2)...(x_n)|$ (A58)

Souriau called "espace jaugé (jauged space)", all vector space E, of finite size, where we have selected a "unit-jauge" defined by vol. If we define a linear operator $A : E \to E$, considered as "affiner" in a jauged space, we can then give definition of:

- ***Determinant of A,*** $\det(A)$ by:

$$\det(A) vol(v_1)(v_2)...(v_n) = vol(Av_1)(Av_2)...(Av_n) \quad \text{(A59)}$$

- ***Adjoint linear operator of A***, $adj(A)$, by:

$$vol(adj(A)v_1)(v_2)...(v_n) = vol(v_1)(Av_2)...(Av_n) \quad \text{(A60)}$$

- ***Trace number of A***, $tr(A)$, by:

$$\begin{aligned} tr(A) vol(v_1)(v_2)...(v_n) &= vol(Av_1)(v_2)...(v_n) + vol(v_1)(Av_2)...(v_n) \\ &\quad ... + vol(v_1)(v_2)...(Av_n) \end{aligned} \quad \text{(A61)}$$

By using the following relation deduced from previous equations:

$$vol(adj(A)Av_1)(v_2)...(v_n) = vol(Av_1)(Av_2)..(Av_n) = \det(A) vol(v_1)(v_2)...(v_n) \quad \text{(A62)}$$

If A is invertible, we recover classical equations:

$$adj(A)A = \det(A) I \text{ and } A^{-1} = [\det(A)]^{-1} adj(A) \quad \text{(A63)}$$

Using these formulas, we can try to invert $[\lambda I - A]$ assuming that $\det(\lambda I - A) \neq 0$. If we use previous determinant definition, we have:

$$\begin{aligned} \det(\lambda I - A) vol(v_1)(v_2)...(v_n) &= vol(\lambda v_1 - Av_1)(\lambda v_2 - Av_2)...(\lambda v_n - Av_n) \\ &= \lambda^n vol(v_1)(v_2)...(v_n) + ... \end{aligned} \quad \text{(A64)}$$

where $\det(\lambda I - A)$ is the characteristic polynomial of A, a polynomial in λ of degree n, with:

$$adj(\lambda I - A)[\lambda I - A] = \det(\lambda I - A)I \Leftrightarrow A.Q(\lambda) = \lambda Q(\lambda) - P(\lambda)I \quad \text{(A65)}$$

(if λ is an eigenvalue of A, the nonzero columns of $Q(\lambda)$ are corresponding eigenvectors). We can then observe that $adj(\lambda I - A)$ is a polynomial of degree *n-1*. We can then define both $P(\lambda)$ and $Q(\lambda)$ by polynomials:

$$P(\lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} k_i \lambda^{n-i} \quad \text{and} \quad Q(\lambda) = adj(\lambda I - A) = \sum_{i=0}^{n-1} \lambda^{n-i-1} B_i \quad \text{(A66)}$$

With $k_0 = 1$, $k_n = (-1)^n \det(A)$, $B_0 = I$ and $B_{n-1} = (-1)^{n-1} adj(A)$ (A67)

By developing equation $adj(\lambda I - A)[\lambda I - A] = \det(\lambda I - A)I$, we can write:

$$\sum_{i=0}^{n} k_i \lambda^{n-i} I = \sum_{i=0}^{n-1} \lambda^{n-i-1} B_i [\lambda I - A] = \lambda B_{n-1} + \sum_{i=1}^{n-1} \lambda^{n-i} [B_i - B_{i-1}A] - B_{n-1}A \quad \text{(A68)}$$

By identification term by term, we find the expression of matrices B_i:

$$\begin{cases} B_0 = I \\ B_i = B_{i-1}A + k_i I \quad , \ i = 1,\dots,n-1 \\ B_{n-1}A + k_n I = 0 \end{cases} \quad \text{(A69)}$$

We can observe that $A^{-1} = -\frac{B_{n-1}}{k_n}$ and also the Cayley-Hamilton theorem:

$$k_0 A^n + k_1 A^{n-1} + \dots + k_{n-1} A + k_n I = 0 \quad \text{(A70)}$$

To go further, we have to use this classical result from analysis on differentiationgiven by $\delta[\det(G)] = tr(adj(G)\delta G)$. If we set $G = (\lambda I - A)$ and $\delta = \frac{d}{d\lambda}$, we then obtain $tr(adj(\lambda I - A)) = \frac{d}{d\lambda}\det(\lambda I - A)$ providing:

$$\sum_{i=0}^{n-1} \lambda^{n-i-1} tr(B_i) = \frac{d}{d\lambda}\left(\sum_{i=0}^{n} k_i \lambda^{n-i}\right) = \sum_{i=0}^{n-1} (n-i) k_i \lambda^{n-i-1} \quad \text{(A71)}$$

We can then deduce that $tr(B_i) = (n-i)k_i$, $i = 0,\dots,n-1$.

As $B_i = B_{i-1}A + k_i I$, $tr(B_i) = tr(B_{i-1}A) + n.k_i$, and then $k_i = -\frac{tr(B_{i-1}A)}{i}$ (A72)

We finally obtain the Souriau Algorithm:

$$k_0 = 1 \text{ and } B_0 = I$$

$$\begin{cases} A_i = B_{i-1}A \quad , \quad k_i = -\frac{1}{i}tr\left(A_i\right),\ i = 1,...,n-1 \\ B_i = A_i + k_i I \quad \text{or} \quad B_i = B_{i-1}A - \frac{1}{i}tr\left(B_{i-1}A\right)I \end{cases} \quad \text{(A73)}$$

$$A_n = B_{n-1}A \text{ and } k_n = -\frac{1}{n}tr\left(A_n\right)$$

Souriau Algorithm to compute exponential map of matrix

Souriau approach of Exponential computation is based on algebraic analogy:

$$\left[\lambda I - A\right]^{-1} = \frac{Q(\lambda)}{P(\lambda)} \Leftrightarrow \left[\lambda I - A\right]Q(\lambda) = P(\lambda)I \quad \text{(A74)}$$

and the differential property (with $\lambda = \frac{d}{dt}$): $\left[I\frac{d}{dt} - A\right]Q\left(\frac{d}{dt}\right) = P\left(\frac{d}{dt}\right)I$ (A75)

If a numeric function γ verifies $P\left(\frac{d}{dt}\right)\gamma = 0$, then:

$$P\left(\frac{d}{dt}\right)\gamma = \sum_{i=0}^{n} k_i \gamma^{(n-i)} = k_0\gamma^{(n)} + k_1\gamma^{(n-1)} + ... + k_{n-1}\gamma^{(1)} + k_n\gamma = 0 \quad \text{(A76)}$$

with $\gamma^{(n)} = \frac{d^n\gamma(t)}{dt^n}$ *n-th* derivative of function γ, with initial conditions:

$$\gamma(0) = \gamma^{(1)}(0) = ... = \gamma^{(n-2)} = 0 \text{ and } \gamma^{(n-1)}(0) = 1 \quad \text{(A77)}$$

In this case, the matrix function $\Phi = Q\left(\frac{d}{dt}\right)\gamma$ is solution of the differential equation $\frac{d\Phi(t)}{dt} = A\Phi(t)$, with initial condition $\Phi\left(0\right) = I$:

$$\Phi = Q\left(\frac{d}{dt}\right)\gamma = \sum_{i=0}^{n-1}\gamma^{(n-i-1)}B_i = \gamma^{(n-1)}B_0 + \gamma^{(n-2)}B_1 + ... + \gamma B_{n-1} \quad \text{(A78)}$$

We can then observe that the exponential map of matrix tA is given by:

$$\begin{cases} e^{tA} = \sum_{i=0}^{n-1}\gamma^{(n-i-1)}B_i = \gamma^{(n-1)}B_0 + \gamma^{(n-2)}B_1 + ... + \gamma B_{n-1} \\ \text{with } B_0 = I \text{ and } B_i = B_{i-1}A - \frac{tr\left(B_{i-1}A\right)}{i}I \end{cases} \quad \text{(A79)}$$

$$\begin{cases} \gamma \text{ such that } k_0\gamma^{(n)} + k_1\gamma^{(n-1)} + ... + k_{n-1}\gamma^{(1)} + k_n\gamma = 0 \\ \text{with } k_i = -\frac{tr\left(B_{i-1}A\right)}{i},\ \gamma(0) = ... = \gamma^{(n-2)} = 0 \text{ and } \gamma^{(n-1)}(0) = 1 \end{cases} \quad \text{(A80)}$$

The solution $\gamma(t)$ of characteristic ordinary differential equation is obtained in $\left[0,h\right]$ of the spectral interval of integration. In the remaining part, the exponential function $\Phi(t)$ is computed by:

$$\Phi(ph) = \left(\Phi(h)\right)^p \quad \text{(A81)}$$

Souriau Algorithm for Exponential Map of Matrix is given by:

$$1)\begin{cases} B_0 = I \text{ and } B_i = B_{i-1}A - \dfrac{tr\left(B_{i-1}A\right)}{i} I \\ k_0 = 1 \;,\; k_i = -\dfrac{tr\left(B_{i-1}A\right)}{i} \quad i = 1,...,n \end{cases}$$

$$2)\begin{cases} \gamma \text{ integrated on } [0,h] \text{ such that} \\ k_0\gamma^{(n)} + k_1\gamma^{(n-1)} + ... + k_{n-1}\gamma^{(1)} + k_n\gamma = 0 \\ \text{with } \gamma(0) = ... = \gamma^{(n-2)} = 0 \text{ and } \gamma^{(n-1)}(0) = 1 \end{cases}$$

$$3) \text{ Computation of } \Phi(t) = e^{tA} = \sum_{i=0}^{n-1} \gamma^{(n-i-1)}(t) B_i \text{ on } [0,h] \quad \text{(A82)}$$

$$4) \text{ Extension of Computation on } [0,ph] \text{ by } \Phi(pt) = \left(\Phi(t)\right)^p$$

$$5) \; X(t) = \Phi(t)X_0 \text{ with } X_0 = X(0)$$

If we observe that $\ln(A) = \int_{-\infty}^{0} [sI - A]^{-1} - [sI - I]^{-1} ds$, this algorithm could be used also to compute $A^s = e^{s\ln(A)}$ such as $A^{1/2}$. This Souriau algorithm to solve $\frac{d\Phi(t)}{dt} = A\Phi(t)$ by computation of exponential $\Phi(t) = e^{tA}$ could be extended to solve $L\frac{d^2\Phi(t)}{dt^2} + M\frac{d\Phi(t)}{dt} + N\Phi(t) = 0$ by substituting $\left[\lambda^2 L + \lambda M + N\right]Q(\lambda) = P(\lambda)I$ to $[\lambda I - A]Q(\lambda) = P(\lambda)I$ through the following algebraic relations:

$$\left(\lambda^2 L + \lambda M + N\right) adj\left(\lambda^2 L + \lambda M + N\right) = \det\left(\lambda^2 L + \lambda M + N\right) I \quad \text{(A83)}$$

$$P(\lambda) = \det\left(\lambda^2 L + \lambda M + N\right) = \sum_{i=0}^{2n} k_i \lambda^{2n-i} , Q(\lambda) = adj\left(\lambda^2 L + \lambda M + N\right) = \sum_{i=0}^{2n-2} \lambda^{2n-i-2} B_i \quad \text{(A84)}$$

Examples of Souriau exponential map algorithm

We can illustrate Souriau algorithm with:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } e^{tJ} = \cos(t)I + \sin(t)J = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \in SO(2) \quad \text{(A85)}$$

$$\left| \begin{array}{l} B_0 = I \text{ and } k_0 = 1 \\ B_1 = B_0 J - tr\left(B_0 J\right) I = J \text{ and } k_1 = -tr\left(B_0 J\right) = 0 \\ B_2 = B_1 J - \dfrac{tr\left(B_1 J\right)}{2} I = J^2 - \dfrac{tr\left(J^2\right)}{2} I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ k_2 = -\dfrac{tr\left(J^2\right)}{2} = 1 \end{array} \right. \quad \text{(A86)}$$

$$\begin{cases} \gamma \text{ on } [0,h] \text{ such that } \dfrac{d^2\gamma(t)}{dt^2} + \gamma = 0 \\ \text{with } \gamma(0) = 0 \text{ and } \gamma^{(1)}(0) = 1 \end{cases} \Rightarrow \gamma(t) = \sin(t) \quad \text{(A87)}$$

$$\begin{cases} \Phi(t) = \dfrac{d\gamma(t)}{dt} B_0 + \gamma(t) B_1 = \cos(t) I + \sin(t) J \text{ on } [0,h] \\ \dfrac{d\Phi}{dt} = J\Phi(t) \end{cases} \quad \text{(A88)}$$

Another example is given by harmonic oscillator:

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} -q \\ p \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=J\begin{pmatrix} p \\ q \end{pmatrix} \text{ with } J^2=-I \tag{A89}$$

then $e^{tJ}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$, rotation in $\begin{pmatrix} p \\ q \end{pmatrix}$-plane. (A90)

Next example, is given for skew-symmetric matrix, corresponding to exponential map for $\mathfrak{so}(3)$, the Lie Algebra of Lie group $SO(3)=\left\{R \,/\, R^{-1}=R^T\right\}$:

$$\omega_\times=\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}=\omega_1 L_1+\omega_2 L_2+\omega_3 L_3 \in \mathfrak{so}(3) \text{ and } \omega=\left(\omega_1,\omega_2,\omega_3\right)\in \mathbb{R}^3 \tag{A91}$$

The generators of $\mathfrak{so}(3)$ correspond to the derivatives of rotation around the each of the standard axes, evaluated at identity. The exponential map that takes skew symmetric matrices to rotation matrices is simply the matrix exponential over a linear combination of the generators. We compute this exponential map by Souriau algorithm:

$$e^{\omega_\times}=\gamma^{(2)}B_0+\gamma^{(1)}B_1+\gamma B_2 \tag{A92}$$

Souriau algorithm provides:

$B_0=I$ and $k_0=1$

$$B_1=I.\omega_\times-\frac{Tr\left(I.\omega_\times\right)}{1}I=\omega_\times \text{ and } k_1=-\frac{Tr\left(I\omega_\times\right)}{1}=0 \tag{A93}$$

$$B_2=B_1.\omega_\times-\frac{Tr\left(\omega_\times.\omega_\times\right)}{2}I=\omega_\times.\omega_\times+\|\omega\|^2 I \text{ and } k_2=-\frac{Tr\left(\omega_\times.\omega_\times\right)}{2}=\|\omega\|^2 \tag{A94}$$

We can observe that $B_2=\omega_\times.\omega_\times+\|\omega\|^2 I=\omega\otimes\omega^T$ and $k_3=0$, and we obtain:

$$e^{\omega_\times}=\gamma^{(2)}I+\gamma^{(1)}\omega_\times+\gamma\omega\otimes\omega^T \tag{A95}$$

The function $\gamma(t)$ should verify:

$$k_0\gamma^{(3)}(t)+k_1\gamma^{(2)}(t)+k_2\gamma^{(1)}(t)+k_3\gamma(t)=0 \text{ with } k_0=1,\ k_1=0,\ k_2=\|\omega\|^2,\ k_3=0 \tag{A96}$$

$$\gamma^{(3)}(t)+\|\omega\|^2\gamma^{(1)}(t)=0 \text{ with } \gamma^{(2)}(0)=1,\gamma^{(1)}(0)=0,\gamma(0)=0 \tag{A97}$$

We can then deduce that:

$$\gamma^{(1)}(t)=\frac{1}{\|\omega\|}\sin\left(\|\omega\|t\right) \text{ and } \gamma(t)=\frac{1}{\|\omega\|^2}\left(1-\cos\left(\|\omega\|t\right)\right) \tag{A98}$$

We can then deduce the exponential map of $\mathfrak{so}(3)$:

$$e^{t.\omega_\times}=\cos\left(\|\omega\|t\right)I+\frac{1}{\|\omega\|}\sin\left(\|\omega\|t\right)\omega_\times+\frac{1-\cos\left(\|\omega\|t\right)}{\|\omega\|^2}\omega\otimes\omega^T \tag{A99}$$

But using the relation $\omega\otimes\omega^T=\omega_\times.\omega_\times+\|\omega\|^2 I$, we recover Rodrigues formula:

$$e^{t.\omega_\times}=I+\frac{1}{\|\omega\|}\sin\left(\|\omega\|t\right)\omega_\times+\frac{1-\cos\left(\|\omega\|t\right)}{\|\omega\|^2}\omega_\times^2 \tag{A100}$$

The exponential map from $\mathfrak{se}(3)$ to $SE(3)=\left\{C \,/\, C=\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R\in SO(3), t\in R^3\right\}$ is the matrix exponential on a linear combination of the generators:

$$\delta=\begin{pmatrix} \omega_\times & u \\ 0 & 0 \end{pmatrix}=u_1G_1+u_2G_2+u_3G_3+\omega_1G_4+\omega_2G_5+\omega_3G_6 \tag{A101}$$

$\delta=\begin{pmatrix} u & \omega \end{pmatrix}\in \mathfrak{se}(3)$ and $\begin{pmatrix} u & \omega \end{pmatrix}^T \in R^6$

$$e^\delta=\exp\begin{pmatrix} \omega_\times & u \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} e^{\omega_\times} & Vu \\ 0 & 1 \end{pmatrix} \text{ with } V=I+\frac{1}{2!}\omega_\times+\frac{1}{3!}(\omega_\times)^2+\ldots \tag{A102}$$

By using the identity, $(\omega_\times)^3=-(\omega^T\omega).\omega_\times=-\|\omega\|^2\,\omega_\times$:

$$V=I+\sum_{i=0}^{\infty}\left[\frac{\omega_\times^{2i+1}}{(2i+2)!}+\frac{\omega_\times^{2i+2}}{(2i+3)!}\right]=I+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)\omega_\times+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+3)!}\right)\omega_\times^2 \tag{A103}$$

$$V=I+\left(\frac{1-\cos(\|\omega\|)}{\|\omega\|^2}\right)\omega_\times+\left(\frac{\|\omega\|-\sin(\|\omega\|)}{\|\omega\|^3}\right)\omega_\times^2 \tag{A104}$$

We can apply Souriau formula for exponential map of $\mathfrak{su}(2)$, the Lie Algebra of Lie group $SU(2)$ through a linear combination of the generators given by the Pauli spin matrices:

$$a.I+i\left(c.\sigma_x+b.\sigma_y+d.\sigma_z\right)=\begin{pmatrix} a+id & b+ic \\ -b+ic & a-di \end{pmatrix} \text{ with } (a,b,c,d)\in R^4 \tag{A105}$$

Last example deals with "Geodesic Shooting" for multivariate Gaussian densities $\aleph(m,R)$. Information Geometry provides an invariant Koszul-Fisher metric and geodesic by Euler-Lagrange equations:

$$\begin{cases} \ddot{R}+\dot{m}\dot{m}^T-\dot{R}R^{-1}\dot{R}=0 \\ \ddot{m}-\dot{R}R^{-1}\dot{m}=0 \end{cases}$$

(A106)

Using Souriau theorem of moment map (geometrization of Noether theorem):

$$\Rightarrow \begin{cases} R^{-1}\dot{R}+R^{-1}\dot{m}m^T=B=cste \\ R^{-1}\dot{m}=b=cste \end{cases}$$

(A107)

This moment map could be computed if we consider the following Lie group action in case of Gaussian densities:

$$\begin{bmatrix} Y \\ 1 \end{bmatrix}=\begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix}=\begin{bmatrix} R^{1/2}X+m \\ 1 \end{bmatrix} \;,\; \begin{cases} (m,R)\in R^n\times Sym^+(n) \\ M=\begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix}\in G_{aff} \end{cases} \tag{A108}$$

$X\approx\aleph(0,I)\rightarrow Y\approx\aleph(m,R)$

With $R^{1/2}$, square root of R, is given by Cholesky decomposition of R. $R^{1/2}$ is the Lie group of triangular matrix with positive elements on the diagonal. Euler-Poincaré equations, reduced equations from Euler-Lagrange equations, are then given by:

$$\begin{cases} \dot{m}=Rb \\ \dot{R}=R\left(B-bm^T\right) \end{cases} \tag{A109}$$

Geodesic shooting is obtained by using equations established by Eriksen for "*exponential map*" using the following change of variables [75-76]:

$$\begin{cases} \Delta(t) = R^{-1}(t) \\ \delta(t) = R^{-1}(t)m(t) \end{cases} \Rightarrow \begin{cases} \dot{\Delta} = -B\Delta + bm^T \\ \dot{\delta} = -B\delta + \left(1 + \delta^T \Delta^{-1} \delta\right) b \\ \Delta(0) = I_p, \delta(0) = 0 \end{cases} \text{ with } \begin{cases} \dot{\Delta}(0) = -B \\ \dot{\delta}(0) = b \end{cases}$$

(A110)

The method based on geodesic shooting consists in iteratively approaching the solution by geodesic shooting in direction $\left(\dot{\delta}(0), \dot{\Delta}(0)\right)$, using Souriau exponential map:

$$\Lambda(t) = \exp\left(tA\right) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \begin{pmatrix} \Delta & \delta & \Phi \\ \delta^T & \varepsilon & \gamma^T \\ \Phi^T & \gamma & \Gamma \end{pmatrix}$$

$$\text{with } \quad A = \begin{pmatrix} -B & b & 0 \\ b^T & 0 & -b^T \\ 0 & -b & B \end{pmatrix}$$

(A111)

$$A^2 = \begin{pmatrix} -B & b & 0 \\ b^T & 0 & -b^T \\ 0 & -b & B \end{pmatrix}^2 = \begin{pmatrix} B^2 + bb^T & -Bb & -bb^T \\ -b^T B & 2b^T b & -b^T B \\ -bb^T & -Bb & B^2 + bb^T \end{pmatrix} \quad \text{(A112)}$$

$k_0 = 1, B_0 = I$ and $k_1 = 0, B_1 = A$ because $tr\left(A\right) = 0$

$$B_2 = A^2 - \frac{tr\left(A^2\right)}{2} I, k_2 = -\frac{tr\left(A^2\right)}{2}, B_i = B_{i-1}A - \frac{tr\left(B_{i-1}A\right)}{i} I, k_i = -\frac{tr\left(B_{i-1}A\right)}{i}$$

(A113)

$$k_0 \gamma^{(n)} + k_1 \gamma^{(n-1)} + \ldots + k_{n-1}\gamma^{(1)} + k_n \gamma = 0 \text{ with } \gamma(0) = \ldots = \gamma^{(n-2)} = 0, \ \gamma^{(n-1)}(0) = 1$$

$$e^{tA} = \sum_{i=0}^{n-1} \gamma^{(n-i-1)}(t) B_i$$

(A114)

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