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THE VIBRATIONAL PROPERTIES OF AN ELECTRON GAS

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*THE VIBRATIONAL PROPERTIES OF AN ELECTRON GAS**

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1. STATEMENT OF THE PROBLEM

In many problems one is dealing with a collection of (positively or negatively) charged particles which can move relatively freely between their neighbors. Such systems are the collection of electrons and ions in discharge tubes (plasma), the ionosphere (Heaviside layer), and the so-called free electrons in metals (electron gas).

Contemporary theories of such systems are based upon the analogy with a normal gas of neutral particles, i.e., they assume that the motion of each charged particle is essentially an inertial motion except for short times when the particles approach one another (collisions). This is, for instance, the classical picture of the electron gas. When considering various processes in a plasma this is the picture one starts with in the theory of the propagation of radiowaves in the ionosphere (Larmor, Foersterling, Lassen, etc.). A consistent adoption of such a point of view must be based upon the integral equation from kinetic theory where one would think one includes the necessary elements determining the properties of the gas studied: 1) inertia, 2) interaction forces between the particles, 3) external forces. From the point of view of the kinetic equation one can only specialize to a system under consideration by fixing the law of interaction between the particles, which in the case considered of charged particles means the Coulomb law. Landau[1] was the first to give such a discussion.

However, the kinetic equation scheme and our concept of a gas connected with it are a well-defined approximation of the many-body problem, based upon taking the interaction into account in a special way. One considers only binary interactions between particles - interactions through collisions. The applicability of such an approximation is not always justified. If the forces are such that it is possible to introduce a "sphere of action," i.e., to neglect changes in the distribution function due to distant transits (compared to the radius of the "sphere of action"), and if for the system of particles considered the mean distance between the particles is large compared to the radius of the "sphere of action," the kinetic equation scheme is sufficient. In the opposite case it is not sufficient to consider merely binary interactions. Each particle interacts simultaneously with a number of others, and the many-body problem can in this case no longer be reduced to the usual kinetic equation scheme.

In the case of Coulomb interactions we are dealing with forces which decrease relatively slowly with distance. We see already, for instance, the insufficiency of taking solely binary interactions into account from the fact that a change in the relative density of positive and negative particles at some place will be connected with the appearance of a space charge which will act upon the motion of the charged particles at considerably larger distances than the average distance between the particles. Taking only binary interactions into account is in this case clearly insufficient. An essential role must be played by the interaction forces at distances larger than the mean interparticle distance (we shall henceforth call these "long-range forces"), the

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action of which can not be taken into account by the usual kinetic equation scheme. This fact also emerges, for instance, from the fact that it is, strictly speaking, impossible to introduce a "sphere of action" for the Coulomb interaction since scattering by a Coulomb center leads to a diverging expression for the total cross-section which is just due to taking into account the action of forces at large distances. The physical cause of the diverging integrals lies in the fact that although distant transits change the trajectory little, the probability for such transits because of the increase of the cross section increases faster so that as a result the magnitude of the "sphere of action" becomes infinite and connected with this the integrals in the kinetic equation are also divergent.

The facts such as those given here compel us to make the statement that for a system of charged particles the kinetic equation method which considers only binary interactions - interactions through collisions - is an approximation which is strictly speaking inadequate, so that in the theory of such systems an essential role must be played by the interaction forces, particularly at large distances and, hence, a system of charged particles is, in essence, not a gas but a distinctive system coupled by long-range forces.

Taking "long-range forces" into account leads to properties which do not occur in a normal gaseous medium, the properties of which are confined to those following from the usual kinetic equation scheme. Among those we must reckon the peculiar vibrational properties of an electron plasma which were briefly mentioned by Rayleigh[2] in 1906 in connection with the special problem of the behavior of a system of electrons in the old Thomson model of the atom and which were studied in a similar way in 1929 by Langmuir and Tonks[3] for a gaseous plasma. Their consideration was basically intended to show only the fact that vibrations were possible and to find their frequency. The presence of vibrations was obtained at once from the following elementary considerations: Let an inhomogeneity of the electron density be created in a homogeneous plasma ($N_+e_+ = N_-e_-$); the problem arises how it changes with time.

Assuming the ions to be immovable (which is possible because of the high vibrational frequency) and that the deviations from the stationary state (in which we have $u = 0$, $E = 0$, $\overline{\rho} = \rho_0 = mN$ for the macroscopic values of the velocity, field and electron charge) are small, we have the following set of equations for the first approximation (the equation of continuity, the equation of motion, and the field equation):

$$\left.\begin{aligned} \dot{\rho}_1 + \rho_0 \operatorname{div} \mathbf{u}_1 = 0, \\ \rho_0 \dot{\mathbf{u}}_1 = \rho_0 \frac{e}{m} \mathbf{E}_1, \\ \operatorname{div} \mathbf{E}_1 = 4\pi\rho_1 \frac{e}{m}. \end{aligned}\right\} \quad \text{(A)}$$

Eliminating $\mathbf{E}_1$ and $\mathbf{u}_1$ we get an oscillator equation for ρ_1

$$\frac{\partial^2 \rho_1}{\partial t^2} + \omega_0^2 \rho_1 = 0, \quad \omega_0^2 = 4\pi N e^2/m,$$

i.e., the change in the density does not relax, as in a normal gas, but oscillates with a well-defined frequency characteristic for a plasma. It is at once clear already in this discussion that in the presence of vibrations it is essential to introduce "long-range forces" because the motions of the charges (set (A)) are connected over long distances via the field $\mathbf{E}$ and an essential role is thus played by interactions over distances larger than the mean interparticle distance. The inclusion of "long-range forces" is necessary for the presence of vibrations; the fact of their existence can not therefore be obtained from the usual kinetic equation scheme.

However, the discussion given has no pretence at completeness and, strictly speaking, is only a rational hint in the proper direction. For instance, it is not at all clear what are the conditions on the temperature and density of the electron gas for which the occurrence of vibrations is possible, what is the role of the temperature of the electron gas which does not appear in the equations given here, what is the role of the interactions at large distances, why do these oscillations not propagate, and so on. A more detailed derivation is necessary.

The aim of the present paper is to determine by means of the setting up of a rational mathematical apparatus, which includes "long range" (as well as "short-range") forces, those properties which are caused by those forces, and especially to give basically an explanation, as complete as possible, of the vibrational properties of an electron gas as a basic consequence of taking "long-range forces" into account.

2. INITIAL EQUATIONS AND THEIR SIMPLIFICATION

In this section we wish to formulate the initial equations for a gaseous plasma (a system of electrons, ions, and neutral particles), although their applicability will have a large degree of generality (see Sec. 5). The state of a gaseous plasma is determined by the values of three distribution functions: for electrons - $f_1(x, y, z, \xi, \eta, \zeta, t)$, for ions - $f_2(x, y, z, \xi, \eta, \zeta, t)$, and for neutral particles - $f_3(x, y, z, \xi, \eta, \zeta, t)$. A change in the number of particles of each kind within a volume element in phase space $dx dy dz d\xi d\eta d\zeta$, due to the translation of particles or to the action of external forces, will be taken into account in the usual way. Since the interaction of the electron and ions with the neutral particles can be described by introducing a "sphere of action," i.e., has the character of a "collision," we can also treat changes in the distribution functions due to this cause by using the usual kinetic equation scheme. Of greatest importance for a plasma is allowance for the interactions between charged particles. Since, according to what we have discussed earlier, it is strictly speaking impossible to describe this interaction through a "collision" (without ignoring "long-range forces"), we divide its consideration into two parts: first the interaction at close distances which are less than the mean interparticle distance, and second interactions at "far" distances, larger than this distance.

The interaction at close distances can be taken into account in an artificial manner - by cutting off the Coulomb interaction, for instance, at half the mean interparticle distance; thanks to such a cut-off it becomes possible to retain the concept of interactions through collisions, i.e., one can take that part of the interaction into account also in the kinetic equation scheme. For the vibrational properties of a plasma which we are investigating the long-range forces are the determining ones. We shall take the interaction at large distances macroscopically into account, i.e., as follows. We assume that in the stationary state for the plasma the density of the positive charges N_+e_+ is equal to the density of the negatives N_-e_-, and also that for the currents $j_+ = e_+N_+u_+ = 0$ and $j_- = e_-N_-u_- = 0$ so that the macroscopic values of the fields $\mathbf{E} = 0$ and $\mathbf{H} = 0$. Any deviation from a stationary distribution connected with the occurrence of charges and currents and also through a field influences the way the distribution function changes. Thus, through the fields **E** and **H** the values of the distribution functions, generally speaking, at any distances are mutually connected. To take this connection into account we shall have, on the one hand, the Maxwell equations where the connection between **E** and **H** and the distribution functions will be established through charges and currents, while on the other hand, to take the inverse influence of the fields present on the distribution functions into account it is sufficient to introduce them into the kinetic equation for the charged particles.

For the initial system of equations we shall thus have

$$\left.\begin{aligned}
&\frac{\partial f_1}{\partial t} + \operatorname{div}_r \mathbf{v} f_1 + \frac{e_1}{m_1}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v f_1 = \\
&\qquad = \left[\frac{\partial f_1}{\partial t}\right]_{11}^{st} + \left[\frac{\partial f_1}{\partial t}\right]_{12}^{st} + \left[\frac{\partial f_1}{\partial t}\right]_{13}^{st},\\
&\frac{\partial f_2}{\partial t} + \operatorname{div}_r \mathbf{v} f_2 + \frac{e_2}{m_2}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v f_2 = \\
&\qquad = \left[\frac{\partial f_2}{\partial t}\right]_{21}^{st} + \left[\frac{\partial f_2}{\partial t}\right]_{22}^{st} + \left[\frac{\partial f_2}{\partial t}\right]_{23}^{st},\\
&\frac{\partial f_3}{\partial t} + \operatorname{div}_r \mathbf{v} f_3 = \left[\frac{\partial f_3}{\partial t}\right]_{31}^{st} + \left[\frac{\partial f_3}{\partial t}\right]_{32}^{st} + \left[\frac{\partial f_3}{\partial t}\right]_{33}^{st},\\
&\operatorname{div}\mathbf{E} = 4\pi\rho, \quad \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j},\\
&\operatorname{div}\mathbf{H} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},\\
&\rho = e_1\int_{-\infty}^{+\infty} f_1\, d\xi\, d\eta\, d\zeta + e_2\int_{-\infty}^{+\infty} f_2\, d\xi\, d\eta\, d\zeta,\\
&\mathbf{j} = e_1\int_{-\infty}^{+\infty} \mathbf{v} f_1\, d\xi\, d\eta\, d\zeta + e_2\int_{-\infty}^{+\infty} \mathbf{v} f_2\, d\xi\, d\eta\, d\zeta.
\end{aligned}\right\} \quad (\mathrm{I})^*$$

The set (I) allows us to state the problem: if at $t = 0$ we give $f_1(x, y, z, \xi, \eta, \zeta, 0)$, $f_2(x, y, z, \xi, \eta, \zeta, 0)$, and $f_3(x, y, z, \xi, \eta, \zeta, 0)$ for all values of the variables and also the boundary conditions, what will be the values of f_1, f_2, f_3 for any t? We must assume that the given system characterizes the plasma sufficiently completely and therefore can be used as a basis to consider the different processes in it. It must include the vibrational properties in which we are interested.

We analyze the role of the terms $[\partial f_1/\partial t]_{11}^{st}$, $[\partial f_2/\partial t]_{12}^{st}, \ldots$ which contain the interactions through collisions. Since the presence of inertia is essential for the existence of vibrational properties, it is necessary that the concentration of charged particles and of the neutral gas be such that the vibrational frequency be larger than the frequency of collisions between charged particles and the neutral gas. It is clear that this may be important, but is not a matter of principle. In the following we shall assume that the vibrational frequency is appreciably larger than the collision frequency so that we can neglect collisions between electrons and the neutral gas. The principal problem is the interaction through "collisions" between charged particles.

The magnitude of this interaction can be estimated if we cut off the impact parameter at some maximum value. If we restrict our considerations to only close transits and just those which are connected with turning through an angle larger than $\pi/2$, we can easily estimate the role of the terms $[\partial f/\partial t]_{11}^{st}, \ldots$. In this case the dependence of the impact parameter b on the angle θ through which the relative velocity v is turned differs little from the one occurring when elastic spheres collide and as a result the terms $[\partial f/\partial t]^{st}$ take in these two cases the same form so that from this comparison one sees easily what role is played by the "sphere of action" for the Coulomb interaction.

For the interaction through collisions between electrons we have (see, e.g.,[4])

$$\left[\frac{\partial f}{\partial t}\right]_{11}^{st} = \int_{-\infty}^{+\infty}\int_0^{\sigma}\int_0^{2\pi} (ff_1 - f'f_1')\, vb\, db\, d\omega_1\, d\varepsilon,$$

where b is the impact parameter and v the magnitude of the relative velocity.

The integration is over all velocities ξ_1, η_1, ζ_1, and over all values of the collision parameters b and ϵ. The post-collision variables ξ', η', ζ', ξ_1', η_1', ζ_1' are functions of ξ, η, ζ, ξ_1, η_1, ζ_1, θ, ϵ which are determined by the conservation laws. The law of the interaction between the particles, however, determines the dependence of the impact parameter on the angle of turning θ and the angle ϵ. Hence, the character of the interaction law between the particles enters (1) only through $b = b(\theta, \epsilon)$. For a collision between elastic spheres

$$b = \sigma\cos(\theta/2),$$
$$\left[\frac{\partial f}{\partial t}\right]_{11}^{st} = \int (ff_1 - f'f_1')\, v\sigma^2\cos(\theta/2)\sin(\theta/2)\, d\theta/2\, d\varepsilon. \tag{1}$$

For the Coulomb interaction

$$b = \frac{2e^2}{mv^2}\operatorname{ctg}\frac{\theta}{2}. \tag{2}$$

For scattering angles θ (θ - an obtuse angle) lying in the interval from $\pi/2$ to π, the behavior of $\cos(\theta/2)$ and of $\cot(\theta/2)$ and of their first derivatives are relatively little different. For $\theta/2 = \pi/2$ the terms $\cos(\theta/2)\sin(\theta/2)$ and $\cot(\theta/2)/\sin^2(\theta/2)$ are the same, and the second one is larger than the first one by a factor ~ 1.08 when, $\theta/2 = 80°$, by ~ 1.7 for $60°$, and by ~ 3 for $45°$. The formulae show for these two cases therefore an analogous behavior, provided we take in the case of the Coulomb interaction instead of the constant diameter σ the variable quantity $2e^2/mv^2$. If we understand by v the constant average speed of the particles (which is allowable for estimates of orders of magnitude) the integrals are in fact practically identical if we make the diameters equal. Thus, the formula

$$\sigma^2 = 4(e^2/mv^2)^2 \tag{3}$$

must determine the order of magnitude of the "sphere of action" for large scattering angles, where we must understand by v the average particle velocity.

For angles less than $\pi/2$ there is no longer an analogy with the elastic-sphere case and we must estimate the role of $[\partial f/\partial t]_{11}^{st}$ differently.

L. Landau[1] has taken into account just the distant transits $(0 < \theta < \pi/2)$ in the framework of the usual

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$

kinetic equation scheme. For the mean free path Landau gives the formula

$$l \cong \frac{(kT)^2}{e^4 LN}, \tag{4}$$

whence we get for the cross-section σ^2

$$\sigma^2 = \frac{1}{\pi l N} \cong \frac{1}{\pi}\left(\frac{e^2}{kT}\right)^2 L,$$

or, as $3kT/2 = mv^2/2$:

$$\sigma^2 = \frac{9}{\pi}\left(\frac{e^2}{mv^2}\right)^2 L, \tag{5}$$

which differs from Eq. (4) only through the logarithmic term $L = \ln b_2/b_1$ where b_1 and b_2 are the minimum and maximum values of the impact parameter. Taking only close or only distant transits into account leads thus, apart from a logarithmic term, to the same expressions for the "sphere of action." For the minimum distance we must assume that it corresponds to $\theta = \pi/2$, i.e.,

$$b_1 = 2e^2/mv^2.$$

The maximum value b_2 must lie between b_1 and half the mean interparticle distance.* Putting $b_2 = \frac{1}{2}N^{1/3}$, we have for L

$$L = \ln \frac{3kT}{4N^{1/3}e^2}.$$

Depending on the relation between density and temperature, L can belarger or less than unity, and in accordance with that the magnitude of the "sphere of action" can essentially be defined both by distant $(b > b_1)$ (5) and by close $(b < b_1)$ (3) transits. The order of magnitude of the sphere of action (taking both "close" and "distant" transits into account) can thus be estimated from the equation

$$\sigma^2 = \alpha (e^2/mv^2)^2, \tag{6}$$

where $a \sim 10$.

In the conditions of the ionosphere $N \sim 10^6$ el/cm^3, we take $T = 300°$K, and we get the following values for the "sphere of action" σ, the mean free path l, and the collision frequency ν_{st}:

$$\sigma \cong \sqrt{3\alpha}\, 10^{-6}\text{cm},$$
$$l = \frac{1}{\pi\sigma^2 N} \cong \frac{1}{3\pi\alpha} 10^6 \cong 10^4\text{cm},$$
$$\nu_{st} = v/l \cong 10^3 \text{ Hz}.$$

For the eigenfrequency ν_0 of the vibrations of the electron plasma with the same concentration we have, on the other hand,

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{4\pi N e^2}{m}} \cong 1.3\cdot 10^7 \text{ Hz}.$$

Hence, as $\nu_0 \gg \nu_{st}$ we can completely neglect the interaction through collisions between the electrons.

For an electron plasma in a discharge tube, putting $N = 10^{10}$ el/cm^3, $T = 10^{4}$°K, we have $\div\ \sigma \cong \sqrt{(27a)}\ 10^{-8} \sim 10^{-7}$ cm, $l \sim 10^3$ cm, $\nu_{st} \sim 10^4$ Hz, $\nu_0 \sim 10^9$ Hz. One sees easily that for the interaction through collisions between electrons and ions the expression for the diameter σ has the same form as (6) provided we take for m the effective mass of these particles so that the estimate for σ, l, and ν_{st} remains practically unaltered also for that case.

Thus, the problem of the vibrational properties allows us to simplify the problem: we can neglect all interactions through "collisions."

The class of problems considered which are connected with high frequencies allow us still one more simplification in the initial equations: because of the large mass of the ions as compared to that of the electrons we can neglect their displacements i.e., we can practically assume the ions to be immovable. Under all those conditions the set of initial equations takes the form

$$\left.\begin{aligned}
&\frac{\partial f}{\partial t} + \operatorname{div}_r \mathbf{v} f + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v f = 0,\\
&\operatorname{div}\mathbf{E} = 4\pi e\left(\int_{-\infty}^{+\infty} f\, d\xi\, d\eta\, d\zeta - N\right),\\
&\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e}{c}\int_{-\infty}^{\infty} \mathbf{v} f\, d\xi\, d\eta\, d\zeta,\\
&\operatorname{div}\mathbf{H} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},
\end{aligned}\right\} \tag{II}$$

where f is the electron distribution function. In the problem considered we are thus led to a set of equations describing the behavior of only one electron gas.

The role of the positive ions enters only in the expression for the total charge density, i.e., it is merely reduced to the compensating part of the electron density, corresponding to a stationary state. The set of equations obtained is non-linear. The non-linearity enters when we take the term

$$e/m\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v f,$$

into account, as **E** and **H** themselves depend on f. In the present paper we restrict ourselves to a study of the linearized system. In the stationary state there is a Maxwellian distribution, as follows in the usual way from the set of Eqs. (I). We assume that the deviations from the stationary state

$$\Phi_0(\xi, \eta, \zeta) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(\xi^2+\eta^2+\zeta^2)/2kT} \tag{7}$$

are small

$$f(x, y, z, \xi, \eta, \zeta, t) = \Phi_0(\xi, \eta, \zeta) + \varphi(x, y, z, \xi, \eta, \zeta, t). \tag{8}$$

Substituting (8) into the set (II) and dropping all terms quadratic in φ we obtain the initial set of linearized equations in the form*

$$\left.\begin{aligned}
&\frac{\partial \varphi}{\partial t} + \operatorname{div}_r \mathbf{v}\varphi + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v \Phi_0 = 0,\\
&\operatorname{div}\mathbf{E} = 4\pi e\int_{-\infty}^{+\infty} \varphi\, d\xi\, d\eta\, d\zeta,\\
&\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e}{c}\int_{-\infty}^{+\infty} \mathbf{v}\varphi\, d\xi\, d\eta\, d\zeta,\\
&\operatorname{div}\mathbf{H} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}.
\end{aligned}\right\} \tag{III}$$

3. SOLUTION OF THE LINEARIZED EQUATIONS

A characteristic peculiarity of the first approximation (the linearized set (III)) is that the solutions for

*To take for b_2 the Debye distance is not fully legitimate since it can also be larger than the mean interparticle distance, and since in that case there will be several particles in the sphere of action of one particle, the concept of a "collision" loses already its meaning and with it also the original usual kinetic equation scheme.

*It is not necessary to neglect the Lorentz force: when solving the linearized system of Eqs. (III) its action vanishes automatically (see Sec. 3).

the fields which can exist in the plasma divide into two independent kinds: irrotational and rotational fields (longitudinal and transverse waves). The general solution is constructed additively from the solutions for the rotational and irrotational fields, taken separately. To show this and to find the equations which describe separately both parts we write - as is always possible - the vectors occurring in the set (III) as a sum of two vectors, a rotational and an irrotational one:

$$\mathbf{E} = \mathbf{E}^{(l)} + \mathbf{E}^{(t)},$$

where

$$\operatorname{rot} \mathbf{E}^{(l)} = 0 \text{ and } \operatorname{div} \mathbf{E}^{(t)} = 0, \quad \mathbf{H} = \mathbf{H}^{(t)},$$

since according to (III) div $\mathbf{H} = 0$. Analogously we also divide the vector of the macroscopic velocity:

$$\mathbf{v} = \mathbf{v}^{(l)} + \mathbf{v}^{(t)};$$

since $\mathbf{v} = \int_{-\infty}^{+\infty} \mathbf{v}\varphi \, d\xi d\eta d\zeta$ this last division will be possible if the distribution function itself is in the form of two parts

$$\varphi = \varphi^{(l)} + \varphi^{(t)},$$

for which

$$\mathbf{v}^{(l)} = \int_{-\infty}^{+\infty} \mathbf{v}\varphi^{(l)} \, d\xi \, d\eta \, d\zeta, \quad \mathbf{v}^{(t)} = \int_{-\infty}^{+\infty} \mathbf{v}\varphi^{(t)} \, d\xi \, d\eta \, d\zeta.$$

To obtain the equations which separately describe the two parts, we consider two particular solutions of the set (III):

1. Let there only be an irrotational field in the plasma, i.e., $\mathbf{E}^{(l)} \neq 0$, $\mathbf{E}^{(t)} = \mathbf{H}^{(t)} = 0$; moreover $\mathbf{v}^{(t)} = 0$, $\mathbf{v}^{(l)} \neq 0$, which is possible, if $\varphi^{(l)} = 0$, $\varphi^{(l)} \neq 0$.

2. There is only a rotational field present: $\mathbf{E}^{(l)} = 0$, $\mathbf{E}^{(t)} \neq 0$, $\mathbf{H}^{(t)} \neq 0$; $\mathbf{v}^{(l)} = 0$, $\mathbf{v}^{(t)} \neq 0$, i.e., $\varphi^{(l)} = 0$, $\varphi^{(t)} \neq 0$.

For the irrotational part (longitudinal waves) we get a set of equations by substituting in the set (III) the conditions formulated sub 1):

$$\left.\begin{aligned} &\frac{\partial\varphi^{(l)}}{\partial t} + \operatorname{div}_{\mathbf{r}} \mathbf{v}\varphi^{(l)} + \frac{e}{m}\mathbf{E}^{(l)} \operatorname{grad}_{\mathbf{v}} \Phi_0 = 0, \\ &\operatorname{div} \mathbf{E}^{(l)} = 4\pi e \int_{-\infty}^{+\infty} \varphi^{(l)} \, d\xi \, d\eta \, d\zeta, \\ &-\frac{\partial \mathbf{E}^{(l)}}{\partial t} = 4\pi e \int_{-\infty}^{+\infty} \mathbf{v}\varphi^{(l)} \, d\xi \, d\eta \, d\zeta, \end{aligned}\right\} \qquad (\mathrm{III}^{(l)})$$

with the condition curl $\mathbf{E}^{(l)} = 0$.

For the rotational part we get, substituting in the set (III) the conditions formulated sub 2)

$$\left.\begin{aligned} &\frac{\partial\varphi^{(t)}}{\partial t} + \operatorname{div}_{\mathbf{r}} \mathbf{v}\varphi^{(t)} + \frac{e}{m}\left(\mathbf{E}^{(t)} + \frac{1}{c}[\mathbf{v}\mathbf{H}^{(t)}]\right) \operatorname{grad}_{\mathbf{v}} \Phi_0 = 0, \\ &\operatorname{rot} \mathbf{H}^{(t)} = \frac{1}{c}\frac{\partial \mathbf{E}^{(t)}}{\partial t} + \frac{4\pi e}{c} \int_{-\infty}^{+\infty} \mathbf{v}\varphi^{(t)} \, d\xi \, d\eta \, d\zeta, \\ &\operatorname{rot} \mathbf{E}^{(t)} = -\frac{1}{c}\frac{\partial \mathbf{H}(t)}{\partial t}, \ \operatorname{div} \mathbf{H}^{(t)} = 0, \ \operatorname{div} \mathbf{E}^{(t)} = 0. \end{aligned}\right\} \qquad (\mathrm{III}^{(t)})$$

The sum of the solutions of the set $(\mathrm{III}^{(l)})$ and $(\mathrm{III}^{(t)})$ will because of the linearity of the equations also be a solution of the initial set (III). Moreover, the sum of solutions of the separate equations can be chosen such that it satisfies any arbitrary initial conditions (as well as the set (III)) and given conditions at infinity, i.e., it will thus be the general solution of the set (III). Indeed, the initial conditions for the field in the plasma can in the general case for the set (III) be split into a rotational and an irrotational part, i.e., it can without limitations be written additively as the sum of the initial conditions for $(\mathrm{III}^{(l)})$ and $(\mathrm{III}^{(t)})$; hence, it is sufficient to solve the separate systems also to satisfy any initial conditions together with the set (III).

Thus, in the approximation considered the possible motions in the electron gas are split into mutually independent rotational and irrotational motions; the properties of the electron gas must essentially be determined by the peculiar properties of these two kinds of independent motions and be included in the set of Eqs. $(\mathrm{III}^{(l)})$ and $(\mathrm{III}^{(t)})$.

We consider the set of Eqs. $(\mathrm{III}^{(l)})$. A peculiarity of this set is that the number of equations is larger than the number of unknowns. Indeed, combining (1) and (2), or (1) and (3) from set $(\mathrm{III}^{(l)})$ we have in both cases a pair of equations which is sufficient to determine the unknown distribution function $\varphi(x, y, z, \xi, \eta, \zeta, t)$. However, comparing the solutions of those two pairs of different equations shows that the solutions are identical. We find first of all an exact solution of the first pair and then of the second pair and show that these solutions are identical. The fact that Eqs. (2) and (3) (of (III)) are equivalent expresses a particular fact: in the problem considered taking into account the Coulomb law (Eq. (2)) is equivalent to taking into account the longitudinal displacement current (Eq. (3).

Let us find the solution of the first pair of equations. They allow separation of variables (separation of the t-dependence). Putting

$$\left.\begin{aligned} \varphi^{(l)}(x, y, z, \xi, \eta, \zeta, t) &= T(t)\,\varphi_0^{(l)}(x, y, z, \xi, \eta, \zeta), \\ \mathbf{E}^{(l)}(x, y, z, t) &= T(t)\,\mathbf{E}_0^{(l)}(x, y, z), \end{aligned}\right\} \qquad (9)$$

and substituting this into the equations studied, we find

$$\frac{dT/dt}{T} = -\frac{\operatorname{div}_{\mathbf{r}} \mathbf{v}\varphi_0^{(l)} + \frac{e}{m}\mathbf{E}_0^{(l)} \operatorname{grad}_{\mathbf{v}} \Phi_0}{\varphi_0^{(l)}} = \mathrm{const} = i\omega.$$

Here ω is an as yet undetermined quantity.

The time dependence is determined at once:

$$T = T(0)\, e^{i\omega t}. \qquad (10)$$

To determine $\varphi_0^{(l)}(x, y, z, \xi, \eta, \zeta)$ we have a set of equations

$$\left.\begin{aligned} &i\omega\varphi_0^{(l)} + \operatorname{div}_{\mathbf{r}} \mathbf{v}\varphi_0^{(l)} + \frac{e}{m}\mathbf{E}_0^{(l)} \operatorname{grad}_{\mathbf{v}} \Phi_0 = 0, \\ &\operatorname{div} \mathbf{E}^{(l)} = 4\pi e \int_{-\infty}^{+\infty} \varphi_0^{(l)} \, d\xi \, d\eta \, d\zeta. \end{aligned}\right\} \qquad (11)$$

Since by definition curl $\mathbf{E}^{(l)} = 0$, we shall look for a solution in the form of an expansion in plane longitudinal waves. Because of the linearity of the equations each separate plane wave must be a solution.

For each separate plane wave we have

$$\left.\begin{aligned} \varphi_0^{(l)}(x, y, z, \xi, \eta, \zeta) &= g_{\mathbf{k}}^{(l)}(\xi, \eta, \zeta)\, e^{-i\mathbf{k}\mathbf{r}}, \\ \mathbf{E}_0^{(l)}(x, y, z) &= \mathbf{k}_{\mathrm{I}} a_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}}, \end{aligned}\right\} \qquad (12)$$

where $\mathbf{k}_{\mathrm{I}}$ is a unit vector in the direction of propagation. Substituting into (11) we get a set of equations to determine the amplitude

$$\left.\begin{aligned} &i\left[\omega-(k_x\xi+k_y\eta+k_z\zeta)\right]g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=\\ &=-\frac{e}{m}\left(k_x\frac{\partial\Phi_0}{\partial\xi}+k_y\frac{\partial\Phi_0}{\partial\eta}+k_z\frac{\partial\Phi_0}{\partial\zeta}\right)\frac{1}{|\mathbf{k}|}a_{\mathbf{k}},\\ &-i|\mathbf{k}|a_{\mathbf{k}}=4\pi e\int_{-\infty}^{+\infty}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \end{aligned}\right\} \tag{13}$$

or, writing $\partial\Phi_0/\partial\xi = \partial\Phi_0/\partial\epsilon\cdot\partial\epsilon/\partial\xi = \partial\Phi_0/\partial\epsilon\cdot m\xi$, $\partial\Phi_0/\partial\eta = \partial\Phi_0/\partial\epsilon\cdot m\eta$, $\partial\Phi_0/\partial\zeta = \partial\Phi_0/\partial\epsilon\cdot m\zeta$, where $\epsilon = m/2(\xi^2+\eta^2+\zeta^2)$:

$$\left.\begin{aligned} &i\left[\omega-(\mathbf{kv})\right]g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=-\frac{e}{|\mathbf{k}|}(\mathbf{kv})\frac{\partial\Phi_0}{\partial\varepsilon}a_{\mathbf{k}},\\ &-i|\mathbf{k}|a_{\mathbf{k}}=4\pi e\int_{-\infty}^{+\infty}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \end{aligned}\right\} \tag{14}$$

whence we have for the determination of $g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)$ the equation

$$g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=\frac{4\pi e^2}{|\mathbf{k}|^2}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\int_{-\infty}^{+\infty}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \tag{15}$$

Formula (15) determines the dispersion law for longitudinal waves and also determines the form of the distribution function for any t, i.e., it gives a complete solution of the problem.

Indeed, integrating on the left and on the right over the velocities, we get

$$\frac{4\pi e^2}{|\mathbf{k}|^2}\int_{-\infty}^{+\infty}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\xi\,d\eta\,d\zeta=1, \tag{IV}$$

i.e., the dispersion law for the waves considered.

Since the integral in Eq. (15) is independent of the velocities it follows that we have found the dependence of $g_{\mathbf{k}}^{(l)}$ on the velocities:

$$g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=c(\mathbf{k})\frac{4\pi e^2}{|\mathbf{k}|^2}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}. \tag{16}$$

The constant $c(\mathbf{k})$ can depend only on $\mathbf{k}$.

Finally, by superposition of partial waves we get the general solution:

$$\begin{aligned} \varphi(x,y,z,\xi,\eta,\zeta,t)&=\int_{-\infty}^{+\infty}e^{i\omega t-i\mathbf{kr}}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta\\ &=\int_{-\infty}^{+\infty}e^{i\omega t-i\mathbf{kr}}c(\mathbf{k})\frac{4\pi e^2}{|\mathbf{k}|^2}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\mathbf{k}. \end{aligned} \tag{V}$$

The form of $c(\mathbf{k})$ is determined by the initial condition for the distribution function

$$\begin{aligned} \varphi(x,y,z,\xi,\eta,\zeta,0)&=\int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\mathbf{k}\\ &=\int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}c(\mathbf{k})\frac{4\pi e^2}{|\mathbf{k}|^2}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\mathbf{k}. \end{aligned}$$

Integrating on the right and on the left over the velocities and using (IV):

$$\rho(x,y,z,0)=\int_{-\infty}^{+\infty}\varphi^{(l)}(x,y,z,\xi,\eta,\zeta,0)\,d\xi\,d\eta\,d\zeta=\int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}c(\mathbf{k})\,d\mathbf{k},$$

where $\rho(x,y,z,0)$ is the initial value of the change in the electron density, so that $c(\mathbf{k})$ is determined as the Fourier component:

$$c(\mathbf{k})=\int_{-\infty}^{+\infty}\rho(x,y,z,0)\,e^{i\mathbf{kr}}\,d\mathbf{r}. \tag{VI}$$

Thus, Eqs. (IV), (V), and (VI) form the solution of the problem.

Let us now go over to solving the other pair of equations of the same set $(\mathrm{III}^{(l)})$ (Eqs. (1) and (3)):

$$\left.\begin{aligned} &\frac{\partial\varphi^{(l)}}{\partial t}+\mathrm{div}_{\mathbf{r}}\,\mathbf{v}\varphi^{(l)}+\frac{e}{m}\mathbf{E}^{(l)}\,\mathrm{grad}_{\mathbf{v}}\,\Phi_0=0,\\ &-\frac{\partial\mathbf{E}^{(l)}}{\partial t}=4\pi e\int_{-\infty}^{+\infty}\mathbf{v}\varphi^{(l)}\,d\xi\,d\eta\,d\zeta. \end{aligned}\right\} \tag{17}$$

Proceeding analogously for the previous and for this pair we separate the time dependence; and also here we get

$$\left.\begin{aligned} &\varphi^{(l)}(x,y,z,\xi,\eta,\zeta,t)=T(t)\,\varphi_0^{(l)}(x,y,z,\xi,\eta,\zeta),\\ &\mathbf{E}^{(l)}(x,y,z,t)=T(t)\,\mathbf{E}_0^{(l)}(x,y,z). \end{aligned}\right\} \tag{18}$$

Substituting into (17), we find for $T(t)$

$$T(t)=T(0)\,e^{i\omega t}, \tag{19}$$

where ω must be determined by the subsequent equations.

To determine $\varphi_0^{(l)}(x,y,z,\xi,\eta,\zeta)$ we have the set

$$\left.\begin{aligned} &i\omega\varphi_0^{(l)}+\mathrm{div}_{\mathbf{r}}\,\mathbf{v}\varphi_0^{(l)}+\frac{e}{m}\mathbf{E}_0^{(l)}\,\mathrm{grad}_{\mathbf{v}}\,\Phi_0=0,\\ &-i\omega\mathbf{E}_0^{(l)}=4\pi e\int_{-\infty}^{+\infty}\mathbf{v}\varphi_0^{(l)}(x,y,z,\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}\right\} \tag{20}$$

We write the solution here also as a superposition of longitudinal waves:

$$\left.\begin{aligned} &\varphi_0^{(l)}(x,y,z,\xi,\eta,\zeta)=g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,e^{-i\mathbf{kr}},\\ &\mathbf{E}_0^{(l)}(x,y,z)=\mathbf{k}_I a_k e^{-i\mathbf{kr}}. \end{aligned}\right\} \tag{21}$$

To determine the amplitude we substitute into the set (20)

$$\left.\begin{aligned} &i\left[\omega-(\mathbf{kv})\right]g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=-\frac{e}{|\mathbf{k}|}(\mathbf{kv})\,a_{\mathbf{k}}\,\partial\Phi_0/\partial\varepsilon,\\ &-i\omega\mathbf{k}_I a_{\mathbf{k}}=4\pi e\int_{-\infty}^{+\infty}\mathbf{v}g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}\right\} \tag{22}$$

Taking the scalar product of the second equation of this set with $\mathbf{k}$ and substituting into the first equation we find

$$g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)=\frac{4\pi e^2}{\omega|\mathbf{k}|^2}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\int_{-\infty}^{+\infty}(\mathbf{kv})\,g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \tag{23}$$

Equation (23), similar to (15), gives the complete solution of the problem. To obtain the dispersion formula, we multiply (23) by $(\mathbf{k}\cdot\mathbf{v})$ on the right and on the left and integrate over all values of the velocities; we get the dispersion law in the form

$$\frac{4\pi e^2}{\omega|\mathbf{k}|^2}\int_{-\infty}^{+\infty}\frac{(\mathbf{kv})^2\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\xi\,d\eta\,d\zeta=1. \tag{IV(a)}$$

We now show that Eqs. (IV) and (IV(a)) are identical, and thus also that the solutions given by (15) and (23) are identical. We choose k along the x-axis so that in order that (IV) and (IV(a)) are identical it is necessary that

$$\frac{1}{\omega}\int_{-\infty}^{+\infty}\frac{\xi^2\partial\Phi_0/\partial\varepsilon}{k\xi-\omega}\,d\xi\,d\eta\,d\zeta=\frac{1}{k}\int_{-\infty}^{+\infty}\frac{\xi\partial\Phi_0/\partial\varepsilon}{k\xi-\omega}\,d\xi\,d\eta\,d\zeta,$$

or, multiplying and dividing by $(k\xi+\omega)$:

$$\frac{1}{\omega}\int_{-\infty}^{+\infty}\frac{\xi^2(k\xi+\omega)\,\partial\Phi_0/\partial\varepsilon}{k^2\xi^2-\omega^2}\,d\xi\,d\eta\,d\zeta=\frac{1}{k}\int_{-\infty}^{+\infty}\frac{\xi(k\xi+\omega)\,\partial\Phi_0/\partial\varepsilon}{k^2\xi^2-\omega^2}\,d\xi\,d\eta\,d\zeta.$$

We need of the function $\Phi_0(\xi,\eta,\zeta)$ only the fact that it is even in the velocities, so that the integrals of odd powers vanish and on the right and the left remain identical terms so that we have verified that the dispersion formulae (IV) and (IV(a)) are identical. As a consequence we get also the identity of the solutions.

To see that we integrate (23) over ξ, η, ζ, and get

$$\int_{-\infty}^{+\infty} g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta = \frac{1}{\omega}\int_{-\infty}^{+\infty}(\mathbf{kv})\,g_{\mathbf{k}}^{(l)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \times\left\{\frac{4\pi e^2}{|\mathbf{k}|^2}\int_{-\infty}^{+\infty}\frac{(\mathbf{kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\xi\,d\eta\,d\zeta\right\},$$

which gives the identity of (15) and (23), if we use $(\mathrm{IV}^{(a)})$ and (IV).

We shall now solve the equations for the transverse waves: the set $(\mathrm{III}^{(t)})$.

As in the case of the longitudinal waves the set of equations allows us also here to separate the variables - we can split off the time-dependence. We get

$$\left.\begin{aligned}\varphi^{(t)}(x,y,z,\xi,\eta,\zeta,t) &= T(t)\,\varphi_0^{(t)}(x,y,z,\xi,\eta,\zeta),\\ \mathbf{E}^{(t)}(x,y,z,t) &= T(t)\,\mathbf{E}_0^{(t)}(x,y,z),\\ \mathbf{H}^{(t)}(x,y,z,t) &= T(t)\,\mathbf{H}_0^{(t)}(x,y,z).\end{aligned}\right\}\tag{24}$$

Substituting into $(\mathrm{III}^{(t)})$ we get for $T(t)$

$$T(t) = T(0)\,e^{i\omega t},$$

where ω must be determined from the dispersion law for transverse waves. To determine $\varphi_0^{(t)}$, $\mathbf{E}_0^{(t)}$, and $\mathbf{H}_0^{(t)}$ we get the set

$$\left.\begin{aligned}&i\omega\varphi_0^{(t)} + \operatorname{div}_r \mathbf{v}\varphi_0^{(t)} + \frac{e}{m}\left(\mathbf{E}_0(t) + \frac{1}{c}[\mathbf{vH}]\right)\operatorname{grad}_v\Phi_0 = 0,\\ &\operatorname{rot}\mathbf{H}_0^{(t)} = \frac{i\omega}{c}\mathbf{E}_0^{(t)} + \frac{4\pi e}{c}\int_{-\infty}^{+\infty}\mathbf{v}\varphi_0^{(t)}\,d\xi\,d\eta\,d\zeta,\\ &\operatorname{rot}\mathbf{E}_0^{(t)} = -\frac{i\omega}{c}\mathbf{H}_0^{(t)},\quad \operatorname{div}\mathbf{E}_0^{(t)} = 0,\quad \operatorname{div}\mathbf{H}_0^{(t)} = 0.\end{aligned}\right\}\tag{25}$$

We look also here for a solution for $\mathbf{E}_0^{(t)}$, $\mathbf{H}_0^{(t)}$, and $\varphi_0^{(t)}$ in the form of a plane wave:

$$\left.\begin{aligned}\mathbf{E}_0^{(t)}(x,y,z) &= \mathbf{e_k}b_{\mathbf{k}}e^{-i\mathbf{kr}},\\ \mathbf{H}_0^{(t)}(x,y,z) &= \mathbf{h_k}c_{\mathbf{k}}e^{-i\mathbf{kr}},\\ \varphi_0^{(t)}(x,y,z,\xi,\eta,\zeta) &= g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,e^{-i\mathbf{kr}}.\end{aligned}\right\}\tag{26}$$

Because

$$\operatorname{div}\mathbf{E}_0^{(t)} = -i(\mathbf{ke_k}) = 0,\quad \operatorname{div}\mathbf{H}_0^{(t)} = -i(\mathbf{kh_k}) = 0$$

the waves are transverse: $\mathbf{e_k}\perp\mathbf{k}$, $\mathbf{h_k}\perp\mathbf{k}$.

The mutual relation between the vectors $\mathbf{e_k}$, $\mathbf{h_k}$, and $\mathbf{k}$ is given by the equation

$$\operatorname{rot}\mathbf{E}_0^{(t)} = -i\omega/c\cdot\mathbf{H}_0^{(t)},$$

which also connects the amplitudes $b_{\mathbf{k}}$ and $c_{\mathbf{k}}$. We have

$$[\mathbf{ke_k}]\,b_{\mathbf{k}} = \mathbf{h_k}c_{\mathbf{k}}\omega/c,\tag{27}$$

whence

$$\mathbf{h_k} = [\mathbf{ke_k}]\frac{1}{|\mathbf{k}|}\tag{28}$$

and

$$b_{\mathbf{k}} = \frac{\omega}{c|\mathbf{k}|}c_{\mathbf{k}}.\tag{29}$$

We use the third equation of the set (25):

$$-i[\mathbf{kh_k}]\,c_{\mathbf{k}} = \frac{i\omega}{c}\mathbf{e_k}b_{\mathbf{k}} + \frac{4\pi e}{c}\int_{-\infty}^{+\infty}\mathbf{v}g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.\tag{30}$$

Using (27) to eliminate from (29) the amplitude $c_{\mathbf{k}}$ we express $b_{\mathbf{k}}$ through $g_{\mathbf{k}}^{(t)}$; we have

$$-\frac{c}{\omega}i[\mathbf{k}[\mathbf{ke_k}]]\,b_{\mathbf{k}} = i\frac{\omega}{c}\mathbf{e_k}b_{\mathbf{k}} + \frac{4\pi e}{c}\int_{-\infty}^{+\infty}\mathbf{v}g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta,$$

but

$$\frac{1}{|\mathbf{k}|^2}[\mathbf{k}[\mathbf{ke_k}]] = \frac{\mathbf{k}}{|\mathbf{k}|^2}(\mathbf{ke_k}) - \mathbf{e_k}(\mathbf{kk})\frac{1}{|\mathbf{k}|^2} = -\mathbf{e_k};$$

(30) becomes

$$i\frac{c^2k^2-\omega^2}{\omega c}\mathbf{e_k}b_{\mathbf{k}} = \frac{4\pi e}{c}\int_{-\infty}^{+\infty}\mathbf{v}g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta,\tag{31}$$

or, multiplying on the right and on the left by $\mathbf{e_k}$, we have

$$b_{\mathbf{k}} = i4\pi e\frac{\omega}{\omega^2-c^2k^2}\int_{-\infty}^{+\infty}(\mathbf{e_kv})\,g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.\tag{32}$$

Thus the amplitudes of the fields $\mathbf{E}_0^{(t)}$ and $\mathbf{H}_0^{(t)}$ are expressed through Eqs. (27) and (32) in terms of the amplitude of the distribution function. To determine it we use the first equation of the set (25). Using

$$\operatorname{grad}_v\Phi_0 = \frac{\partial\Phi_0}{\partial\varepsilon}\left(\frac{\partial\varepsilon}{\partial\xi}\mathbf{i} + \frac{\partial\varepsilon}{\partial\eta}\mathbf{j} + \frac{\partial\varepsilon}{\partial\zeta}\mathbf{k}\right) = m\mathbf{v}\frac{\partial\Phi_0}{\partial\varepsilon},$$

we get by substituting (26)

$$i[\omega-(\mathbf{kv})]\,g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta) = -e\left\{(\mathbf{e_kv})\,b_{\mathbf{k}} + \frac{1}{c}([\mathbf{vh_k}]\,\mathbf{v})\,c_{\mathbf{k}}\right\}\frac{\partial\Phi_0}{\partial\varepsilon}.$$

As $[\mathbf{v}\times\mathbf{h_k}]\cdot\mathbf{v} = 0$ the action of the Lorentz force $(e/c)[\mathbf{v}\times\mathbf{H}]$ automatically drops out of the calculations. Expressing on the right-hand side $b_{\mathbf{k}}$ in terms of the unknown amplitude $g_{\mathbf{k}}^{(t)}$ through (32), we get

$$g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta) = 4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\frac{(\mathbf{e_kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\int_{-\infty}^{+\infty}(\mathbf{e_kv})\,g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta\tag{33}$$

(cf. the analogous expression (23) for longitudinal waves).

We obtain the dispersion law for the transverse waves by multiplying (33) by $(\mathbf{e_k}\cdot\mathbf{v})$ and integrating over all values of the velocities ξ, η, ζ:

$$4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\int_{-\infty}^{+\infty}\frac{(\mathbf{e_kv})^2\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\xi\,d\eta\,d\zeta = 1\tag{VII}$$

(compare (IV), $(\mathrm{IV}^{(a)})$).

We obtain the general solution in the same way as for the longitudinal waves, viz.; noting that the integral in Eq. (33) is independent of the velocities

$$g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta) = 4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\frac{(\mathbf{e_kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}c(\mathbf{k}),\tag{34}$$

to obtain the general solution we must superpose partial waves with the amplitude found from (34):

$$\begin{aligned}\varphi^{(t)}(x,y,z,\xi,\eta,\zeta,t) &= \int_{-\infty}^{+\infty}e^{i\omega t-i\mathbf{kr}}g_{\mathbf{k}}^{(t)}(\xi,\eta,\zeta)\,d\mathbf{k} \\ &= \int_{-\infty}^{+\infty}e^{i\omega t-i\mathbf{kr}}c(\mathbf{k})\,4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\frac{(\mathbf{e_kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\mathbf{k}.\end{aligned}\tag{VIII}$$

The form of $c(\mathbf{k})$ is determined by the initial conditions for the distribution function

$$\varphi(x,y,z,\xi,\eta,\zeta,0) = \int_{-\infty}^{+\infty}e^{-i\mathbf{kr}}c(\mathbf{k})\,4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\frac{(\mathbf{e_kv})\,\partial\Phi_0/\partial\varepsilon}{(\mathbf{kv})-\omega}\,d\mathbf{k},\tag{35}$$

whereas in the case of longitudinal waves $c(\mathbf{k})$ was expressed in terms of the initial distribution of the electron density $\rho(x, y, z, 0)$ (Eq. (VI)); in the case considered where only transverse waves are present $\rho(x, y, z, 0) = 0$ and $c(\mathbf{k})$ is now expressed in terms of the initial current distribution. Indeed, multiplying (35) by the velocity vector $\mathbf{v}$ and integrating over all values of the velocities we get on the left

$$\mathbf{j}(x, y, z, 0) = \int_{-\infty}^{+\infty} \mathbf{v}\varphi(x, y, z, \xi, \eta, \zeta, 0)\, d\xi\, d\eta\, d\zeta.$$

On the right we write the multiplying vector out in its components

$$\mathbf{v} = (\mathbf{e_k v})\,\mathbf{e_k} + (\mathbf{k_1 v})\,\mathbf{k_1} + (\mathbf{h_k v})\,\mathbf{h_k}.$$

The following integrals occur

$$\int_{-\infty}^{+\infty} \frac{(\mathbf{e_k v})^2\, \partial\Phi_0/\partial\varepsilon}{(\mathbf{kv}) - \omega}\, d\xi\, d\eta\, d\zeta,$$

$$\int_{-\infty}^{+\infty} \frac{(\mathbf{e_k v})(\mathbf{k_1 v})\, \partial\Phi_0/\partial\varepsilon}{(\mathbf{kv}) - \omega}\, d\xi\, d\eta\, d\zeta,$$

$$\int_{-\infty}^{+\infty} \frac{(\mathbf{e_k v})(\mathbf{h_k v})\, \partial\Phi_0/\partial\varepsilon}{(\mathbf{kv}) - \omega}\, d\xi\, d\eta\, d\zeta.$$

The first integral is the same as the integral in the dispersion formula (VII) and is determined by it. The last two are exactly zero; for the second one, for instance, we have, choosing $x \parallel \mathbf{k}$ and $y \parallel \mathbf{e_k}$

$$\int_{-\infty}^{+\infty} \frac{\xi\eta\, \partial\Phi_0/\partial\varepsilon}{k\xi - \omega}\, d\xi\, d\eta\, d\zeta = \int_{-\infty}^{+\infty} \frac{\xi\, d\xi}{k\xi - \omega} \int_{-\infty}^{+\infty} \eta\, \partial\Phi_0/\partial\varepsilon\, d\eta\, d\zeta = 0$$

because $\partial\Phi_0/\partial\epsilon$ is even in the velocities. Similarly for the third one. Hence, we have finally

$$\mathbf{j}(x, y, z, 0) = \int_{-\infty}^{+\infty} \mathbf{v}\varphi(x, y, z, \xi, \eta, \zeta, 0)\, d\xi\, d\eta\, d\zeta = \int_{-\infty}^{+\infty} e^{-i\mathbf{kr}}\mathbf{e_k} c(\mathbf{k})\, d\mathbf{k}, \tag{IX}$$

so that $c(\mathbf{k})$ is determined as a Fourier component.

Equation (VII), (VIII), and (IX) give us the solution of the problem for the case considered (transverse waves).

4. DISPERSION OF LONGITUDINAL WAVES

We consider the dispersion relation for longitudinal waves (IV). Without loss of generality we choose a system of coordinates with the x axis along **k**. The dispersion relation then has the form*

$$\frac{4\pi e^2}{k} \int_{-\infty}^{+\infty} \frac{\xi\, \partial\Phi_0/\partial\varepsilon}{k\xi - \omega}\, d\xi\, d\eta\, d\zeta = 1. \tag{36}$$

The distribution function $\Phi_0(\xi, \eta, \zeta)$ occurring in the dispersion relation is the distribution function for the stationary state. In the present section we consider an electron gas which is characterized in the stationary state by a Maxwell distribution function

$$\Phi_0(\varepsilon) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\varepsilon/kT}, \tag{37}$$

where $\epsilon = \frac{1}{2}m(\xi^2 + \eta^2 + \zeta^2)$. Introducing (37) into (36) and integrating over η and ζ we get

$$-\frac{4\pi e^2 N}{k}\sqrt{\frac{m}{2\pi (kT)^3}} \int_{-\infty}^{+\infty} \frac{\xi e^{-m\xi^2/2kT}}{k\xi - \omega}\, d\xi = 1. \tag{38}$$

It is convenient to introduce dimensionless quantities.

As unit of frequency we choose the eigenfrequency of the electron gas

$$\omega_0 = \sqrt{4\pi N e^2/m},$$

as unit of wave number the inverse of the Debye distance D:

$$D = 1/\varkappa, \quad \varkappa = \sqrt{4\pi e N^2/kT}.$$

The unit of velocity is then

$$\omega_0/\varkappa = \sqrt{kT/m}.$$

In these units $\omega^* = \omega/\omega_0$, $k^* = k/\kappa$, $\xi^* = \xi\kappa/\omega_0 = \xi\sqrt{(m/kT)} = x$. Introducing all this into (38) we get the dispersion relation in the form

$$\int_{-\infty}^{+\infty} \frac{x e^{-x^2/2}\, dx}{x - \omega^*/k^*} = -\sqrt{2\pi}k^{*2}. \tag{39}$$

The fact that in the formula obtained the quantities, e, m, N, and T which determine the electron plasma do not occur explicitly but only through the chosen units of measurement shows that for the system considered the time interval defined by the frequency ω_0 and the spatial interval defined by the Debye distance are the natural dimensions characteristic for the system considered.

This is also indicated by the fact that the Debye formula for the static polarization is automatically included in the dispersion relation obtained and is a particular case of it. Indeed, putting in (39) $\omega = 0$, we get $k^{*2} = -1$, i.e.,

$$k = \pm i\varkappa. \tag{40}$$

Hence, the spatial dependence of the solutions (12) will in that case be determined by the terms $e^{\pm\kappa r}$ which are identical with the Debye ones.

We are interested in how an inhomogeneity created in the electron gas is dissipated; we shall thus assume that k is given in the dispersion relation.

We restrict our considerations to the physically most interesting case where the macroscopic inhomogeneity created in the electron plasma is large compared with the Debye distance, i.e.,

$$|k^*| = |k/\varkappa| = 2\pi\,|D/\lambda| < 1.$$

In that case the integrand can be written as a power series in k

$$\frac{x}{x - \omega^*/k^*} = -\frac{x}{\omega^*}k^*\left[1 + \frac{x}{\omega^*}k^* + \left(\frac{x}{\omega^*}\right)^2 k^{*2} + \left(\frac{x}{\omega^*}\right)^3 k^{*3} + \ldots\right], \tag{41}$$

which converges and uniformly converges in the interval $|x| < \omega^*/k^*$. Substituting (41) into (39) and for the present restricting ourselves to the first non-vanishing term, we have

$$-\left(\frac{k^*}{\omega^*}\right)^2 \int x^2 e^{-\frac{1}{2}x^2}\, dx = -\sqrt{2\pi}k^{*2},$$

or for ω^* we get

$$-(k^*/\omega^*)^2\sqrt{2\pi} = -\sqrt{2\pi}\,k^{*2}, \tag{42}$$

whence it follows that we get a constant value for the angular frequency

$$\omega^{*2} = 1, \tag{43}$$

Thus, in the approximation considered the dispersion relation is such that the angular frequency ω is independent of the wave number and equal to the con-

*In the paper "On the Vibrations of an Electron Plasma" (J. Phys. USSR **10**, 25 (1946)) L. D. Landau has shown that when the problem is solved more rigorously by the Laplace transform method, it is necessary to go around the singularity in the integrand in (36) in the complex plane. He then found a new effect of a specific damping of the waves due to their interaction *with resonance* electrons ("Landau damping"). (Note by editor to Usp. Fiz. Nauk).

stant ω_0 which characterizes the plasma. This indicates an anomalously strong dispersion, namely such a one that the magnitude of the group velocity for such a dispersion law vanishes, i.e., in this approximation there is no propagation; a macroscopic inhomogeneity once produced does not relax as in the usual case neither does it propagate, and it vibrates with a large frequency.

The dispersion law determines the way the solutions given by (V) depend on the time. For the change of the electron density $\rho(x, y, z, t)$ we have from (V)

$$\rho(x, y, z, t) = \int_{-\infty}^{+\infty} e^{i\omega t - i\mathbf{k}\mathbf{r}} c(\mathbf{k})\, dk, \tag{44}$$

where the amplitudes $c(k)$ are determined by the initial condition for ρ

$$\rho(x, y, z, 0) = \int_{-\infty}^{+\infty} e^{-i\mathbf{k}\mathbf{r}} c(\mathbf{k})\, dk. \tag{45}$$

Using (43) we have thus from (44) and (45)

$$\rho(x, y, z, t) = \rho(x, y, z, 0)\, e^{i\omega_0 t}, \tag{46}$$

which is the same as the formula obtained from elementary considerations (see Sec. 1).

We now turn to the next approximation - taking the first two non-vanishing terms in the expansion (41) into account; we have

$$-\left\{\left(\frac{k^*}{\omega^2}\right)^2 \int x^2 e^{-\frac{1}{2}x^2} dx + \left(\frac{k^*}{\omega^*}\right) \int x^4 e^{-\frac{1}{2}x^2} dx\right\} = -\sqrt{2\pi}\, k^{*2}.$$

Also here we can extend the integration to infinity without appreciable error. We get

$$-\{(k^*/\omega^*)^2 \sqrt{2\pi} + (k^*/\omega^*)^4 \cdot 3\sqrt{2\pi}\} = -\sqrt{2\pi}\, k^{*2}. \tag{47}$$

Hence, to determine ω as function of k we have the equation

$$1/\omega^{*2} + 3k^{*2}/\omega^{*4} = k^{*2},$$

or

$$\omega^{*4} - \omega^{*2} - 3k^{*2} = 0. \tag{48}$$

Since the initial Eq. (47) is true only when $|k| < 1$ (in second order) it is necessary to write the solution of Eq. (48) as a series in k and to restrict ourselves to the first two non-vanishing terms. Thus, we have

$$\omega^{*2} = 1/2 \pm 1/2\,(1 + 12k^{*2})^{1/2} \cong 1/2 \pm 1/2\,(1 + 6k^{*2}).$$

For the root with the positive sign we have

$$\omega^{*2} = 1 + 3k^{*2}, \tag{49}$$

i.e., a further refinement of Eq. (42) or, changing to the usual units,

$$\omega^2 = \omega_0^2 + 3\frac{kT}{m}k^2 = \omega_0^2 + 3\frac{kT}{m}\left(\frac{2\pi}{\lambda}\right)^2. \tag{50}$$

The root with the negative sign does not agree with the original assumption $\omega^*/k^* > 1$ and can thus be dropped.

In the case when the given wavenumber is real and as usual larger than the Debye one, Eq. (49) shows that in contradistinction to the first approximation a given macroscopic inhomogeneity oscillates but with a frequency which now depends on the magnitude of the macroscopic inhomogeneity λ.

The magnitude of this correction to the frequency is determined by the temperature and the density of the electron gas.*

In the case when the frequency is given it is convenient to write Eq. (50) in the form

$$k^2 = (\omega^2 - \omega_0^2)/v^2, \quad v^2 = 3kT/m. \tag{X}$$

Equation (X) is to a fair approximation also the dispersion relation for longitudinal waves. It is similar to the well-known dispersion relation for the propagation of transverse electromagnetic waves in the ionosphere (and also the dispersion relation for the free motion wave function in quantum mechanics) differing essentially by the fact that the role of the light velocity c^2 is here played by the thermal velocity $v^2 = 3kT/m$.

We note some consequences of the dispersion relation:

1. If $\omega > \omega_0$, k will be real, i.e., we have propagation in this case - the presence of longitudinal waves. However, in the case $\omega < \omega_0$, k is imaginary and there is no propagation; the spatial dependence of the solution is in that case similar to that of the static polarization.

2. For the group velocity of the longitudinal waves we have from (50)

$$v_{gr} = d\omega/dk = 3kT/m \cdot k/\omega;$$

and the connection between the phase and the group velocities will thus be

$$v_{gr} v_{ph} = v^2, \quad v^2 = 3kT/m. \tag{51}$$

To find the values of the quantities v_{gr} and v_{ph} themselves, restricting ourselves to the first non-vanishing terms (considering as before an inhomogeneity larger than the Debye one: $|k/\kappa| = v/\omega_0 |k| < 1$), we have from (50)

$$v_{gr} \cong \frac{v^2}{\omega_0} k = \frac{3kT}{\sqrt{4\pi e^2 m N}} k, \qquad v_{ph} \cong \frac{\omega_0}{k} = \sqrt{4\pi e^2 N/m}\,\frac{1}{k}; \tag{52}$$

since, by definition, $|k| v/\omega_0 < 1$, the group velocity is always less and the phase velocity always larger than the thermal velocity of the electrons.

Due to the small mass of the electrons the velocities (52) are nevertheless relatively large. For the ionosphere, for instance (see Sec. 2), for a train with wavelength 1 m, $v_{gr} \sim 10^5$ cm/s and for 1 cm a hundred times larger (the Debye distance for the ionosphere is ~ 1 mm).

3. The dispersion law determines the change with the time and coordinates of the electron density as is

*We must bear in mind that since this correction is by definition less than the oscillation frequency, allowance for collisions will influence it more strongly. In the case when the collision frequency ω^{st} (see Sec. 2) is larger than this correction its influence is swamped by collisions. Its presence shows up most purely if

$$\Delta\omega/\omega^{st} = \pi\sqrt{3}\, Dl/\lambda^2 \sim 10^6 T^{5/2}/N^{3/2}\lambda^2 \gg 1.$$

For this case, to which strictly speaking our calculation in second approximation refers one can completely neglect collisions and this is always allowed for sufficiently high temperatures.

When, e.g., $N = 10^6$ el/cm^3, $N = 10^8$ el/cm^3, $N = 10^{10}$ el/cm^3; $T = 10^3$ °K, $T = 10^4$ °K, $T = 10^5$ °K — when $\lambda = 10^2$ cм, $\dfrac{\Delta\omega}{\omega^{st}} \sim 10$,

i.e., under those conditions we can neglect collisions even up to meter waves (the Debye distance for these cases ranges from 1 to 0.1 mm).

clear from the general solution (44) for $\rho(x, y, z, t)$.

Restricting ourselves also here to considering the case where the dimensions of the macroscopic inhomogeneity are large compared with the Debye distance, we can find the general character of the change with time of the electron density. Substituting thus into Eq. (44) the expression for $\omega(k)$

$$\omega(k)=\sqrt{\omega_0^2+v^2k^2}\cong\omega_0+1/2\,v^2/\omega_0k^2, \qquad (53)$$

we have

$$\rho(x, y, z, t)=e^{i\omega_0 t}\int e^{\frac{i}{2}\frac{v^2}{\omega_0}k^2t+i\mathbf{kr}}c(\mathbf{k})\,dk=e^{i\omega_0 t}f(x, y, z, t), \qquad (54)$$

where $c(k)$ is determined by the initial density distribution. The integral in (54) has the same form as the one occurring in the well-known problem in wave mechanics of the spreading out of wave packets in free motion, which is also clear from the fact that the function $f(x, y, z, t)$ is the general solution of the equation

$$\nabla^2 f=\frac{2i\omega_0}{v^2}\frac{\partial f}{\partial t}, \qquad (55)$$

which is similar to the wave equation for free motion with only $m/\hbar$ instead of ω_0/v^2. We need thus not perform any calculations but can at once use the well-known formula for the speed of the spreading out, changing in our problem $m/\hbar$ to ω_0/v^2. We shall thus have for the effective width of the inhomogeneity in the electron distribution at time t

$$\delta_t^2=\delta_0^2+(v^2/\delta_0\omega_0)^2t^2, \qquad (56)$$

where δ_0 is the linear dimension of the inhomogeneity at the initial moment.

It thus follows that a given density inhomogeneity in the electron distribution oscillates with frequency ω_0 and additionally spreads out. The speed of the spreading can be characterized by the time in which the dimensions are increased by a factor two; for this time τ we shall have

$$\tau=3\delta_0^2\frac{\omega_0}{v^2}=\delta_0^2\frac{\sqrt{4\pi e^2mN}}{kT}; \qquad (57)$$

since by definition, we consider an inhomogeneity larger than the Debye one, automatically the time of dissipation τ is larger than the oscillation period T, because the last requirement is equivalent to the first one:

$$\frac{\tau}{T}=\frac{3}{2\pi}\frac{\omega_0^2}{v^2}\delta_0^2=\frac{1}{2\pi}\left(\frac{\delta_0}{D}\right)^2.$$

Thus, a given inhomogeneity in the electron density oscillates and slowly (compared with the oscillation period) dissipates.

5. DISPERSION OF LONGITUDINAL WAVES IN AN ELECTRON GAS WITH A FERMI DISTRIBUTION FUNCTION

In the present section we wish to show that the presence of oscillationary properties also occurs for an electron gas in a state which is close to the degenerate one with characteristics which are different from the previous case of the Maxwell distribution.

When the conditions under which the original Eqs. (III) are valid are satisfied the specialization to the case of a Fermi gas consists merely in the specialization in the original equations for the distribution function for the stationary state $\Phi_0(\xi, \eta, \zeta)$. For the case considered we must take for it the Fermi distribution.

To fix our ideas we shall consider the electron gas in metals, and especially such metals (mainly alkali metals) for which the free electron hypothesis is a fair approximation. The conditions necessary for the existence of vibrations (the condition for the applicability of Eqs. (III), see Sec. 2) consist in the conditions preserving the manifestation of inertia of the charged particles during the period of their oscillations. For this it is necessary that the collision frequency of the charged particles with the neutral particles and with one another be less than the oscillation frequency. In the opposite case an inhomogeneity produced in the electron plasma will aperiodically relax without oscillations.

The frequency in a metal of collisions of the electrons with the crystal lattice will be

$$\nu^{st}\cong v/l,$$

where l is the mean free path and v the mean velocity.

We can estimate this quantity from the empirical value of the electrical conductivity μ:

$$\mu=\frac{e^2N}{m}\frac{l}{v}, \quad \nu^{st}\cong\frac{e^2N}{m}\frac{1}{\mu},$$

for silver, e.g., $\mu = 5\cdot10^{17}$, $N = 5.9\cdot10^{22}$ and hence $\nu^{st} = 2.9\cdot10^{12}$ Hz. Expecting for the oscillation frequency the same value as in the classical case,

$$\nu_0=\frac{1}{2\pi}\sqrt{\frac{4\pi Ne^2}{m}},$$

we have, for instance, for silver $\nu_0 \cong 2.10^{15}$ Hz.

The oscillation frequency is larger than the collision frequency by about a factor one hundred. Because of this we can practically neglect for the problem considered the interaction of the electrons with the lattice.

The presence of a high density of the electron gas in metals makes the question of the role of interactions through "collisions" between the electrons themselves important for the problem considered. For the total scattering cross-section we have (see Sec. 2)

$$\sigma^2=\alpha(e^2/mv^2)^2,$$

where v is as to order of magnitude equal to the mean electron velocity and $\alpha \sim 10$. The mean free path is

$$l=(\pi\sigma^2N)^{-1}.$$

Hence we have for the collision frequency

$$\nu_{el}^{st}\cong v/l\cong\alpha\pi(e^2/m)^2N/v^3. \qquad (58)$$

When the degeneracy is complete

$$v^3=\left(\frac{2\bar{\epsilon}_{av}}{m}\right)^{3/2}=\left(\frac{3}{5}\right)^{3/2}\frac{h^3}{m^3}\cdot\frac{3N}{8\pi}, \qquad (59)$$

where $\bar{\epsilon}_{av}$ is the mean kinetic energy of the electrons in the case of complete degeneracy. Substituting (59) into (58) we get

$$\nu_{el}^{st}\cong\left(\frac{5}{3}\right)^{3/2}\frac{8\pi^2\alpha}{3}\left(\frac{me^4}{h^3}\right)\sim 8\cdot10^{15}\,\text{Hz},$$

which is independent of the density. This particular fact is caused by the fact itself that there is degeneracy.

Whereas ν_{el}^{st} is independent of the density, the oscillation frequency ν_0 is essentially determined by it. Because of this there can also be for sufficiently high density realizable conditions for which the vibrational properties must occur in their purest form.

For silver $\nu_0 \sim \nu_{el}^{st}$. Thus, it is possible to talk about the vibrational properties also for the electron gas inside metals.

We consider the dispersion relation for longitudinal waves applied to the case considered. We use (IV) and (36), where we do not yet make a special choice for the form of the distribution function $\Phi_0(\xi, \eta, \zeta)$; for the case considered we must take for it the Fermi function.

It is convenient to introduce also here dimensionless quantities. For the unit of frequency we choose also here the eigenfrequency of an electron plasma $\omega_0 = \sqrt{4\pi Ne^2/m}$. In the case of classical statistics we took for the unit of length the Debye distance

$$D = \varkappa^{-1}, \text{ where } \varkappa = \sqrt{4\pi Ne^2/kT}.$$

In the case where there is degeneracy the role of κ must be played by the quantity

$$\sqrt{\frac{4\pi Ne^2}{2/3\bar{\varepsilon}}}, \tag{60}$$

i.e., the role of kT is played by 2/3 times the maximum energy $\bar{\epsilon}$ of the Fermi-distribution. Indeed, we define the quantity k for the static case. To do this we put $\omega = 0$ in the dispersion relation. For the case of complete degeneracy $-\partial\Phi_0/\partial\epsilon$ has the character of a δ-function which is non-vanishing only on the boundary of the Fermi distribution:

$$\frac{\partial\Phi_0}{\partial\varepsilon} = -\frac{2m^3}{h^3}\,\delta(\varepsilon-\bar{\varepsilon}), \tag{61}$$

where $\bar{\epsilon}$ is the maximum energy of the Fermi distribution. Substituting (61) into (36), putting $\omega = 0$, and integrating:

$$\frac{4\pi e^2}{k^2}\int_{-\infty}^{+\infty}\frac{\partial\Phi_0}{\partial\varepsilon}\,d\xi\,d\eta\,d\zeta = -\frac{4\pi e^2}{k^2}\left(\frac{2m^3}{h^3}\right)\int_{-\infty}^{+\infty}\delta(\varepsilon-\bar{\varepsilon})\cdot\frac{4\pi}{m}\sqrt{\frac{2\varepsilon}{m}}\,d\varepsilon = -\frac{(4\pi)^2}{k^2}\frac{e^2(2m)^{3/2}}{h^3}\,\bar{\varepsilon}^{1/2} = 1, \tag{62}$$

or, using the expression for the maximum energy

$$\bar{\varepsilon}^{3/2} = \frac{h^3}{(2m)^{3/2}}\cdot\frac{3N}{8\pi}, \tag{63}$$

we have

$$-\frac{3}{2}\,\frac{4\pi e^2 N}{k^2\bar{\varepsilon}} = 1,$$

whence we get

$$k^2 = k_0^2 = -\frac{4\pi e^2 N}{2/3\bar{\varepsilon}},$$

i.e., Eq. (60).

For the unit of wave number we take thus the quantity (60). The unit of velocity will be

$$\omega_0/k_0 = \sqrt{2/3\bar{\varepsilon}/m}.$$

In these units

$$\omega^* = \frac{\omega}{\omega_0}, \quad k^* = \frac{k}{k_0}, \quad \varepsilon^* = \frac{\varepsilon}{2/3\bar{\varepsilon}}.$$

Introducing this all into (36) we get the dispersion relation in the form

$$\int_{-\infty}^{+\infty}\frac{\xi^*\,\partial\Phi_0/\partial\varepsilon^*}{\xi^*-\omega^*/k^*}\,d\xi^*\,d\eta^*\,d\zeta^* = 2\pi\sqrt{3}\,k^{*2}, \tag{64}$$

where all quantities occurring, including Φ_0, are dimensionless.

We are also here interested in how a density inhomogeneity created in the electron gas dissipates; we shall therefore assume that k is given in the dispersion relation. Since $|k_0|$ is very large, for silver, e.g., $N = 5.9\cdot10^{22}$, $\bar{\epsilon} = 8.5\cdot10^{-12}$ erg, $|k_0|^2 = 3\cdot10^{16}$ cm^2, and hence λ_0 ($2\pi/\lambda_0 = k_0$) is of the order of magnitude of 10^{-8} cm so that the case of physical interest to us is the one where the macroscopic inhomogeneity is large compared with that distance. We assume therefore that

$$|k^*| = |k/k_0| = \lambda_0/\lambda \ll 1,$$

and we can expand the integrand in (64) in a power series in k^*:

$$\frac{\xi^*}{\xi^*-\omega^*/k^*} = -\frac{\xi^*}{\omega^*}k^*\left(1+\frac{\xi^*}{\omega^*}k^*+\left(\frac{\xi^*}{\omega^*}\right)^2 k^{*2}+\dots\right). \tag{65}$$

As $\partial\Phi_0/\partial\epsilon^*$ is even in the velocities the integrals with odd powers of ξ^* vanish. Restricting ourselves to the first two non-vanishing terms we have

$$-\left\{\left(\frac{k^*}{\omega^*}\right)^2\int\xi^{*2}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\xi^*\,d\eta^*\,d\zeta^* + \left(\frac{k^*}{\omega^*}\right)^4 \times\int\xi^{*4}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\xi^*\,d\eta^*\,d\zeta^*\right\} = 2\pi\sqrt{3}\,k^{*2}. \tag{66}$$

Because of the spherical symmetry of $\partial\Phi_0/\partial\epsilon$ in the velocities we put

$$\xi^2 = 1/3v^2 = \frac{2}{3}\frac{\varepsilon}{m},$$

or

$$\xi^{*2} = 1/3\varepsilon^*.$$

In dimensionless quantities we have also

$$d\xi^*\,d\eta^*\,d\zeta^* = 4\pi v^{*2}\,dv^* = 2\pi\sqrt{\varepsilon^*}\,d\varepsilon^*.$$

Substituting into (66) we get

$$-\left\{\left(\frac{k^*}{\omega^*}\right)^2\int\varepsilon^{*3/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\varepsilon^* + \frac{1}{3}\left(\frac{k^*}{\omega^*}\right)^4\int\varepsilon^{*5/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\varepsilon^*\right\} = 3\sqrt{3}\,k^{*2}. \tag{67}$$

The integrals occurring here are the same as some well-known integrals in Fermi statistics. For the case of complete degeneracy

$$\left.\begin{aligned} -\int\varepsilon^{*3/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\varepsilon^* &= \int\varepsilon^{*3/2}\delta(\varepsilon^*-3)\,d\varepsilon^* = 3\sqrt{3}, \\ -\int\varepsilon^{*5/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}\,d\varepsilon^* &= \int\varepsilon^{*5/2}\delta(\varepsilon^*-3)\,d\varepsilon^* = 3^2\sqrt{3}. \end{aligned}\right\} \tag{68}$$

Restricting ourselves to the first term in the expansion we have

$$\omega^{*2} = 1, \tag{69}$$

or, in the usual units

$$\omega^2 = \omega_0^2 = 4\pi e^2 N/m. \tag{70}$$

We obtain the same result as in the classical case.

Hence, the vibrational properties also occur in the degenerate gas. The frequency of the eigenvibrations is the same as in the case of the Maxwell distribution.

Taking the first two terms into account, we have

$$(k^*/\omega^*)^2 + (k^*/\omega^*)^4 = k^{*2}.$$

Proceeding as before (see Sec. 4), we get

$$\omega^{*2} = 1 + k^{*2}, \tag{71}$$

or, changing to the usual units,

$$\omega^2 = \omega_0^2 + \frac{\omega_0^2}{k_0^2}k^2 = \omega_0^2 + \frac{2}{3}\frac{\bar{\varepsilon}}{m}k^2, \tag{72}$$

or

$$k^2 = (\omega^2-\omega_0^2)/v^2, \tag{73}$$

i.e., we find that the dispersion law also in Fermi statistics is the same as in the Maxwellian case; the difference lies in the velocity v, the role of the thermal velocity $\sqrt{3kT/m}$ is here played by two thirds of the limiting (T-independent) velocity of the Fermi distribution.

The change in frequency occurring in the second approximation,

$$\Delta\omega \cong \frac{1}{3}\frac{\bar{\varepsilon}}{m\omega_0}k^2$$

must be taken into account only under such conditions when it will not be completely swamped by the electron collisions. Of principal importance are here the collisions of the electrons with one another. To take this change into account it is at least necessary that $\Delta\omega$ is comparable to ω_{el}^{st} of (58); since the latter is independent of frequency, while $\Delta\omega$ is determined by it, in principle the required condition can be satisfied for sufficiently high density. However, for the electron gas inside a metal the frequency of collisions between electrons is already comparable with ω_0 and hence always exceeds the small change $\Delta\omega$, if we restrict ourselves as before to inhomogeneities large compared with the distance (60) which here plays the role of the Debye one. Hence we can only make a statement about the presence of vibrations and in practice it is now impossible to talk about their propagation in an electron gas inside a metal.

If we do not restrict ourselves to the case of complete degeneracy, then we can in the general case express the integrals occurring in (67) in terms of the number of particles (first integral) and the energy (second integral) per unit volume as follows

$$-\int \varepsilon^{*3/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}d\varepsilon^* = \frac{1}{(^1/_3\bar{\varepsilon}^*)^{3/2}}\cdot\frac{3}{2}\,\mathrm{const}\cdot N,$$
$$-\int \varepsilon^{*5/2}\frac{\partial\Phi_0}{\partial\varepsilon^*}d\varepsilon^* = \frac{1}{(^1/_3\bar{\varepsilon}^*)^{5/2}}\cdot\frac{5}{2}\,\mathrm{const}\cdot E,$$
$$\mathrm{const} = \frac{h^3}{4\pi(2m)^{3/2}}.$$

Substitution into (67) gives

$$\left(\frac{k^*}{\omega^*}\right)^2\cdot\frac{3}{2}\,\mathrm{const}\cdot N\frac{1}{\bar{\varepsilon}^{3/2}}+\left(\frac{k^*}{\omega^*}\right)^4\cdot 5\,\mathrm{const}\cdot E\frac{1}{\bar{\varepsilon}^{5/2}} = k^{*2},$$

whence, as before, ω^* is determined by the equation

$$\omega^{*2} = \frac{3}{2}\,\mathrm{const}\cdot N\frac{1}{\bar{\varepsilon}^{3/2}}+5\,\mathrm{const}\cdot E\frac{1}{\bar{\varepsilon}^{5/2}}k^{*2},$$

or, in the usual units

$$\omega^2 = \frac{3}{2}\,\mathrm{const}\cdot N\frac{\omega_0^2}{\bar{\varepsilon}^{3/2}}+2\frac{\mathrm{const}\cdot E}{m}\frac{1}{^3/_5\bar{\varepsilon}^{3/2}}k^2.$$

Using for simplicity

$$\bar{\varepsilon}^{3/2} = {}^3/_2\,\mathrm{const}\cdot N \text{ and } E_0 = {}^3/_5\bar{\varepsilon}N,$$

where E_0 is the energy per unit volume in the case of complete degeneracy, we get

$$\omega^2 = \frac{4\pi e^2N}{m}+\frac{2}{3}\frac{\bar{\varepsilon}}{m}\frac{E}{E_0}k^2. \tag{74}$$

We have thus for the critical frequency the old dependence. For the velocity v occurring in the dispersion relation (73) we have

$$v^2 = \frac{2}{3}\frac{\bar{\varepsilon}}{m}\frac{E}{E_0}.$$

Equation (74) determines the dispersion relation for longitudinal waves in Fermi statistics - in the general case of an arbitrary state of the Fermi gas. In that sense it is general. In the case of complete degeneracy the equations obtained go over into the previous ones. When there is a small departure from the degenerate state

$$E = E_0+\gamma T^2, \quad \gamma = \frac{3\pi^2k^2}{5}\frac{N}{\bar{\varepsilon}^2}.$$

We have for the velocity v

$$v^2 = \frac{2}{3}\frac{\bar{\varepsilon}}{m}(1+\alpha T^2), \quad \alpha = \frac{\gamma}{E_0} = \frac{1}{2}\frac{\pi^2k^2}{\bar{\varepsilon}^2}.$$

6. DISPERSION OF TRANSVERSE WAVES

In conclusion we consider in this section the dispersion relation (VII) for transverse waves. In the usual elementary theory which does not use the kinetic equation scheme including long-range forces those simplifications which lie at the basis of the derivation do not have an entirely clear character. It is therefore desirable to analyze within the framework of the considerations given here the dispersion relation (VII).

Considering dispersion does not lead to a result differing from what is known. To a good approximation the dispersion relation for transverse waves reduces to the well-known Larmor formula for the ionosphere. The influence of the temperature of the electron gas on the way dispersion occurs appears only in the second order in the ratio v/c, where v is the thermal velocity of the electrons.

We consider a linearly polarized wave: $\mathbf{e_k} \parallel y$, $k \parallel x$. The dispersion relation (VII) takes the form

$$I(\omega, k) = 4\pi e^2\frac{\omega}{\omega^2-c^2k^2}\int_{-\infty}^{+\infty}\frac{\eta^2\,\partial\Phi_0/\partial\varepsilon}{k\xi-\omega}d\xi\,d\eta\,d\xi = 1. \tag{75}$$

For a Maxwell distribution

$$\frac{\partial\Phi_0}{\partial\varepsilon} = -\frac{1}{kT}\Phi_0 = -\frac{1}{kT}N\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-\frac{m(\xi^2+\eta^2+\zeta^2)}{2kT}}.$$

For the integral in $I(\omega, k)$ we have hence

$$-\frac{1}{kT}\left(\frac{m}{2\pi kT}\right)^{3/2}N\int_{-\infty}^{+\infty}\frac{e^{-m\xi^2/2kT}}{k\xi-\omega}d\xi\int_{-\infty}^{+\infty}\eta^2e^{-m\eta^2/2kT}d\eta\int_{-\infty}^{+\infty}e^{-m\zeta^2/2kT}d\zeta;$$

since

$$\int_{-\infty}^{+\infty}\eta^2e^{-m\eta^2/2kT}d\eta = \left(\frac{kT}{m}\right)^{3/2}\sqrt{2\pi}, \quad \int_{-\infty}^{+\infty}e^{-m\xi^2/kT}d\xi = \left(\frac{kT}{m}\right)^{1/2}\sqrt{2\pi},$$

we get

$$I(\omega, k) = \frac{4\pi e^2N/m}{\omega^2-c^2|k|^2}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{m}{kT}}\int_{-\infty}^{+\infty}\frac{e^{-m\xi^2/2kT}d\xi}{1-\xi k/\omega} = 1. \tag{76}$$

We introduce here also dimensionless quantities. For the unit of frequency we take here also the eigenfrequency of the electron plasma. The unit of wavenumber will be the quantity

$$k_0 = \sqrt{4\pi Ne^2/mc^2},$$

which will be clear from a comparison of the formulae obtained with the earlier ones (49) for the case of longitudinal waves. In these units

$$\omega^* = \omega/\omega_0, \quad k^* = k/k_0.$$

Introducing this into the dispersion relation (76) we get

$$\omega^{*2} = k^{*2}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}x^2}dx}{1-\frac{k^*}{\omega^*}\frac{v}{c}x}, \tag{77}$$

where v is the electron thermal velocity, $v = \sqrt{kT/m}$.

Assuming that $v/c \ll 1$ we can write the integrand in a power series similar to the case of the longitudinal waves:

$$\frac{1}{1-\frac{k^*}{\omega^*}\frac{v}{c}x}=1+\left(\frac{k^*}{\omega^*}\frac{v}{c}x\right)+\left(\frac{k^*}{\omega^*}\frac{v}{c}x\right)^2+\dots \quad (78)$$

In the case of longitudinal waves the validity of a similar expansion was limited by the value $|k^*|<1$. In the case considered, however, because of the factor v/c which occurs the applicability of the expansion is wider. It is only necessary that

$$\frac{k^*}{\omega^*}\frac{v}{c}\ll 1, \quad (79)$$

or, in the usual units

$$\omega/k\gg v. \quad (80)$$

In the final Eq. (83).

$$(\omega/k)^2=c^2+(\omega_0/k)^2.$$

Substituting (78) into (77) and restricting ourselves to the first two terms, we have

$$\omega^{*2}=k^{*2}+\frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}x^2}dx+\left(\frac{k^*}{\omega^*}\frac{v}{c}\right)^2\frac{1}{\sqrt{2\pi}}\int x^2e^{-\frac{1}{2}x^2}dx. \quad (81)$$

The integration can here only be taken up to $|x| < (\omega^*/k^*)(c/v)$ but because of (79) one can extend the limits of integration to infinity for the calculations without appreciable error.

Under those conditions, if we also restrict ourselves to the first term in the expansion we get

$$\omega^{*2}=1+k^{*2}, \quad (82)$$

or, changing to the usual units

$$k^2=(\omega^2-\omega_0^2)/c^2, \quad (83)$$

i.e., the Larmor formula.

The second term in the expansion is of order $(v/c)^2$ and must be taken into account by a relativistic calculation when it is required.

7. SUMMARY AND CONCLUSION

1. Taking "long-range forces" into account leads to the possibility that longitudinal waves (connected with a change in the electron density) can propagate in an electron plasma with a large dispersion.

2. We have investigated the conditions as to temperature and density under which they can occur.

3. We determined the dispersion law for longitudinal waves for an electron gas with a Maxwell distribution. The dispersion law leads to the fact that if the relation between the temperature and the density is such that the macroscopic inhomogeneity is appreciably larger than the Debye distance, in first approximation in these relations a given macroscopic inhomogeneity vibrates with frequency ω_0 without changing its shape. In second approximation there occurs in addition dissipation, the behavior of which is governed by the temperature and the density.

The dispersion law is such that for frequencies less than the oscillation frequency ω_0 longitudinal waves do not propagate; for larger frequencies propagation occurs. The velocity of propagation depends on the temperature and density of the electron gas.

4. We found a solution of the linearized equations for the distribution function of the electrons both for the case of longitudinal and for the case of transverse waves.

5. In first approximation (restricting ourselves to the linearized equations) the presence of longitudinal waves is not connected with radiation (longitudinal and transverse waves superpose in this approximation).

6. We determined the conditions for the existence of vibrational properties for a degenerate Fermi gas. We found also for this case the dispersion law for longitudinal waves.

7. In Sec. 6 we found the dispersion relation for transverse waves.

In conclusion we must note that since there is an appreciable change in the dispersion of longitudinal waves and the dispersion includes the basic characteristics of an electron plasma, the possibility of observing longitudinal waves experimentally is clearly of well-defined interest from the point of view of a possible method of analyzing the properties of a plasma. Allowance for the damping of longitudinal waves should also be included in the considerations, and also the processes of interaction through "collisions" between charged particles and between charged particles and the neutral gas.

In our considerations we have restricted ourselves to an analysis of the linearized equations. In this case the presence of longitudinal waves is not connected with the transverse ones, and vice versa. Since the original equations are non-linear, interaction between them must occur already in the next approximation. The presence of the one kind must cause that of the other. This fact indicates the limit of applicability of the contemporary theory of the propagation of radiowaves in the ionosphere which does not assume that longitudinal waves are present. Such a consideration is possible only in the framework of the linearized equations. Among these circumstances one must clearly also view the nature of non-linear effects which occur in the practice of radiotelegraphy and which apparently appear as the result of the interaction of transverse waves with longitudinal ones which must arise under the influence of the transverse waves and also of the transverse waves with one another. The solution of this problem must be included in the next approximation of the initial non-linear set (II). This remark refers also to the optics of metals (mainly the alkali metals) in the ultraviolet where electron inertia occurs and where one constructs a theory similar to the one for the ionosphere.

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