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Source: *Annals of Mathematics*, Second Series, Vol. 135, No. 3 (May, 1992), pp. 411-468

Published by: Mathematics Department, Princeton University

Stable URL: <https://www.jstor.org/stable/2946572>

Accessed: 14-05-2020 19:55 UTC

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The existence of noncollision singularities in newtonian systems

By ZHIHONG XIA*

Introduction

In this paper we solve a long-standing problem in celestial mechanics proposed by Painlevé and Poincaré in the last century. The problem, which concerns the nature of the singularities in the n -body problem, asks whether there exists a noncollision singularity in the newtonian n -body problem? Here we give an affirmative answer to this problem by proving the existence of noncollision singularities in the 5-body problem.

We consider n point-masses moving in a euclidean space \mathbb{R}^3 . Let the mass of the i^{th} particle be $m_i > 0$, let its position be $\mathbf{q}_i \in \mathbb{R}^3$ and let $\dot{\mathbf{q}}_i \in \mathbb{R}^3$ be its velocity. According to Newton's law,

$$(0.1) \quad m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} (\mathbf{q}_i - \mathbf{q}_j) = \frac{\partial U}{\partial \mathbf{q}_i},$$

where the double dot denotes the second derivative with respect to time and U is the negative newtonian potential energy or the self-potential

$$U = \sum_{j < i} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}.$$

The basic problem in celestial mechanics is to describe the solutions of system (0.1).

*Supported in part by the National Science Foundation and a Sloan Fellowship. This paper was submitted while at Harvard University. The main result of this paper first appeared in my Thesis for Northwestern University as partial fulfillment of a Ph.D. degree. I am most indebted to my advisor, Professor Don Saari, for his continual encouragement, advice and support over all these years. It is my pleasure to thank Professors K. Meyer, D. Schmidt and C. Robinson for their support and help during my thesis research. I am grateful to Professors C. Fefferman and J. Mather for their interest in this work and for pointing out some errors in the original manuscript, in particular, J. Mather, who read the entire manuscript and made many useful suggestions and comments. Many people have expressed interest in this work and I would like to thank all of them, especially Professors J. Moser, S. T. Yau and R. McGehee for their encouragement.

A solution of a system of differential equations is said to experience a singularity at time $\sigma < \infty$ if the solution cannot be analytically extended beyond σ . The equations of motion (0.1) are real analytic everywhere, except where two or more of the particles occupy the same point in the physical space. More precisely let

$$\Delta_{ij} = \{ \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in (\mathbb{R}^3)^n \mid \mathbf{q}_i = \mathbf{q}_j \},$$

$$\Delta = \bigcup_{i < j} \Delta_{ij}.$$

The self-potential U is a real-analytic function on $(\mathbb{R}^3)^n \setminus \Delta$. The standard existence and uniqueness theory of ordinary differential equations yields the following result:

THEOREM 0.1. *Given $\mathbf{q}(0) \in (\mathbb{R}^3)^n \setminus \Delta$ and $\dot{\mathbf{q}}(0) \in (\mathbb{R}^3)^n$, there exists a unique solution $\mathbf{q}(t)$ defined for all $0 \leq t < \sigma$, where σ is maximal.*

Definition 0.1. If $\sigma < \infty$, then the solution $\mathbf{q}(t)$ is said to experience a *singularity* at σ .

In other words, a solution experiences a singularity at σ if the standard existence and uniqueness theory of ordinary differential equations no longer extends the solution. One sees that the singularities of the solution must be related to the singularities of the potential function U . In fact, a classical theorem states that the minimum distance between all pairs of particles must approach zero at a singularity.

THEOREM 0.2 (Painlevé, 1895). *If $\mathbf{q}(t)$ experiences a singularity at σ , then*

$$\mathbf{q}(t) \rightarrow \Delta \quad \text{as } t \rightarrow \sigma.$$

A proof of this theorem can be found in the book by Siegel and Moser [20].

It is natural to ask whether $\mathbf{q}(t)$ must approach a definite point on Δ as $t \rightarrow \sigma$? A priori $\mathbf{q}(t)$ might oscillate wildly while approaching Δ , or it might become unbounded as the distance to Δ goes to zero. If $\mathbf{q}(t)$ does approach a point \mathbf{q}^* as $t \rightarrow \sigma$, then each of the particles has some limiting position at time σ . Since $\mathbf{q}^* \in \Delta$, at least two of these limiting positions must coincide, which means that these particles must collide as $t \rightarrow \sigma$. Therefore we have the following definition:

Definition 0.2. Suppose that $\mathbf{q}(t)$ has a singularity at σ ; this singularity is called a *collision singularity* if there exists a $\mathbf{q} \in \Delta$ such that $\mathbf{q}(t) \rightarrow \mathbf{q}$ as $t \rightarrow \sigma$. Otherwise the singularity is called a *noncollision singularity*.

A full understanding of the nature of singularities in the n -body problem of classical celestial mechanics has eluded mathematicians to this day. While the existence of collision singularities is more or less trivial, the question whether there exist noncollision singularities has been open since the time of Painlevé a century ago. The following theorem concerning the 3-body problem is due to Painlevé:

THEOREM 0.3. *For $n = 3$, all singularities are collision singularities.*

What remains, then, is to establish whether there exist noncollision singularities for $n \geq 4$.

In this paper we will solve Painlevé's problem by affirming that there exists a noncollision singularity. We prove its existence in a 5-body problem. A modification of our approach shows that such behavior exists in the n -body system for $n > 5$.

First, however, we give a brief historical survey of this problem.

An important step toward providing an answer to Painlevé's question was taken by von Zeipel [25] in 1908. He showed that if the positions of all the particles remain bounded as $t \rightarrow \sigma$, then the singularities must be due to a collision. In other words, a noncollision singularity can occur only if the system of particles becomes unbounded in finite time.

To be precise let

$$I = \sum_{i=1}^n \frac{1}{2} m_i |\mathbf{q}_i|^2,$$

where I is the moment of inertia that measures the size the system. Differentiating I twice, we have the Lagrange–Jacobi equation

$$\ddot{I} = U + 2h.$$

Observe as $\mathbf{q} \rightarrow \Delta$ that $U \rightarrow \infty$, and thus that $\ddot{I} \rightarrow \infty$. The following proposition can be easily proved by this fact and Theorem 0.2 of Painlevé.

PROPOSITION 0.1. *Let σ be a singularity (collision or noncollision); then $\lim I \leq \infty$ exists as $t \rightarrow \sigma$.*

Now we may state the theorem of von Zeipel [25].

THEOREM 0.4 (von Zeipel). *If σ is a singularity, and $\lim I < \infty$ as $t \rightarrow \sigma$, then σ is a collision singularity. On the other hand, if σ is a noncollision singularity, then $\lim I = \infty$ as $t \rightarrow \sigma$.*

The first proof of this theorem can be found in [25]. For a historical survey related to this theorem and a modern version of the proof, see McGehee [9]. In fact, this theorem was also proved by several others, e.g., Chazy in 1920

[2], Sperling in 1970 [24] and Saari in 1973 [16]. Saari essentially extended von Zeipel's result to show that a noncollision singularity cannot occur if the moment of inertia is "slowly varying." Recent interest in the subject of singularities in celestial mechanics is due largely to the work of Saari and Pollard in the early 1970s (see [13]-[17]).

Von Zeipel's theorem is a remarkable result; it says that the only way a noncollision singularity can occur is for the system of particles to explode to infinity in finite time. This makes the existence of noncollision singularities seemingly impossible, since a particle escaping to infinity in finite time would have to acquire an infinite amount of kinetic energy. However, since the potential energy U is not bounded from below, there is no a priori upper bound on the kinetic energy of a particle. Indeed recent work of McGehee and Mather ([6]-[8]) make an affirmative answer to Painlevé's question less doubtful than people originally thought. In fact, McGehee's techniques of blowing up the collision singularities will be our major tool in constructing the noncollision singularity.

Observe that, by Theorem 0.2, if $\mathbf{q}(t)$ experiences a noncollision singularity at σ , then $\mathbf{q}(t) \rightarrow \Delta$ as $t \rightarrow \sigma$. This suggests that, in any neighborhood of a noncollision singularity in the physical space, there must be some collision singularities. The binary collisions are essentially algebraic branch points that can be regularized analytically or geometrically and thus treated as elastic bounces. Intuitively we seem to know that, in any neighborhood of a noncollision singularity, there must be some collision singularity with three or more colliding particles. Therefore fully understanding a collision singularity with three or more particles colliding is essential to the study of noncollision singularities.

Collision singularities have been studied by Wintner [26], Siegal and Moser [20], Saari and Pollard [13]-[17] and others. They obtained some important results concerning the limiting behavior of colliding particles. In 1974, McGehee [7] introduced a remarkable new set of coordinates for the study of triple collisions. As we noticed earlier, U is not defined everywhere, and there are "holes" in the phase space where the vector fields are not defined. Certain orbits of the system reach this singularity in finite time, while others begin at the singularity. Moreover the local behavior of the system near these "holes" can be very complicated. However McGehee's coordinates allow us to read off the behavior of these solutions with relative ease.

McGehee uses "polar coordinates" to blow up the singularity set and to replace it with an invariant boundary called the collision manifold. The dynamical system extends smoothly (after a rescaling of time) over this boundary. So we get a new flow on an augmented phase space. It turns out that this new flow, restricted to the boundary, is extremely simple to understand

and usually is a “gradient-like” Morse–Smale flow. So on the boundary of a complicated dynamical system we find a simple system. And this fact enables us to understand so readily the behavior of solutions near the singularity.

McGehee’s coordinates have been used extensively in the study of triple collisions in the collinear 3-body problem (see [7],[6],[19]), the isosceles 3-body problem (see [3],[4]; [10],[11]), and the anisotropic Kepler problem (see [4]). A great many new results have come out of these works, and some classical results have been re-proved with relative ease as well. The most important feature of McGehee’s coordinates is that collision singularities now correspond to some hyperbolic rest points in the collision manifold, and the orbits that reach collision in finite time now approach these rest points as the new rescaled time approaches infinity. Therefore one can use standard methods in general dynamical systems and find a rich orbit structure by studying the stable and unstable manifolds of these rest points. Erratic and chaotic solutions arise naturally.

Among these new discoveries is a remarkable example of Mather and McGehee [6], which sheds some light on the final answer to Painlevé’s problem of the existence of noncollision singularities. Mather and McGehee constructed an unbounded solution in finite time in the collinear 4-body problem for a Cantor set of initial condition; i.e., they constructed the solutions such that $I \rightarrow \infty$ as $t \rightarrow \sigma < \infty$ for some σ . However, as binary collisions in the collinear 4-body problem are inevitable, their solution contains an infinite number of binary collisions that are extended by elastic bounces. Based on their ideas, Anosov [1] suggested that the noncollision singularity might exist in the neighborhood of the example constructed by Mather and McGehee in the planar 4-body problem, but this approach has not been proved to be successful.

By a different approach and after the original version of this paper (in thesis form) was given, Gerver [5] asserted the existence of a noncollision singularity in a planar $3N$ -body problem, where N is very large.

In the following section we will state our main results and describe the noncollision-singularity solution of the 5-body problem. And we will also give some intuitive ideas as to why the motion should exist. The rest of the paper will be devoted to proving its existence.

1. Main theorems and an outline of their proofs

We consider five point-masses m_1, m_2, \dots, m_5 , moving in a euclidean space \mathbb{R}^3 . Let $m_1 = m_2$ and $m_4 = m_5$. Choose the initial conditions such that m_3 always stays on the z axis, m_1 and m_2 are always symmetric to one another

with respect to the z axis, and m_4 and m_5 are symmetric to one another with respect to the z axis. If we fix the center of masses at the origin, this defines a dynamical system with six degrees of freedom.

This system admits the energy integral and the angular-momentum integral. For our purpose we restrict ourselves to the zero angular-momentum hypersurface, an algebraic variety of dimension 11. Let Ω be any energy hypersurface of this manifold so that Ω is 10 dimensional.

It is well known that binary collisions can be regularized by either the analytic method or Easton's block regularization. Throughout this paper we shall speak of flow as though the orbits were extended through binary collisions. Our first construction for unbounded solutions in finite time involves the possibility of binary collisions. Only later will we prove that, among these unbounded solutions in finite time, there are orbits that do not go through any binary collision. In this way we establish the existence of a noncollision singularity.

The triple collisions between m_1, m_2, m_3 and m_3, m_4, m_5 are of special importance to us. There are several different ways for m_1, m_2 and m_3 to reach a triple collision. Classical results show that as particles approach a collision, they must approach special configurations called central configurations. In setting this special 5-body problem, we find that there are three central configurations for m_1, m_2 and m_3 : One is the collinear configuration with m_3 in the middle of m_1 and m_2 , and the other two configurations are where m_1, m_2 and m_3 form equilateral triangles—one with m_3 below m_1 and m_2 , denoted by E_+ , and the other one with m_3 above m_1 and m_2 , denoted by E_- . Let Σ_1 be the subset of Ω consisting of all the initial conditions such that the corresponding trajectories end in triple collisions of the 1st, 2nd and 3rd particles with the limiting configuration E_+ . We show that Σ_1 is a codimension-2 immersed manifold. Similarly let Σ_4 be the subset of Ω consisting of the initial conditions such that the corresponding trajectories end in triple collisions of the 3rd, 4th and 5th particles, which have a limiting configuration similar to that of E_- . Then Σ_4 is also a codimension-2 immersed submanifold.

The goal of this paper is to prove the following two theorems. We remind readers that the binary collisions are regularized.

THEOREM 1.1. *There exist positive masses $m_1 = m_2, m_3, m_4 = m_5$ such that the following holds: For $x^* \in \Sigma_1$ let t^* be the time when the trajectory starting from x^* ends. There exist choices of x^* so that $\mathbf{q}_4(x^*, t^*) = \mathbf{q}_5(x^*, t^*)$, i.e., for the trajectory starting from x^* , when m_1, m_2, m_3 collide at t^* , then m_4 and m_5 also have a binary collision at t^* . For some 3-dimensional hypersurface Π in Ω crossing Σ_1 at x^* there is an uncountable set Λ of points on Π having the following property: Let $x \in \Lambda$; there exist $t_\infty > 0, t_\infty < \infty$, such that*

the trajectory with the initial condition x is defined for all $0 \leq t < t_\infty < \infty$ (possibly with binary collisions) and satisfies

$$z_1(t) = z_2(t) \rightarrow \infty, \quad z_4(t) = z_5(t) \rightarrow -\infty \quad \text{as } t \rightarrow t_\infty.$$

While the above theorem gives the existence of an unbounded solution in finite time, the following theorem asserts the existence of a noncollision singularity:

THEOREM 1.2. *Let x^* , Π , Λ be that of Theorem 1.1. There exist $x^* \in \Sigma_1$, $\Pi \subset \Omega$ such that the following holds: There is an uncountable subset Λ_0 of Λ such that, for all $x \in \Lambda_0$, $\mathbf{q}_1(t) \neq \mathbf{q}_2(t)$, $\mathbf{q}_4(t) \neq \mathbf{q}_5(t)$ for all $0 \leq t < t_\infty$; i.e., the solutions starting from Λ_0 experience noncollision singularities.*

The proof of the above theorem is long and fairly complicated. Therefore, before giving the formal proofs of above two theorems, we shall outline, in an informal fashion, some of the basic ideas.

As we mentioned earlier, the objective of this paper is to prove the existence of the newtonian motion for the 5-body problem that is unbounded in physical space in finite time. Moreover we must show that this motion exists without the benefit of an accumulation of infinitely many binary collisions. The basic idea behind this dynamical behavior is fairly simple. Consider a 5-body problem consisting of four particles in two pairs, the particles in each pair having the same masses. The four particles form two binaries. Each binary is in a highly elliptical orbit with a plane of motion parallel to the x - y plane. What differs is that one binary is far above the x - y plane with a rotation in one direction, while the other binary is far below the x - y plane rotating in the opposite direction. This difference in rotation permits the total angular momentum of the system to be zero and also permits the two binaries to have arbitrarily small but nonzero angular momentum.

Meanwhile the fifth particle is restricted to the z -axis. The oscillation of this fifth particle actually drives the system and creates this unbounded motion. To see the behavior of this system, imagine the following scenario: Suppose that the oscillating particle passes through the plane of motion defined by a binary just when the two particles in the binary are nearing their closest approach. Because the particles in the binary are almost at their closest point, they also are very close to the fifth particle. Simó [21] has analyzed the motion for such near-collision orbits (see also [3] and [10]) by using McGehee's collision manifold. The interesting point is that, for certain mass ratios (for example, if the fifth particle is significantly heavier than the masses of the binary), the above proximity among the particles imposes a considerable force on the fifth particle that is directed back toward the plane of motion of the binary. This

forces the fifth particle to return through the binary's plane of motion when the binary starts to separate. This separation effect reduces the retaining force on the fifth particle. In return, this permits the fifth particle to move, at a very fast rate, toward the other binary system. In due course, the action-reaction effect of this 3-body system causes the original binary to move further away from the x - y plane.

The motion, then, is obtained by iteration of this scenario. Each time the fifth particle approaches one binary, the timing is such that the close approach of these particles provides the force to accelerate the fifth particle back toward the other binary. The difficulty is verifying that this scenario actually occurs. One interesting conclusion to be drawn is that this behavior occurs with a reasonably massive singleton.

The timing sequence is accomplished with a symbolic dynamics argument. But before this argument can be used, several other elements about the dynamics of interaction need to be established. In particular, if this motion is to become unbounded in finite time, then clearly the acceleration effects on the oscillating particle must become infinitely large. Consequently, the close approaches of each of the binaries must become infinitesimally small. But the only manner in which this can occur is as a by-product of 3-body interactions. Therefore we devote a major portion of this paper to developing a theory to explain the dynamics of this kind of 3-body interaction. This material is given in Sections 2 and 3. Actual triple collisions and the triple-collision manifold are considered in Section 2, while the emphasis in Section 3 is on near-collision orbits.

The infinitesimally small, close approaches of the binaries create another worry. Do these binaries collide? They do not if the angular momentum of the binaries is nonzero for all time. To handle this we took a portion of the problem of avoiding collisions and embedded it into our symbolic dynamics argument. Thus symbolic dynamics needs to include information about the rate of expulsion of the singleton from the binaries and the (rotating and elliptical) status of the binary. Furthermore it must connect the two "3-body" problems to show that the singleton can oscillate between the two binaries. As we discovered, the symbols turn out to be characterized by regions in the phase space. These regions, which are "wedges," are determined by the nearness of different orbits to a particular stable manifold (corresponding to a triple collision) and by their relationship to an unstable manifold. One can envision "wedges" for each close approach of the binaries, because the wedges need to correspond to when there is an appropriate near approach of a particular binary and the oscillating particle. These wedges are introduced in Section 4. In Section 5 the symbolic dynamics argument is carried out and the motion of unbounded solutions in finite time is established.

Finally, in Section 6, we establish the existence of the noncollision singularity.

In the following sections we will study the isosceles 3-body problem and especially the solutions that pass close to a triple collision.

2. The isosceles 3-body problem—triple-collision manifolds

In this section we restrict ourselves to looking at a special type of 3-body problem, the so-called isosceles 3-body problem. This problem concerns the motion of three particles in \mathbb{R}^3 with certain symmetries. Let $m_1 = m_2 = m$, $m_3 = m_3$. Choose the initial conditions so that m_3 remains on the z -axis for all time and so that the pair m_1, m_2 will always be symmetric to each other with respect to the z -axis. This is a mechanical system with three degrees of freedom when the center of mass is fixed at the origin.

Let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ and $\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2, \dot{\mathbf{q}}_3$ be the position vectors and velocity vectors of m_1, m_2, m_3 , respectively. If $\mathbf{q}_1 = (x, y, z) \in \mathbb{R}^3$, $\dot{\mathbf{q}}_1 = (\dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^3$, then by symmetry we have $\mathbf{q}_2 = (-x, -y, z) \in \mathbb{R}^3$ and $\dot{\mathbf{q}}_2 = (-\dot{x}, -\dot{y}, \dot{z}) \in \mathbb{R}^3$. Since the center of mass is at the origin, $2mz + m_3z_3 = 0$ and $2m\dot{z} + m_3\dot{z}_3 = 0$, where $\mathbf{q}_3 = (0, 0, z_3)$ and $\dot{\mathbf{q}}_3 = (0, 0, \dot{z}_3)$. If T and U are the kinetic and self-potential of the system, then

$$(2.1) \quad \begin{aligned} T &= m(\dot{x}^2 + \dot{y}^2) + m(1 + 2\alpha)\dot{z}^2, \\ U &= \frac{1}{2}mm_3 \left[\alpha(x^2 + y^2)^{-1/2} + 4(x^2 + y^2 + (1 + 2\alpha)^2 z^2)^{-1/2} \right], \end{aligned}$$

where $\alpha = m/m_3$ is the mass ratio of m_1 (or m_2) and m_3 .

Let M be the 3×3 diagonal matrix $\text{diag}[2m, 2m, 2m(1 + 2\alpha)]$ and define

$$\xi = M^{1/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \eta = M^{1/2} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

These new variables satisfy Hamilton's equation with the Hamiltonian function

$$H = \frac{1}{2}|\eta|^2 - U(\xi),$$

where

$$U(\xi) = \frac{1}{\sqrt{2}}m^{3/2}m_3 \left[\alpha(\xi_1^2 + \xi_2^2)^{-1/2} + 4(\xi_1^2 + \xi_2^2 + (1 + 2\alpha)\xi_3^2)^{-1/2} \right].$$

And the equations of motion then read:

$$\dot{\xi} = \frac{\partial H}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial H}{\partial \xi}.$$

Our objective in this section is to develop the necessary tools for understanding the motion close to a triple collision. For this it is convenient to introduce McGehee's variables [8]. Let $r = |\xi|$, $\mathbf{s} = r^{-1}\xi$, $\mathbf{z} = r^{1/2}\eta$ and change the time variable t to τ by $dt = r^{3/2}d\tau$. The resulting equations of motion are

$$(2.2) \quad \begin{aligned} r' &= (\mathbf{s} \cdot \mathbf{z})r, \\ \mathbf{s}' &= \mathbf{z} - (\mathbf{s} \cdot \mathbf{z})\mathbf{s}, \\ \mathbf{z}' &= \nabla U(\mathbf{s}) + \frac{1}{2}(\mathbf{s} \cdot \mathbf{z})\mathbf{z}. \end{aligned}$$

Here the “'” denotes the differential with respect to the new time variable τ . By definition, $|\mathbf{s}| = 1$. This suggests that we use spherical coordinates. Let

$$\mathbf{s} = (s_1, s_2, s_3) = (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi).$$

Now the vectors

$$(2.3) \quad \begin{aligned} \mathbf{u}_1 &= \mathbf{s}, \\ \mathbf{u}_2 &= \frac{\partial \mathbf{s}}{\partial \theta} = (-\sin \theta \cos \phi, \cos \theta \cos \phi, 0), \\ \mathbf{u}_3 &= \frac{\partial \mathbf{s}}{\partial \phi} = (-\cos \theta \sin \phi, -\sin \theta \sin \phi, \cos \phi), \end{aligned}$$

form an orthogonal (but not orthonormal) basis for \mathbb{R}^3 when $\phi \neq \pm\pi/2$. Therefore we may decompose \mathbf{z} on this basis. Let $\mathbf{z} = v\mathbf{u}_1 + w_2\mathbf{u}_2 + w_3\mathbf{u}_3$ (where $v = \mathbf{z} \cdot \mathbf{u}_1 = \mathbf{z} \cdot \mathbf{s}$, $w_2 = (\mathbf{z} \cdot \mathbf{u}_2)/(\mathbf{u}_2 \cdot \mathbf{u}_2)$, $w_3 = \mathbf{z} \cdot \mathbf{u}_3$) and use (2.2) to find the equations for the variables $(r, \theta, \phi, v, w_2, w_3)$. In this manner we obtain:

$$(2.4) \quad \begin{aligned} r' &= vr, \\ \theta' &= w_2, \\ \phi' &= w_3, \\ v' &= \frac{1}{2}v^2 + w_2^2 \cos^2 \phi + w_3^2 - U(\phi), \\ w_2' &= -\frac{1}{2}vw_2 + 2w_2w_3 \tan \phi, \\ w_3' &= U'(\phi) - \frac{1}{2}vw_3 - w_2^2 \cos^2 \phi \tan \phi, \end{aligned}$$

where $U(\phi) = \frac{1}{\sqrt{2}}m^{3/2}m_3[\alpha \sec \phi + 4(1 + 2\alpha \sin^2 \phi)^{-1/2}]$.

Notice that the potential energy $U(\phi)$ is independent of θ and that the right-hand side of (2.4) does not contain θ . This reflects both the invariance of U , with respect to the rotation of a configuration, and the resulting conservation of angular momentum. The equation for w_2 does not involve U , so one can easily verify that $c = r^{1/2}w_2 \cos^2 \phi$ is a constant of motion. In fact c is the total angular momentum of the system. Therefore we can use this

angular-momentum integral to eliminate w_2 from the equation. Also we can safely ignore the equation for θ , because the vector field does not involve θ . (This is equivalent to projecting the phase space onto the r, u, v, w_2, w_3 space.) Finally we end up with a system with two degrees of freedom on each angular-momentum surface. But in doing so, because c has a troublesome factor $r^{1/2}$, we find that the angular-momentum zero surface is a variety that consists of two components designated as $r = 0$ and $w_2 = 0$ (when $c \neq 0$, this surface is a manifold). This variety complicates our analysis of the motion for small values of c . Therefore we retain the equation for w_2 , but ignore θ . What remains is a system of differential equations of order five (r, ϕ, v, w_2, w_3). Again we have an energy integral:

$$\frac{1}{2}(v^2 + w_3^2 + w_2^2 \cos^2 \phi) - U(\phi) = rh.$$

In our new system there is a singularity at $\phi = \pm\pi/2$ due to binary collisions of m_1 and m_2 . This singularity can be removed by a change of variables. Let $w = w_3 \cos \phi$, $u = w_2 \cos^2 \phi$ and multiply the resulting vector field by $\cos \phi$. Still, using the $'$ to denote differentiation with respect to this new independent variable, we have

$$\begin{aligned} r' &= vr \cos \phi, \\ \phi' &= w, \\ v' &= U(\phi) \cos \phi - \frac{1}{2}v^2 \cos \phi + 2rh \cos \phi, \\ w' &= U'(\phi) \cos^2 \phi - \frac{1}{2}vw \cos \phi \\ &\quad - (2U(\phi) + 2rh - v^2) \sin \phi \cos \phi, \\ u' &= -\frac{1}{2}vu \cos \phi, \end{aligned} \tag{2.5}$$

with the energy relation

$$\frac{1}{2}(v^2 \cos^2 \phi + w^2 + u^2) - U(\phi) \cos^2 \phi = rh \cos^2 \phi.$$

Now the vector field (2.5) is analytic everywhere, since the functions $U(\phi) \cos \phi$ and $U'(\phi) \cos^2 \phi$ are analytic for all $\phi \in [-\pi/2, \pi/2]$.

From the equation for r : $r' = rv \cos \phi$ we see that $r = 0$ is an invariant manifold for the flow. Notice that the flow has been extended analytically to $r = 0$ and, replacing the triple-collision singularity, that it is an analytic manifold. The orbits that end up in a triple collision now tend to this manifold as τ tends to infinity. The orbits ejected from a triple collision now tend to this triple-collision manifold as τ tends to negative infinity. The motion on this manifold is, of course, fictitious, but the orbits that pass close to

this manifold must mimic the fictitious orbits on this manifold. Therefore understanding the flow on this manifold provides considerable information about near-collision orbits. Since the unbounded solution in finite time must pass repeatedly and very closely to the triple collision in our problem, we find it is of great importance to study the flow on this triple collision for our construction of a noncollision singularity.

Let M denote this $r = 0$ manifold; we call it the “triple-collision manifold for a nonplanar, isosceles, 3-body problem.” In the rest of this section we concentrate on the flow on M , especially on the rest points and their stable and unstable manifolds.

For $r = 0$ equations (2.5) become

$$(2.6) \quad \begin{aligned} \phi' &= w, \\ v' &= U(\phi) \cos \phi - \frac{1}{2}v^2 \cos \phi, \\ w' &= U'(\phi) \cos^2 \phi - \frac{1}{2}vw \cos \phi - (2U(\phi) - v^2) \sin \phi \cos \phi, \\ u' &= -\frac{1}{2}vu \cos \phi, \end{aligned}$$

and the energy relation is then

$$\frac{1}{2}(v^2 \cos^2 \phi + w^2 + u^2) - U(\phi) \cos^2 \phi = 0.$$

Therefore, for any energy h , M is determined by this same equation, and the flow on M is determined by (2.6).

Topologically M is a 3-sphere minus four points.

From equations (2.6), when $u = 0$, we have $u' = 0$; so it follows that $u = 0$ is an invariant submanifold of M . Denote this manifold by M_0 . (Recall that u is the variable measuring rotation in the x - y plane.) Now M_0 is given by the equation

$$\frac{1}{2}(v^2 \cos^2 \phi + w^2) - U(\phi) \cos^2 \phi = 0.$$

The flow on M_0 is defined by

$$(2.7) \quad \begin{aligned} \phi' &= w, \\ v' &= U(\phi) \cos \phi - \frac{1}{2}v^2 \cos \phi, \\ w' &= U'(\phi) \cos^2 \phi - \frac{1}{2}vw \cos \phi - (2U(\phi) - v^2) \sin \phi \cos \phi. \end{aligned}$$

In fact M_0 is the triple-collision manifold for the planar, isosceles, 3-body problem for a plane containing the z -axis. Usually one obtains this manifold

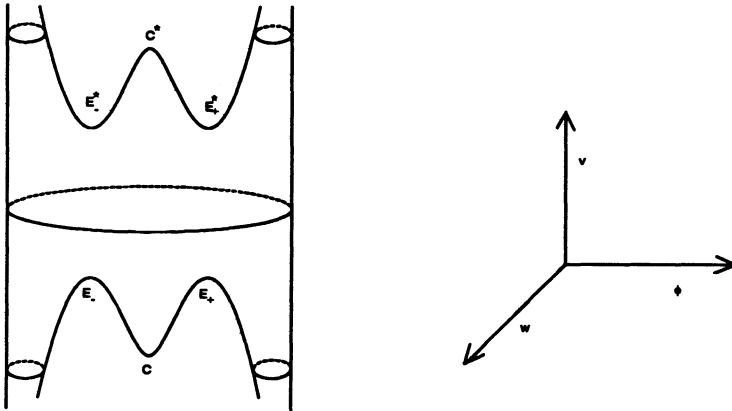


FIGURE 1. The triple-collision manifold

by first setting the angular momentum to zero and then blowing up the triple-collision singularity. The flow on this manifold has been explored extensively by Devaney [3],[4], Moeckel [10],[11] and others.

Topologically M_0 is a 2-sphere minus four points. Figure 1 represents its graph. There are six rest points for the flow on M_0 . The flow is gradient-like with respect to v ; i.e., v is strictly increasing on every orbit of M_0 except on these six rest points. (Recall that $v' \geq 0$ from equations (2.7) and the energy relation.)

Three central configurations correspond to the isosceles 3-body problem: two equilateral triangles (one with z_3 positive and one with z_3 negative) and one collinear central configuration with m_3 in the middle of m_1 and m_2 . Here E_+^* , E_+ , E_-^* , E_- correspond to the two equilateral central configurations and C , C^* correspond to the collinear central configuration. The stable manifolds of E_+ , E_- and C are those orbits ending up in a triple collision, and the unstable manifolds of E_+^* , E_-^* and C^* are those orbits beginning with a triple collision.

On M_0 , E_+ , E_- , E_+^* and E_-^* there are saddle points; C is a source and C^* is a sink. In our problem we are particularly interested in the stable and unstable manifolds for E_+ and E_- because of the important roles they play later on.

Note that there are three major possibilities for the unstable manifolds of E_+ and E_- . Since v must increase along these orbits, a typical branch of the unstable manifold may run up the left or right "arm" of the triple-collision manifold M_0 , or else it may end up in the sink. A degenerate possibility is that the unstable manifold might match up exactly with the stable manifold of one of the saddles.

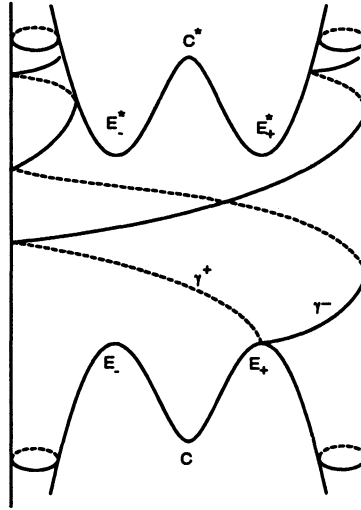


FIGURE 2. The unstable manifold of E_+ in M_0

Let γ^+ be the branch of the unstable manifold of E_+ in M_0 with a positive w -coordinate near E_+ and let γ^- be the other branch of the unstable manifold of E_+ . Figure 2 shows the typical behavior of γ^+ and γ^- . We will call a set of masses *allowable* if γ^+ and γ^- run up the arms of the manifold with $\theta = \pi/2$ and $\theta = -\pi/2$, respectively. There is a large open set of masses which is allowable (see Devaney [3], Simó [21]).

One last observation about M_0 is that, by symmetry, the unstable manifold for E_- and the stable manifolds for E_+^* , E_-^* are completely determined by γ^+ and γ^- .

From now on assume that all sets of masses we discuss are allowable.

Let us now turn our attention to M , which is a 3-dimensional manifold embedded in a 4-dimensional space \mathbb{R}^4 with M_0 as an invariant submanifold. It is difficult to depict M in \mathbb{R}^3 in a manner one can visualize. However, since we are only interested in the part of M that is close to M_0 , we can divide M into two symmetric parts, one with $u \geq 0$ and the other with $u \leq 0$; both parts are invariant under the flow and M_0 is their common boundary. Now regard each part as the solid in \mathbb{R}^3 enclosed by M_0 . On this solid, u is determined by the energy relation:

$$u = \pm \sqrt{2U(\phi) \cos^2 \phi - (v^2 \cos^2 \phi + w^2)}.$$

Next consider the flow on M . We easily see that u must be zero at the rest points. Therefore the original six rest points in M_0 remain the only rest points of M . Again the flow on M is gradient-like with respect to v . To understand the local structure of these rest points in M we linearize the vector fields at

the rest points. Let $(\phi, v, u, w) = (\phi^*, v^*, 0, 0)$ be the values at each rest point. Then by equations (2.6),

$$\begin{pmatrix} \delta\phi' \\ \delta v' \\ \delta u' \\ \delta w' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}v^* \cos \phi^* & 0 \\ U''(\phi) \cos^2 \phi^* & 0 & 0 & -\frac{1}{2}v^* \cos \phi^* \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta v \\ \delta u \\ \delta w \end{pmatrix},$$

where $(\delta\phi, \delta v, \delta u, \delta w) \in TM|_{(\phi^*, v^*, 0, 0)}$, and the tangent space at the rest points is spanned by $\delta\phi$, δu and δw . Note that $\delta v = 0$ on $TM|_{(\phi^*, v^*, 0, 0)}$.

The eigenvalues of this linearized equation at each rest point are easy to find. At C and C^* , $U''(\phi) < 0$, again we see that C is a source and C^* is a sink. As for E_+ , E_- , E_+^* and E_-^* , they are still saddle points. For E_+ , besides having one stable and one unstable direction in M_0 , we have another repelling direction in M . This new unstable direction is in the direction of δu . From the matrix it follows that the new eigenvalue is weaker than the original one in M_0 . The stable-manifold theorem leads to E_+ having a 1-dimensional stable manifold and a 2-dimensional unstable manifold in M .

By symmetry, the local structure of E_- , E_+^* , E_-^* follows from that of E_+ . For instance, the local structure of E_- and E_+^* exactly reflects that of E_+ through the v -axis and the $v = 0$ -plane, respectively.

Now let $\text{Un}(E_+)$ be the 2-dimensional unstable manifold of E_+ . Then γ^+ and γ^- are on this manifold. Since γ^+ and γ^- end up in the two arms of M_0 with $\theta = -\pi/2$ and $\theta = \pi/2$, respectively, some small neighborhood of γ^+ and γ^- will also end up in the two arms.

For any orbit that dies in one of the two arms, as $\tau \rightarrow \infty$, we have $v \rightarrow \infty$, $u \rightarrow 0$, $w \rightarrow 0$ and $\phi \rightarrow \pi/2$. Physically this corresponds to where m_3 separates from the binary m_1 and m_2 ; this leads to two fictitious 2-body problems. One is the binary m_1 and m_2 , and the other is m_3 relative to the center of the masses of m_1 and m_2 . The variables v , u , w and ϕ define only the second 2-body problem in the limiting position. Therefore some new variables are needed to define the limiting position for the first binary, i.e., m_1 and m_2 . There is an obvious choice: since the limiting position is an elliptic orbit, we may use the major axis of the ellipse and the eccentricity of the ellipse. The ignored θ variable reflects the rotational symmetry about the z -axis of the system; so we only need to define a variable that gives the eccentricity of the elliptical orbit. Toward this end let $w_{12} = |h_{12}c_{12}|c_{12}$, where h_{12} and c_{12} are the energy and the angular momentum of the binary m_1 and m_2 , respectively. It may seem that this is not well defined on M , where $r = 0$, as h_{12} is not well defined for $r = 0$. However w_{12} can be extended to $r = 0$; i.e., w_{12} is a well-defined function on M . This is because w_{12} is independent of r even though h_{12} and c_{12} both involve r .

In fact,

$$w_{12} = \left| \frac{1}{2}(v \cos \phi + w \tan \phi)^2 + \frac{1}{2}u \sec^2 \phi - m^{3/2} \sec \phi \right| |u|u.$$

There still remains a singularity at $\phi = \pm\pi/2$ due to binary collision. But this singularity can be removed by the use of the energy integral in standard fashion.

Since $u = 0$ is an invariant submanifold in M , we observe that if u is ever positive, then it remains positive for all time, but that u approaches zero as $\tau \rightarrow \infty$. The following lemma shows that this does not happen to w_{12} :

LEMMA 2.1. *Consider any orbit γ in M such that γ runs up one of the two arms of M . If $w_{12}(\gamma(\tau_0)) \neq 0$ for some τ_0 , then $w_{12}(\gamma(\tau)) \neq 0$ for all τ . Furthermore the limit $\lim w_{12}(\gamma(\tau))$, as $\tau \rightarrow \infty$, exists and*

$$w_{12}^\infty(\gamma) = \lim_{\tau \rightarrow \infty} w_{12}(\gamma(\tau)) \neq 0.$$

Proof. The direct way to prove this lemma is to start from the equation and to estimate the changes of $w_{12}(\tau)$ along the orbit for large values of τ . We use a different approach, however, based on one of the features of McGehee's transformation; i.e., namely the r variable can be separated from the rest of the coordinates. (We will explore this relationship further in the next section.)

The vector field on M is defined by equations (2.5) when we set $r = 0$ and ignore the first equation. However, if $h = 0$ and we ignore the equation for r , notice that we have exactly the same system of differential equations. To be precise let ϕ_0, w_0, v_0 and u_0 be the initial values with $u_0 \neq 0$ that satisfy

$$\frac{1}{2}(v^2 \cos^2 \phi + w^2 + u^2) - U(\phi) \cos^2 \phi = 0.$$

Then vector field (2.6) determines a unique orbit on M . Now, by adding an arbitrary initial value $r = r_0$ and solving equations (2.5), we have the same solution for ϕ, w, v and u . Thus the solution given by (2.6) is an orbit for the isosceles 3-body problem with $h = 0$ and $c \neq 0$ ($u_0 \neq 0$ and $c = r_0^{1/2}u = r_0^{1/2}u_0$). This means that, for fixed $r_0 \neq 0$, there is a one-to-one correspondence between orbits for $h = 0$ and orbits on M . Any orbit with $h = 0$ that ends up in one of the arms of M represents an orbit of hyperbolic-elliptic type, where the eccentricity of the limiting elliptic orbit of the binary m_1 and m_2 is strictly less than 1 (i.e., $w_{12}^\infty \neq 0$) should $c \neq 0$. This is because $c_{12} = c \neq 0$ remains fixed, and from [2] it is known that h_{12} approaches a negative constant. Hence $w_{12} = |h_{12}c_{12}|c_{12}$ has a nonzero limit. Therefore the corresponding orbit in M has the same property. This proves the lemma. \square

We remark that the limiting value of h_{12} above continuously depends on the initial value. Therefore $w_{12}^\infty(x)$ is a continuous function.

We also emphasize that, although the flow on M does not admit a physical interpretation, the one-to-one correspondence between the flow on M and that of the $h = 0$ -energy hypersurface enables us to visualize the flow on M . Conversely the information obtained from the collision manifold enriches our understanding of the motion on the zero-energy hypersurface.

Now we return to the unstable manifold of E_+ , $\text{Un}(E_+)$. First, we need to introduce a local coordinate for $\text{Un}(E_+)$ near γ^+ (or γ^-). The intersection of $\text{Un}(E_+)$ with the $v = 0$ -plane is a smooth closed curve, and each orbit intersects with the $v = 0$ -plane exactly once. (This is because $v' > 0$.) Let $\psi \in [0, 2\pi)$ be a parameter on this closed curve with $\psi(\gamma^-) = 0$ and $\psi(\gamma^+) = \pi$. We use ψ to identify each orbit on $\text{Un}(E_+)$.

Observe that, as $\tau \rightarrow \infty$, the orbit near γ^+ (or γ^-) eventually must run up one of the arms. By Lemma 2.1, the limit of w_{12} for the points on $\text{Un}(E_+)$ near γ^+ (or γ^-) will have a nonzero value and, furthermore, from the remark at the end of the proof of Lemma 2.1, w_{12}^∞ is a continuous function. For ψ away from 0 or π , the value of the function w_{12}^∞ is not clear. However there is an open set of ψ such that its corresponding orbits die in the sink C^* , for which w_{12}^∞ equals zero.

We conclude this section by pointing out that, for any set of allowable masses, there exist a $w_{12}^* > 0$, and two intervals of ψ , $[-\psi_1^*, \psi_1^*]$ and $[\pi - \psi_2^*, \pi + \psi_2^*]$ with $\psi_1^* > 0$, $\psi_2^* > 0$, such that $|w_{12}^\infty|$ assumes the value w_{12}^* at both ends of both intervals and $w_{12}^\infty(\psi) \leq w_{12}^*$ for all $\psi \in [-\psi_1^*, \psi_1^*] \cap [\pi - \psi_2^*, \pi + \psi_2^*]$.

3. The isosceles 3-body problem—near-collision orbits

The purpose of studying the triple-collision manifold is to understand the dynamical behavior of those solutions that pass close to triple collisions. In doing so, in the last section, we exploited the one-to-one correspondence between the motion on M and the motion of the $h = 0$ manifold. Here we continue to use this correspondence to obtain sharper results and to study some general solutions.

One very nice and important feature of McGehee's transformation, among others, is that the variable r , the scale of the system, can be separated from the rest of the variables. This provides a simple way to reduce by one the dimension of the total system via the energy integral. By rewriting the last four equations of (2.5), where the energy relationship was substituted for the

terms r and h , we obtain:

$$\begin{aligned}
 \phi' &= w, \\
 v' &= U(\phi) \cos \phi - \frac{1}{2}v^2 \cos \phi \\
 &\quad + \frac{1}{\cos \phi}(v^2 \cos^2 \phi + w^2 + u^2 - 2U(\phi) \cos^2 \phi), \\
 w' &= U'(\phi) \cos^2 \phi - \frac{1}{2}vw \cos \phi - (2U(\phi) - v^2) \sin \phi \cos \phi, \\
 u' &= -\frac{1}{2}vu \cos \phi.
 \end{aligned}
 \tag{3.1}$$

These equations are defined on $\mathbb{R}^3 \times (-\pi/2, \pi/2)$ and do not involve either r or h . The orbits of this vector field are the projections of the orbits of our original system and vice versa. In other words, any orbit of our original system can be obtained simply by the use of the energy integral, if $h \neq 0$, or by the integration of the first equation of (2.5),

$$r' = vr \cos \phi,$$

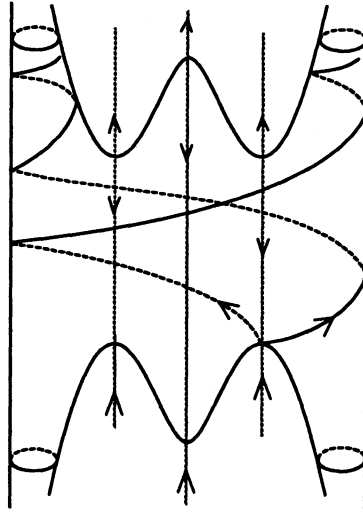
if $h = 0$.

One may see that, again, there is a singularity for the vector field (3.1) at $\phi = \pm\pi/2$, due to the binary collision of m_1 and m_2 . This is not a singularity for equations (2.5), because they were regularized by the energy relation. Note that, with the projection, it re-appears. However, since this is not an essential singularity, we can remove it by using Easton’s method of block regularization. We do this so that, near the singular set, the orbits of the new vector field are still the projections of the orbits of the vector field defined by (2.5). Once this is done, denote this new regularized system by N , where, for simplicity of notation, N is also used to denote its underlying manifold. Equations (3.1) still define this vector field with the simple assumption that, at $\phi = \pm\pi/2$, the vector field is used in the sense of a regularized system. We remark that the entire procedure of regularization depends only on equations (3.1). It does not depend on any specific value of h or r .

From the original energy relation we have

$$\frac{1}{2}(v^2 \cos^2 \phi + w^2 + u^2) - U(\phi) \cos^2 \phi = 0
 \tag{3.2}$$

being an invariant submanifold in N . Because this is the equation for the triple-collision manifold M , we still denote this manifold as M . However it is important to emphasize that, in the present setting, M is *not only* the triple-collision manifold, but also the $h = 0$ -invariant manifold, and that the flow on this manifold corresponds to the projections of the original system. Thus, with this dual identification, the pre-image of this new M contains not only

FIGURE 3. Flow on N_0

the triple-collision manifold M with $r = 0$, but also orbits with $h = 0$ and $r > 0$. Identifying the invariant manifold of $r = 0$ with that of $h = 0$ is due to the fact that only the product of r and h appears in the energy relation.

The flow on M has been discussed in the last section. To see the flow on N , first let us define a submanifold N_0 by letting u be zero on N . This invariant submanifold can be identified with solutions that do not rotate about the z -axis ($u = 0$). Here N_0 is a 3-dimensional manifold, and we may think of N_0 as $\mathbb{R}^2 \times (-\pi/2, \pi/2)$ with two lines attached to it, where the two lines correspond to the binary collisions. The flow on N_0 is shown in Figure 3.

Let $M_0 = N_0 \cap M$. Then M_0 is an invariant submanifold of N_0 and, of course, the flow on M_0 is interpreted differently from that of the triple-collision manifold M_0 of the last section. Here M_0 divides the space into two disjoint segments. "Inside" of M_0 , i.e., the set given by

$$(3.3) \quad \frac{1}{2}(c^2 \cos^2 \phi + w^2) - U(\phi) \cos^2 \phi < 0,$$

corresponds to the projection of the original system with $h < 0$; and similarly the "outside" of M_0 is identified with the set of orbits that are the projections of the original system with $h > 0$. Both the flows outside and on M_0 are gradient-like with respect to v . However the flow inside M_0 does not possess this nice property; this corresponds to the well-known complications in the study of the 3-body problem with $h < 0$.

There are several straight-line orbits in N_0 —the lines joining $E_+^* E_+$, $E_-^* E_-$ and $C^* C$ —that correspond to homothetic solutions. In terms of [26], a

homothetic solution is a solution that begins with and ends at a total collision and whose configuration stays at a central configuration.

Returning to the analysis of N , we encounter the problem that the energy integral and angular-momentum integrals are not well defined, because both integrals involve r . To resolve this difficulty we combine these two constants of motion to obtain an integral for N . Let $e = |hc|c$; i.e.,

$$(3.4) \quad e = u \left| u \frac{\frac{1}{2}(v^2 \cos^2 \phi + w^2 + u^2) - U(\phi) \cos^2 \phi}{\cos^2 \phi} \right|.$$

The value e is independent of r . Being well defined on N_0 and the product of two integrals, it is a constant of motion for equations (3.1).

It is interesting to note that, besides the six rest points of M_0 , there is a new pair of rest points for N . These two rest points correspond to where m_3 stays at the origin and m_1 and m_2 move in a circular orbit with different rotational directions.

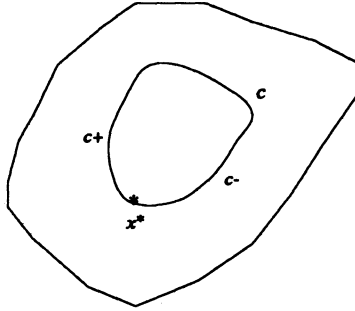
Now consider the rest point E_+ in N (and similarly for E_- , E_+^* , E_-^*). One checks easily that E_+ is a saddle point with a 2-dimensional stable manifold and a 2-dimensional unstable manifold. Both of the unstable directions are in M , as shown in the last section. The 2-dimensional stable directions are in N_0 , and they are shown in Figure 3. The stable and unstable manifolds of E_+ lie in the invariant variety defined by $e = 0$.

We now are prepared to discuss the near-collision orbit. Let $x^* \in \text{St}(E_+)$. That is, this is an orbit that ends up in a triple collision. And $\text{St}(E_+)$, the stable manifold of E_+ , is a codimension-2 manifold. Consider a 2-dimensional cross section Γ , which is transverse to $\text{St}(E_+)$ at x^* .

Notice that the hyperplane $u = 0$ intersects with Γ in a line nearby x^* . This line corresponds to the coplanar problem, and x^* must lie in this line.

Now think of Γ as a set of initial conditions and consider the orbits starting from Γ . In studying these orbits, we use the fact that an orbit starting sufficiently close to the stable manifold of a rest point will follow the unstable manifold of that point arbitrarily far. Therefore, on Γ , an orbit starting close to x^* will follow one of the orbits on the unstable manifold of E_+ , $\text{Un}(E_+)$. Of course, which specific orbit on the unstable manifold will be shadowed depends on the position of the starting point relative to x . In particular the orbit starting from the line $u = 0$ on one side of x^* will closely follow γ^+ ; if it starts on the other side of x^* , it will closely follow γ^- .

Let us trace the local structure of E back along the x^* -orbit to Γ . Corresponding to each orbit in $\text{Un}(E_+)$, we have a curve in Γ ending at x^* such that, for an initial condition along this curve that is close to x , the orbit approaches the corresponding orbit in $\text{Un}(E_+)$.

FIGURE 4. The wedge formed by c^+ and c^-

To understand these orbits we need the variable w_{12} , which was defined on the triple-collision manifold M (see §2). The definition of w_{12} extends to N , and the continuity of w_{12} and w_{12}^∞ is easily established.

We are especially interested in those curves on Γ that correspond to the orbits of $\text{Un}(E_+)$ with $\psi = \pm\psi_1^*$, where the ψ_1^* are defined at the end of the last section. As x approaches x^* along these curves, an x -orbit (i.e., the orbit that starts with x) will approach the corresponding orbits of $\text{Un}(E_+)$ with $\psi = \pm\psi_1^*$. Therefore $w_{12}^\infty(x)$ will approach $\pm w_{12}^*$ as x approaches x^* along these curves.

Recall for an orbit on the $u = 0$ submanifold that $w_{12}^\infty(x) = 0$. This is because the system is coplanar when $u = 0$, and thus $w_{12} \equiv 0$.

Let w_{12}^+ be a positive number such that $w_{12}^+ < w_{12}^*$. Using the above arguments, we see that there are two curves c^+, c^- in Γ such that

(1) each curve starts from x^* , and $u > 0$ on c^+ , $u < 0$ on c^- ; and

(2) for all $x \in c^+$, $w_{12}^\infty(x) = w_{12}^+$, and for all $x \in c^-$, $w_{12}^\infty(x) = -w_{12}^+$. (See Figure 4 above.)

We remark that, by the above construction of the curves, for x in the region formed by c^+, c^- and c , where c is a curve close to x^* connecting c^+ and c^- , the orbit of x will tend toward the arm of M with $\phi = \pi/2$.

The wedge we constructed above can be very small. To study more closely the orbits that start from the wedge, we introduce a cross section to the manifold N .

Let v^+ be a large positive number, $v^+ \gg 0$. Consider the section on N with $v = v^+$; this section is 3-dimensional. We require that $v^+ \gg v(C^*)$, where C^* is the rest point of N with a collinear central configuration, and $v(C^*) > 0$. Recall that the flow on N_0 is gradient-like and that every orbit of N_0 meets this section $v = v^+$ exactly once, except those which end in C^* and other rest points. Therefore γ^+ intersects with this section $v = v^+$ exactly once, and the intersection is transversal. Recall that γ^+ is the orbit of the unstable manifold of E_+ , which is in N_0 and ends up in the arm of N_0 with

$\phi = \pi/2$. Since $\text{Un}(E_+)$ is a 2-dimensional manifold in N , the orbits of $\text{Un}(E_+)$ nearby γ^+ also have to intersect the section $v = v^+$. Thus the intersection of $v = v^+$ with $\text{Un}(E_+)$ locally nearby γ^+ is a smooth curve, and the intersection is transversal.

From the energy integral we have

$$rh = \frac{1}{2} \left(v^2 + \frac{w^2}{\cos^2 \phi} + u^2 \right) - U(\phi).$$

The left side of the above equation has a factor of r , so it is no longer a valid integral in N . However the right side is a well-defined function in N (recall that N is the manifold after regularization). We denote this function by g ; i.e., let

$$g = \frac{1}{2} \left(v^2 + \frac{w^2}{\cos^2 \phi} + u^2 \right) - U(\phi).$$

We must emphasize that g is a continuous function defined on N and that the equation $g = 0$ defines an invariant set, which is exactly N_0 . Here N_0 separates the manifold N into two pieces: one with $g > 0$ corresponds to the system with positive energy and the other one with $g < 0$ corresponds to the system with negative energy. When the energy of the system h is given and is nonzero, the value of g can be used to obtain r . This fact will be used later to get some estimates on the values of r .

Let $T_0 \subset \{v = v^+\}$ be a small compact neighborhood of the intersection of γ^+ with the section $v = v^+$. The intersection of $\text{Un}(E_+)$ with T_0 is a small smooth curve if T_0 is small. By choosing $w^+ > 0$ small, where w^+ is the number given at the end of the last section, we may assume that T_0 is cylinder shaped such that, on top of T_0 , designated as S^+ , we have $w_{12}^\infty = w^+$ and, on the bottom of T_0 , denoted by S^- , we have $w_{12}^\infty = -w^+$ and $|w_{12}^\infty| \leq w^+$ for all points in T_0 . Also, for fixed $g_0 > 0$ small, we may assume, for all points of T_0 , that $|g| < g_0$. The following lemma can be easily proved, and we omit the proof.

LEMMA 3.1. *Let T_0 be the solid cylinder, defined above. There exists $M > 0$ such that whenever $v(\tau) \geq M$, then $|w_{12}(x, \tau)| \leq \frac{4}{3}w^+$ for all $x \in T_0$ and $\frac{2}{3}w^+ \leq |w_{12}(x, \tau)| \leq \frac{4}{3}w^+$ for all $x \in S^+ \cup S^-$.*

From now on we assume that, in the definition of T_0 , we have chosen $v^+ > M$. Hence the following is always true:

$$\begin{aligned} |w_{12}(x, \tau)| &\leq \frac{4}{3}w^+, & \text{for all } x \in T_0 \text{ and } \tau \geq 0; \\ \frac{2}{3}w^+ &\leq |w_{12}(x, \tau)| \leq \frac{4}{3}w^+, & \text{for all } x \in S^+ \cup S^- \text{ and } \tau \geq 0. \end{aligned}$$

From the definition of $g(x)$, for any fixed $\epsilon > 0$ and fixed g_0 , observe that v^+ can be made large enough such that, for all $x \in T_0$ and $\tau \geq 0$, we have $0 \leq \pi/2 - \phi(\tau) \leq \epsilon$ and $\phi(\tau) \rightarrow \pi/2$, as $\tau \rightarrow \infty$. This is possible, because for the trajectory starting from T_0 , the corresponding solutions are of the elliptic-hyperbolic type. Thus for the orbits starting from T_0 , we know that m_3 is far away from m_1 and m_2 for all the time that $\tau \geq 0$. Therefore, for all $x \in T_0$, the 3-body system can be well approximated by a couple of 2-body problems: one with m_1 and m_2 and the other with m_3 and the center of mass of m_1 and m_2 .

We now return to the surface Γ . With the aid of the solid cylinder T_0 , the wedge in Figure 4 now can be better understood. Let $x^* \in \text{St}(E_+) \cap \Gamma$. Then the orbits starting close to x^* will follow the orbits on the unstable manifold of E_+ ; in particular some orbits will follow γ^+ and its nearby unstable manifold of E^+ . Therefore the image of Γ , $\{\Gamma(\tau) \mid \tau > 0\}$, under the flow will intersect with T_0 . And, since both $S^+ \cap \text{Un}(E_+)$ and $S^- \cap \text{Un}(E_+)$ are nonempty sets, we have nonempty intersections $\Gamma(\tau) \cap S^+$ and $\Gamma(\tau) \cap S^-$. The curves c^+ and c^- in Figure 4 are exactly the pre-images on Γ of those intersections.

In the next section we will consider the 5-body problem and construct unbounded solutions in finite time, which pass through the solid cylinder T_0 infinitely many times.

4. The 5-body problem

In this section we consider a special 5-body problem. For this problem we will construct a noncollision singularity or, as the first step, an unbounded solution in finite time. The main result of this section is Theorem 4.4.

Let m_1, m_2, \dots, m_5 be five point-masses moving in a euclidean space \mathbb{R}^3 and let $\mathbf{q}_i, \dot{\mathbf{q}}_i \in \mathbb{R}^3$ be the positions and velocity vectors for $m_i, i = 1, 2, \dots, 5$. Further let $m_1 = m_2$ and $m_4 = m_5$. Then choose an initial condition such that the following symmetries are preserved under the subsequent motion:

$$\begin{aligned}
 \mathbf{q}_1 &= (x_1, y_1, z_1), \\
 \mathbf{q}_2 &= (-x_1, -y_1, z_1), \\
 \mathbf{q}_3 &= (0, 0, z_3), \\
 \mathbf{q}_4 &= (x_4, y_4, z_4), \\
 \mathbf{q}_5 &= (-x_4, -y_4, z_4).
 \end{aligned}
 \tag{4.1}$$

In other words, for all time, m_5 will be on the z -axis. Both particles m_1 and m_4 are, respectively, symmetric to m_2 and m_5 with respect to the z -axis (see Figure 5). Fix the center of the masses at the origin, and then the

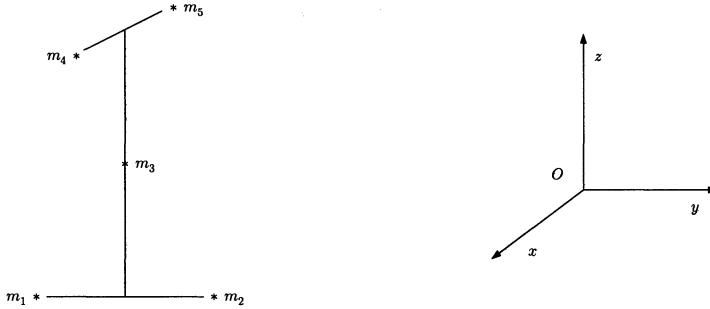


FIGURE 5. The 5-body problem

following equation holds:

$$(4.2) \quad 2m_1z_1 + 2m_4z_4 + m_3z_3 = 0.$$

The resulting system has six degrees of freedom and it is uniquely determined by $\mathbf{q}_1 = (x_1, y_1, z_1)$, $\mathbf{q}_4 = (x_4, y_4, z_4)$ and their derivatives.

Let T and U be the kinetic and potential energy, respectively. With these variables the functions are:

$$\begin{aligned}
 T &= m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + m_4(\dot{x}_4^2 + \dot{y}_4^2 + \dot{z}_4^2) \\
 &\quad + \frac{1}{2}m_3 \left(\frac{2m_1\dot{z}_1}{m_3} + \frac{2m_4\dot{z}_4}{m_3} \right)^2, \\
 U &= \frac{m_1^2}{2(x_1^2 + y_1^2)^{\frac{1}{2}}} + \frac{m_4^2}{2(x_4^2 + y_4^2)^{\frac{1}{2}}} \\
 (4.3) \quad &+ \frac{2m_1m_4}{[(x_1 - x_4)^2 + (y_1 - y_4)^2 + (z_1 - z_4)^2]^{\frac{1}{2}}} \\
 &+ \frac{2m_1m_4}{[(x_1 + x_4)^2 + (y_1 + y_4)^2 + (z_1 - z_4)^2]^{\frac{1}{2}}} + \frac{2m_1m_3}{[x_1^2 + y_1^2 + (z_1 - z_3)^2]^{\frac{1}{2}}} \\
 &+ \frac{2m_4m_3}{[x_4^2 + y_4^2 + (z_4 - z_3)^2]^{\frac{1}{2}}}.
 \end{aligned}$$

Note that z_3 appears in the equation for U . However z_3 is a function of z_1 and z_4 given by equation (4.2). The Hamiltonian function for this system uses the variables

$$\begin{aligned}
 p_{x_1} &= 2m_1\dot{x}_1, \quad p_{x_4} = 2m_4\dot{x}_4, \quad p_{y_1} = 2m_1\dot{y}_1, \quad p_{y_4} = 2m_4\dot{y}_4, \\
 p_{z_1} &= 2m_1\dot{z}_1 + 2m_1 \left(\frac{2m_1\dot{z}_1}{m_3} + \frac{2m_4\dot{z}_4}{m_3} \right), \\
 p_{z_4} &= 2m_4\dot{z}_4 + 2m_4 \left(\frac{2m_1\dot{z}_1}{m_3} + \frac{2m_4\dot{z}_4}{m_3} \right).
 \end{aligned}$$

Once the kinetic energy T is expressed as a function of $p_{x_i}, p_{y_i}, p_{z_i}$ ($i = 1, 3$), then $x_i, y_i, z_i, p_{x_i}, p_{y_i}, p_{z_i}$ ($i = 1, 3$) satisfy Hamilton's equation with the Hamiltonian function

$$H(\mathbf{p}, \mathbf{q}) = T(\mathbf{p}) - U(\mathbf{q}).$$

Besides the energy integral $H(\mathbf{p}, \mathbf{q}) = h$, this system also admits the angular-momentum integral $c_{12} + c_{45} = c$, where c_{12} and c_{45} are the angular momentum possessed by m_1, m_2 and m_4, m_5 , respectively:

$$(4.4) \quad \begin{aligned} c_{12} &= 2m_1(x_1\dot{y}_1 - y_1\dot{x}_1) = x_1p_{y_1} - y_1p_{x_1}, \\ c_{45} &= 2m_4(x_4\dot{y}_4 - y_4\dot{x}_4) = x_4p_{y_4} - y_4p_{x_4}. \end{aligned}$$

Observe that c_{12}, c_{45} are not constants of motion. Nonetheless their sum, the total angular momentum, is conserved under the motion. We are only interested in the subsystem with zero total angular momentum. From now on assume that $c = 0$; i.e., that

$$c_{12} + c_{45} = 0.$$

In the subsequent analysis we consider the system as having three components whenever the three particles m_1, m_2 and m_3 are close to one another relative to the distance of these three particles to m_4 and m_5 :

- (A) the isosceles 3-body system m_1, m_2, m_3 ;
- (B) the 2-body system m_4 and m_5 ; and
- (C) the 2-body system composed of the center of m_1, m_2, m_3 and the center of m_4, m_5 .

When m_3, m_4, m_5 are close together, we use a similar decomposition.

Our first objective is to show that the triple-collision manifold M of Section 2 persists under the presence of m_4 and m_5 . To do this we need to change some variables. Let $\bar{z} = (2m_1z_1 + m_3z_3)/(2m_1 + m_3)$. Then $(0, 0, \bar{z})$ is the center of the masses of m_1, m_2 and m_3 . Let $x_1 = x, y_1 = y, z = z_1 - \bar{z}$ and $\bar{z}_3 = z_3 - \bar{z}$. Following the transformations of Section 2, we obtain a system of equations that is similar to equations (2.2). In particular we obtain

$$(4.5) \quad \begin{aligned} r' &= (\mathbf{s} \cdot \mathbf{z})r, \\ \mathbf{s}' &= \mathbf{z} - (\mathbf{s} \cdot \mathbf{z})\mathbf{s}, \\ \mathbf{z}' &= \nabla U_1(\mathbf{s}) + \frac{1}{2}(\mathbf{s} \cdot \mathbf{z})\mathbf{z} + r^2 \frac{\partial U_2(r\mathbf{s}, \mathbf{q}_4)}{\partial(r\mathbf{s})}, \\ \mathbf{q}'_4 &= r^{3/2} H_{\mathbf{p}_4}(r, \mathbf{s}, \mathbf{z}, \mathbf{p}_4, \mathbf{q}_4), \\ \mathbf{p}'_4 &= -r^{3/2} H_{\mathbf{q}_4}(r, \mathbf{s}, \mathbf{z}, \mathbf{p}_4, \mathbf{q}_4), \end{aligned}$$

where the U_1 are the terms of U , which contains only \mathbf{q}_1 and \mathbf{q}_3 , and where $U_2 = U - U_1$. That is,

$$U_1(s) = \frac{1}{\sqrt{2}} m_1^{3/2} m_3 \left(m_1 (s_1^2 + s_2^2)^{1/2} / m_3 + 4 (s_1^2 + s_2^2 + (1 + 2m_1/m_3) s_3^2)^{-1/2} \right).$$

Again $'$ denotes differentiation with respect to τ , and $d\tau/dt = r^{-3/2}$.

If $z_4 \neq 0$, $H_{\mathbf{p}_4}$, then $H_{\mathbf{q}_4}$ and U_2 are smooth functions even at $r = 0$, except when m_4 and m_5 experience binary collisions. When the simultaneous triple and binary collisions occur, the situation becomes more complicated (this case will be discussed later). However we remark here that all the binary-collision orbits can be continuously extended through collisions and that, on $\{r = 0\}$, the equations for r , \mathbf{s} and \mathbf{z} are independent of the variables associated with the binary m_4 and m_5 .

For a moment let us assume that m_4 and m_5 are not at a binary collision when $r = 0$. Then the flow extends to $\{r = 0\}$, where we obtain a flow on a manifold; for simplicity of notation this is

$$M \times S^1 \times \mathbb{R}^5 \times \{\mathbb{R} \setminus \{0\}\}$$

given by

$$(4.6) \quad \begin{aligned} \mathbf{s}' &= \mathbf{z} - (\mathbf{s} \cdot \mathbf{z})\mathbf{s}, \\ \mathbf{z}' &= \nabla U_1(\mathbf{s}) + \frac{1}{2}(\mathbf{s} \cdot \mathbf{z})\mathbf{z}, \\ \mathbf{q}'_4 &= 0, \\ \mathbf{p}'_4 &= 0. \end{aligned}$$

The above flow is a product of the identity flow on $\mathbb{R}^5 \times \{\mathbb{R} \setminus \{0\}\}$ with a flow on $M \times S^1$. Here S^1 gives the θ coordinate, which was eliminated when we were deriving the triple-collision manifold M in Section 2. The flow on M is exactly the flow studied in that section; i.e., the effect of m_4 and m_5 disappears in the limit as one approaches triple collision among the particles m_1 , m_2 and m_3 .

Vector field (4.6) still has singularities due to the binary collision between m_1, m_2 and m_4, m_5 . It is a known fact that orbits can be extended through these binary collisions either by some transformation or by modification of the vector field in a neighborhood of these singularities, known as Easton's regularization. We do not carry out this regularization procedure here, but we shall speak of the flow as though the orbits are extended through binary collisions. There is another kind of singularity in this problem: the simultaneous binary collisions of m_1 and m_2 and of m_4 and m_5 . For these collisions, Saari

[18] showed that these singularities are algebraic branch points and hence that they are not essential singularities. Recently Simó and Lacomba [22] showed that they also can be C^0 -block regularized, in the sense of Easton. In fact, as we shall see, the solutions constructed here can be made far away from the simultaneous binary collisions. Consequently we are not concerned with the smoothness and the “regularizability” of the simultaneous binary collisions.

Let Σ_1 be the collection of all orbits that end in a triple collision with the equilateral central configuration E_+ . The manifold structure of Σ_1 follows from the computations in Section 2. In fact, Σ_1 is exactly the stable manifold of the invariant set

$$\{E_+\} \times S^1 \times \mathbb{R}^5 \times \{\mathbb{R} \setminus \{0\}\}.$$

In Sections 2 and 3, we showed that E_+ has two attracting and two repelling directions. Therefore Σ_1 is of codimension 2 (9 dimensional).

It is worth noting that there are several invariant submanifolds for this 5-body system for which $c_{12} \equiv 0 \equiv c_{45}$; i.e., both binaries m_1, m_2 and m_4, m_5 are restricted to moving in fixed planes containing the z -axis. The two planes that m_1, m_2 and m_4, m_5 move in must be parallel or perpendicular to one another. This is because, by computation, we have

$$\dot{c}_{12} \equiv -\dot{c}_{45} \equiv 2m_1m_4(x_1y_4 - x_4y_1)(r_{14}^{-3} - r_{15}^{-3}).$$

For $c_{12} \equiv 0$ (or $c_{45} \equiv 0$) either $x_1y_4 - y_4x_1 \equiv 0$ or $r_{14} \equiv r_{15}$.

From now on we require that the masses $m_1 = m_2, m_3$ be allowable, in the sense of Section 2. We also require that the choice of masses $m_4 = m_5, m_3$ be allowable. These are our only requirements for the masses.

Our next step is to show that some of the properties of the 3-body problem, described in the last two sections, persist in this 5-body problem. What we have now is a perturbation problem. We want to show that, under certain conditions, the solutions with initial values at T_0 still have similar properties to those described in the last section. First, we want to analyze the local structure of the invariant set $\{E_+\} \times S^1 \times \mathbb{R}^5 \times \{\mathbb{R} \setminus \{0\}\}$.

Let $p \in S^1 \times \mathbb{R}^5 \times \mathbb{R}^-$ and consider the rest point $\{E_+\} \times \{p\}$. The point p describes the limiting position of m_4 and m_5 ($\mathbb{R}^5 \times \mathbb{R}^-$) as well as the limiting orientation (S^1) of the particles m_1, m_2 and m_3 in the triple collision. Note that here by \mathbb{R}^- we mean that $z_4 < 0$. The case where $z_4 > 0$ can be discussed similarly. One can obtain the local properties of the flow near $\{E_+\} \times \{p\}$ by equations (4.5). Direct computation shows that we have two attracting directions and two repelling directions; all other directions are on the invariant set $\{E_+\} \times S^1 \times \mathbb{R}^5 \times \{\mathbb{R} \setminus \{0\}\}$, where every point is a rest point, and hence they are neutral directions. Therefore we have a 2-dimensional stable manifold at $\{E_+\} \times \{p\}$ and a 2-dimensional unstable manifold at each

point $\{E_+\} \times \{p\}$, for any $p \in S^1 \times \mathbb{R}^5 \times \mathbb{R}^-$. This is exactly what we had for the 3-body problem.

Next we study these stable and unstable manifolds and the solutions nearby.

The unstable manifold of $\{E_+\} \times \{p\}$ lies in the invariant manifold, defined by $r = 0$. The flow on this invariant manifold is given by equations (4.6). As we pointed out earlier, the flow on $r = 0$ is the product of the identity flow and a flow on $M \times S^1$, where M is exactly the triple-collision manifold of the last two sections. Therefore we may say that the unstable manifold of $\{E_+\} \times \{p\}$ is exactly the unstable manifold of $\{E_+\}$ in M . Hence Lemma 2.1 holds and the analysis at the end of Section 2 applies here.

Now we go back to the cylinder T_0 , defined in the last section. First, we define a function π , which projects any point in the phase space into the subspace spanned by the ϕ, u, v, w coordinates. Define a set T by

$$(4.7) \quad T = \{x \in \pi^{-1}(T_0) \mid |\bar{z}_3| \leq A; z_4 \leq -B; \dot{z}_4 \leq -B\},$$

where $A > 0$ is a positive number and will remain fixed throughout, and $B > 0$ is chosen sufficiently large such that, whenever $|\bar{z}_3| \leq A$, we have $z_3 > 0$. This is always possible, because the center of mass is fixed at the origin. Recall that $\bar{z}_3 = z_3 - \bar{z}$.

Similarly, corresponding to the boundaries S^+ and S^- of T_0 , we define the boundaries of T . Without confusion we can use the same notation to denote these new sets.

Consider the solutions starting from T . For this, only the first three equations of (4.5) are needed:

$$\begin{aligned} r' &= (\mathbf{s} \cdot \mathbf{z})r, \\ \mathbf{s}' &= \mathbf{z} - (\mathbf{s} \cdot \mathbf{z})\mathbf{s}, \\ \mathbf{z}' &= \nabla U_1(\mathbf{s}) + \frac{1}{2}(\mathbf{s} \cdot \mathbf{z})\mathbf{z} + r^2 \frac{\partial U_2}{\partial(rs)}. \end{aligned}$$

Without the presence of the term $r^2 \partial U_2 / \partial(rs)$, this system is exactly the 3-body problem already discussed. The perturbation term of m_4 and m_5 on the triple m_1, m_2 and m_3 depends on the distance between m_4 and the triple m_1, m_2 and m_3 . One easily computes that

$$\left| \frac{\partial U_2}{\partial(rs)} \right| \leq \frac{c_1}{|z_4|^2}$$

for $|z_4| > B$, and c_1 constant depending on B .

Before going further, we need to take certain precautions about $w_{12}^\infty(x)$, which was defined on a subset of N in the last section. Note that $w_{12}^\infty(x)$ is not a continuous function in the 5-body problem. This situation is due to

the perturbation of m_4 and m_5 . When r becomes large, either m_3 may come back to m_1 and m_2 , or the binary m_1 and m_2 may come close to m_4 and m_5 . In both cases, the function w_{12} may vary erratically, and this may occur infinitely often. However, as long as r is small and m_4 and m_5 far away, the perturbation of m_4 and m_5 is very small. This difficulty can be overcome by the use of $w_{12}(x(t))$ for some t instead of $t = \infty$. For this purpose let t_1 be the time when \bar{z}_3 first reaches $-A$; i.e., let

$$t_1(x) = \inf\{t > 0 \mid \bar{z}_3 = -A, x \in T\}.$$

We use the convention that $t_1(x) = \infty$ when \bar{z}_3 first reaches A before reaching $-A$.

LEMMA 4.1. *Let T be the above-defined set. Then there exists $B_1 > 0$ such that if $|z_4| \geq B_1$ for all $0 \leq t \leq t_1(x)$ and if $t_1(x) \leq 1$, then for all $x \in S^+ \cup S^-$*

$$\frac{1}{2}w^+ \leq |w_{12}(x, t_1)| \leq \frac{3}{2}w^+.$$

Proof. From the definition of S^+ and S^- , for any $x \in S^+ \cup S^-$, we have

$$\frac{2}{3}w^+ \leq |w_{12}(x)| \leq \frac{4}{3}w_{12}.$$

Furthermore, if $B_1 = \infty$, then

$$\frac{2}{3}w^+ \leq |w_{12}(x, t)| \leq \frac{4}{3}w^+, \quad \text{for all } t > 0.$$

For a point in S^+ and S^- , as pointed out earlier, m_3 is far away from m_1 and m_2 , compared to the distance between m_1 and m_2 . The triple m_1, m_2 and m_3 is much like a couple of 2-body systems: one with m_1 and m_2 and the other with m_3 and the center of masses of m_1 and m_2 . The particle m_3 moves away from m_1 and m_2 with a velocity proportional to $v^+r^{-1/2}$, while the change in $w_{12}(x, t)$ is due to the perturbation of m_3 as well as m_4 and m_5 . The perturbation of m_3 is of the limited amount $(w^+/3)$, and the influence of m_4 and m_5 is of order $\Delta t/(B_1)^2 \leq 1/(B_1)^2$. Therefore, by choosing B_1 sufficiently large, we have $\frac{1}{2}w^+ \leq |w_{12}(x)| \leq \frac{3}{2}w^+$. This proves the lemma. \square

Observe that, in Lemma 4.1, the only restriction on r is $|z_3 - \bar{z}| \leq A$. In fact r may be arbitrarily small.

From now on we assume in the definition of T that $B \geq B_1$. Let

$$\tilde{t}(x) = \sup\{t < 0 \mid z_1(x, t) = z_3(x, t), x \in T\}.$$

Observe that $\tilde{t}(x)$ is defined for the orbit of $\gamma^+ \cap T$ and, by the choosing of w^+ small, it is also defined for all $x \in \text{Un}(E_+) \cap T$. Thus $\tilde{t}(x)$ is defined for all $x \in T$, provided that T is sufficiently small. Let us assume that this is true.

LEMMA 4.2. Consider the orbits starting from T . Let r_0 be the initial value of r . There exists $B_2 > 0$ such that if $z_4 \geq B_2$ for all $\tilde{t} \leq t \leq t_1$, then $|\tilde{t}| \leq d_1 r_0^{3/2}$ and $|t_1| \leq d_2 r_0^{1/2}$ for some constants $d_1 > 0$ and $d_2 > 0$. In particular $\tilde{t} \rightarrow 0$ and $t_1 \rightarrow 0$ as $r \rightarrow 0$.

Proof. First observe for any $x \in T$ that $|\tilde{z}_3| \leq A$. Thus there exists a constant c_1 , depending only on A and the masses such that $r_0 \leq c_1$. Let $\tilde{\tau} < 0$ be the corresponding time τ for \tilde{t} . From $dt = r^{3/2}d\tau$ and $r' = rv$, we have $\tilde{\tau} = \int_0^{\tilde{\tau}} r^{3/2}d\tau$ and

$$(4.8) \quad r(\tau) = r_0 \exp\left(\int_0^{\tilde{\tau}} v(\tau)d\tau\right);$$

therefore

$$\tilde{t} = r_0^{3/2} \int_0^{\tilde{\tau}} \exp\frac{3}{2}\left(\int_0^{\tilde{\tau}} v(\tau)d\tau\right) d\tau.$$

For any $x \in T$ and $r_0 \leq c_1$ there is $B(x, r_0) > 0$ such that if $B \geq B(x, r_0)$, for all $\tilde{t} \leq t \leq 0$, then

$$(4.9) \quad \int_0^{\tilde{\tau}} \exp\frac{3}{2}\left(\int_0^{\tilde{\tau}} v(\tau)d\tau\right) d\tau \leq d_1(x, r_0)$$

for some $d_1(x, r_0)$. From the compactness of T_0 and $0 \leq r_0 \leq c_1$, we can find some $B_2 > 0$ and $d_1 > 0$ such that $B_2 \geq B(x, r_0)$ and $d_1 \geq d_1(x, r_0)$ for all $x \in R$ and $0 \leq r_0 \leq c_1$. Thus $|\tilde{t}| \leq d_1 r_0^{3/2}$ and $\tilde{t} \rightarrow 0$, as $r_0 \rightarrow 0$.

For the second part of the lemma notice at T that $v = v^+$ and $\dot{\tilde{z}}_3 > c_4 r_0^{-1/2} v^+$ for some $c_4 > 0$. The motion of m_3 is close to that of a 2-body problem with m_3 and the center of the mass of m_1 and m_2 . The distance between m_3 and the center of the mass of m_1 and m_2 is of order r_0 ; thus the escaping velocity of m_3 is of order $r_0^{-1/2}$. Therefore, if B_2 is chosen sufficiently large, then t_1 exists for all $x \in T$ and

$$t_1 \leq \frac{A}{c_5 r_0^{-1/2}} = r_0^{1/2} A/c_5 = d_2 r_0^{1/2}$$

for some $c_5 > 0$, $d_2 > 0$. In particular $t \rightarrow 0$ as $r_0 \rightarrow 0$. This proves the lemma. □

Using the above argument, one can easily prove the following lemma:

LEMMA 4.3. There is such a $B_3 \geq B_2 > 0$ that for all the orbits starting from T , if $z_4 \geq B$ for all $t \leq t_1$, then there exist d_3 and d_4 , $0 < d_3 < d_4$, so that

$$d_3 r_0^{-1/2} \leq |\dot{\tilde{z}}_3(x, t_1)| \leq r_0^{-1/2} d_4 \quad \text{for all } x \in T.$$

From now on let us assume that $B > B_3$.

We are ready to prove our first major theorem, which will serve as a crucial tool for constructing the noncollision singularity.

Fix a point $x^* \in \Sigma_1$. There exists a point $p \in S^1 \times \mathbb{R}^5 \times \mathbb{R}^-$ such that $x^* \in \text{St}(\{p\} \times \{E_+\})$, where $\text{St}(\{p\} \times \{E_+\})$ is the stable manifold of the rest point $\{p\} \times \{E_+\}$. Let Γ be a 2-dimensional smooth surface that intersects Σ_1 transversally at x^* . Let $t^* > 0$ be the time when the orbit, starting from x^* , ends in a triple collision of m_1, m_2 and m_3 . Without loss of generality assume that $|z_4(x^*, t^*)| \geq 2B$. For all $x \in \Gamma$ let

$$\begin{aligned}
 \tilde{t}_1(x) &= \inf\{t > 0 \mid z_3(x, t) = z_1(x, t)\}, \\
 t_1(x) &= \inf\{t > \tilde{t}_1(x) \mid \bar{z}_3(x, t) = -A\}, \\
 \bar{t}_1(x) &= \inf\{t > t_1(x) \mid z_3(x, t) = 0\}, \\
 \tilde{t}_2(x) &= \inf\{t > \bar{t}_1(x) \mid z_3(x, t) = z_4(x, t)\}, \\
 &\dots\dots
 \end{aligned}
 \tag{4.10}$$

while for the collision orbit x^* let

$$\tilde{t}_1(x^*) = t_1(x^*) = \bar{t}_1(x^*) = \tilde{t}_2(x^*) = \dots = t^*.$$

Then we have following theorem:

THEOREM 4.4. *Let Γ be the above-mentioned 2-dimensional surface. Consider the solutions starting from Γ . There is a wedge Δ with vertex at x^* and with boundaries c^+, c^- and c such that the following hold:*

(1) *For all x in the wedge, $\tilde{t}_1(x), t_1(x), \bar{t}_1(x)$ and $\tilde{t}_2(x)$ are well-defined and continuous functions. In particular $\tilde{t}_1(x), t_1(x), \bar{t}_1(x)$ and $\tilde{t}_2(x)$ approach t^* as x in the wedge approaches x^* . As a consequence $\dot{z}_3(t_1)$ and $\dot{z}_3(\bar{t}_1) \rightarrow \infty$ as $x \rightarrow x^*$.*

(2) *For all $x \in c^+, \frac{1}{4}w^+ \leq w_{12}(x, \tilde{t}_2) \leq 2w^+$, and for all $x \in c^-, \frac{1}{4}w^+ \leq -w_{12}(x, \tilde{t}_2) \leq 2w^+$.*

(3) *There is $K > 1$, where K depends only on the masses such that for any $\delta > 0$, and for all $x \in \Delta$,*

$$\begin{aligned}
 z_1(x, \tilde{t}_2) &\geq K z_1(x, \tilde{t}_1) > 0, \\
 |z_4(x, \tilde{t}_2) - z_4(x, \tilde{t}_1)| &< \delta.
 \end{aligned}$$

Proof. First, observe for a small neighborhood of x^* that $\tilde{t}_1(x)$ exists and is a continuous function on the initial values. Without loss of generality assume that this is true for all $x \in \Gamma$. Let

$$\Delta_0 = \{x \in \Gamma \mid x(t) \in T, \quad \text{for some } t > 0\}$$

and, for any $x \in \Delta_0$, let $t_0(x)$ be the time such that $x(t_0) \in T$. As we showed earlier, Δ_0 is a nonempty set. Consider the value of $r(x, t_0)$. Since the

unstable manifold of E_+ is in the triple-collision manifold on which $r = 0$ and the orbits starting near x^* tend to follow the unstable manifold of E_+ , we have $r(x, t_0) \rightarrow 0$ as $x \rightarrow x^*$ for $x \in \Delta_0$. Following Lemma 4.2, we find that there exists a smaller wedge $\Delta \subset D$ such that t_1 exists for all $x \in \Delta$ and $t_1 \rightarrow t^*$ as $x \rightarrow x^*$, $x \in \Delta$. Now from Lemma 4.3, $\dot{z}_3(x, t_1) \rightarrow -\infty$ as $x \rightarrow x^*$. Therefore, for x sufficiently close to x^* , eventually m_3 reaches the $z = 0$ -plane and then takes over m_4 and m_5 . Thus \bar{t}_1 and \tilde{t}_2 are defined for all x sufficiently close to x^* , $x \in \Delta$ and $\bar{t}_1, \tilde{t}_2 \rightarrow t^*$ as $x \rightarrow x^*$. By making Δ small, we may assume that the above is true for all $x \in \Delta$.

For the second part of the theorem observe that, from Lemma 4.1, for all $x \in c^+$ and c^- , where c^+ and c^- are the parts of the boundary of Δ such that

$$c^+ = \{x \in \Delta \mid x(t_0) \in S^+\},$$

$$c^- = \{x \in \Delta \mid x(t_0) \in S^-\},$$

the following is true (we may make Δ smaller if necessary):

$$\frac{1}{2}w^+ \leq |w_{12}(x, t_1)| \leq \frac{3}{2}w^+.$$

The rate of change for w_{12} depends on the distance of m_1 and m_2 from the rest of the particles and, for all $t_1 \leq t \leq \tilde{t}_2$, on $|d(w_{12}(x, t))/dt| \leq c_3/A^2$ for some $c_3 > 0$. Since $\tilde{t}_2 - t_1$ can be made arbitrarily small, the wedge Δ can be made small enough such that for all $x \in \Delta$, and $t_1 \leq t \leq \tilde{t}_2$, we have $|w_{12}(x, t) - w_{12}(x, t_1)| \leq \frac{1}{4}w^+$.

The third part of the theorem comes from conservation of momentum. We only need to take note of the following facts:

- (1) Let P_{123} be the momentum of the subsystem m_1, m_2 , and m_3 :

$$P_{123} = 2m_1z_1 + m_3z_3.$$

There exist positive numbers M_1 and M_2 such that

$$|\dot{P}_{123}(x, \tilde{t}_1)| \leq M_1, |\ddot{P}_{123}| \leq M_2$$

for all $t \in [\tilde{t}_1, \bar{t}_1]$ and $x \in \Delta$.

- (2) It follows from the first part of the theorem that $\bar{t}_1 - \tilde{t}_1 \rightarrow 0$ as $x \rightarrow x^*$, $x \in \Delta$. Therefore, for any $\epsilon > 0$, by making Δ small enough, we have

$$|P_{123}(x, t) - P_{123}(x, t')| \leq \epsilon,$$

for all $t, t' \in t \in [\tilde{t}_1, \bar{t}_1]$ and $x \in \Delta$.

Following from the above facts, for all $x \in \Delta$, is

$$\begin{aligned} 2m_1z_1(x, \tilde{t}_2) &> 2m_1z_1(x, \bar{t}_1) = P_{123}(x, \bar{t}_1) - m_3z_3(x, \bar{t}_1) \\ &= P_{123}(x, \bar{t}_1) > P_{123}(x, \tilde{t}_1) - \epsilon \\ &= (2m_1 + m_3)z_1(x, \tilde{t}_1) - \epsilon. \end{aligned}$$

Therefore

$$z_1(x, \tilde{t}_2) \geq \left(1 + \frac{m_3}{2m_1}\right) z_1(x, \tilde{t}_1) - \frac{\epsilon}{2m_1}.$$

Let K be any number such that $1 < K < 1 + m_3/(2m_1)$. Then ϵ can be made small enough that

$$z_1(x, \tilde{t}_2) \geq K z_1(x, \tilde{t}_1) > 0$$

for all $x \in \Delta$. The inequality $|z_4(x, \tilde{t}_2) - z_4(x, \tilde{t}_1)| < \delta$ follows immediately from the first part of the theorem, thereby proving Theorem 4.4.

Now we would like to make several remarks concerning this theorem.

(1) The constant A is prefixed, and its value does not change. However the choice of A is quite arbitrary. In fact A can be chosen arbitrarily small and, in this way, one easily sees that B , whose value depends on A , can be made arbitrarily small.

(2) The triple collisions considered so far are for the central configuration E_+ . In fact the triple collisions with the limited central configuration E_- can be treated equally the same.

(3) All the above discussions apply just as well to the motion of m_3 , m_4 , and m_5 . With some appropriate changes of constants, one may assume that the lemmas and the theorem also apply to the triple m_3 , m_4 , and m_5 .

In the next section we shall use the last theorem to set up symbolic dynamics to construct unbounded solutions in finite time.

5. Unbounded solutions in finite time

We first consider the initial values on the 2-dimensional wedge Δ of Theorem 4.4 of the last section. Later we shall consider the 3-dimensional wedge W described in Section 2.

For all $x \in \Delta$ let $N(x)$ and $n(x)$ be the number of maxima and minima that r_{12} has reached in the time interval $[\tilde{t}_1, \tilde{t}_2]$. Here $N(x)$ and $n(x)$ are the extended notions of the number of complete revolutions that m_1 and m_2 have made in $[\tilde{t}_1, \tilde{t}_2]$. Because the value of w^+ may be made arbitrarily small, we may assume that $w^+ \leq 1/10$.

LEMMA 5.1. $N(x)$, $n(x)$ are continuous for all $x \in \Delta$, except at those points x that $r_{12}(x, t)$ reaches a maximum or minimum at either $\tilde{t}_1(x)$ or $\tilde{t}_2(x)$.

Proof. Suppose that $r_{12}(x, t)$ reaches a minimum or maximum at t' . Then either $\dot{r}_{12}(x, t') = 0$ or $\dot{r}_{12}(x, t')$ is undefined. For the latter case, a binary or triple collision must occur. Thus $r_{12}(x, t') = 0$. We want to show for all $x \in \Delta$ and for all $\tilde{t}_1 \leq t \leq \tilde{t}_2$ that all zeroes of $\dot{r}_{12}(x, t)$ are nondegenerate; i.e., for all t' , $t' \in [\tilde{t}_1, \tilde{t}_2]$ such that whenever $\dot{r}_{12}(x, t') = 0$, we have $\ddot{r}_{12}(x, t') \neq 0$.

Notice for the binary or triple collisions, $r_{12}(x, t') = 0$, that t' is easily seen to be a nondegenerate local minimum for $r_{12}(x, t)$. This means that all zeroes of $\dot{r}_{12}(x, t)$ are either a nondegenerate maximum or a nondegenerate minimum of $r_{12}(x, t)$. From this we can deduce, by the continuity of $r_{12}(x, t)$, that $N(x)$ and $n(x)$ are continuous functions for all $x \in \Delta$, except at those points x of Δ such that r_{12} reaches a maximum or minimum at $\tilde{t}_1(x)$ and $\tilde{t}_2(x)$.

We will prove this in the more general setting of a perturbed 2-body problem. Let

$$(5.1) \quad \ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} + \mathbf{f},$$

where \mathbf{f} is a small perturbation term. Due to the \mathbf{f} term, the energy h and angular momentum c of the system are not constants. Thus

$$h = \frac{1}{2}\dot{r} + \frac{1}{2r^2}c^2 - \frac{1}{r}$$

is a function of time. We only consider the case where $h < 0$. In this case,

$$c^2|h| = c^2 \left| \frac{1}{2}\dot{r} + \frac{1}{2r^2}c^2 - \frac{1}{r} \right|.$$

If $\dot{r} = 0$ for some t , then

$$c^2|h| = c^2 \left| \frac{1}{2r^2}c^2 - \frac{1}{r} \right|.$$

Let $w_{12} = c^2|h|$; when $\dot{r} = 0$,

$$w_{12} = c^2 \left| \frac{1}{2r^2}c^2 - \frac{1}{r} \right|.$$

Solving this equation for c , we obtain either

$$c^2 = r + (r^2 - 2r^2w_{12})^{1/2}$$

or

$$c^2 = r - (r^2 - 2r^2w_{12})^{1/2}.$$

On the other hand, $r^2 = \mathbf{r} \cdot \mathbf{r}$; thus $r\dot{r} = \mathbf{r} \cdot \dot{\mathbf{r}}$ and $r\ddot{r} + \dot{r}^2 = \mathbf{r} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, where we write

$$\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{c^2}{r}.$$

These equations together with the equation for motion yield

$$(5.2) \quad \ddot{r} = \frac{c^2 - r}{r^3} + \frac{\mathbf{r}}{r} \cdot \mathbf{f}.$$

Therefore, corresponding to the two solutions for c obtained above, we have either

$$\ddot{r} = \frac{(1 - 2w_{12})^{1/2}}{r^2} + \frac{\mathbf{r}}{r} \cdot \mathbf{f}$$

or

$$\ddot{r} = -\frac{(1 - 2w_{12})^{1/2}}{r^2} + \frac{\mathbf{r}}{r} \cdot \mathbf{f}.$$

Thus $\ddot{r} \neq 0$, provided that

$$(5.3) \quad |\mathbf{f}r^2| < (1 - 2w_{12})^{1/2}.$$

Now we go back to our original system. First consider the perturbation term \mathbf{f} . In the 5-body problem, the perturbations on the binary m_1 and m_2 are due to m_3 , m_4 and m_5 . It is easy to show that there are some positive numbers M_1 and M_2 such that

$$|\mathbf{f}| \leq \frac{M_1}{r_{13}^2} + \frac{M_2}{r_{14}^2},$$

$$|\mathbf{f}r_{12}^2| \leq \frac{M_1 r_{12}^2}{r_{13}^2} + \frac{M_2 r_{12}^2}{r_{14}^2}.$$

Now we divide the interval $[\tilde{t}_1, \tilde{t}_2]$ into two separate subintervals $[\tilde{t}_1, t_1^*]$ and $[t_1^*, \tilde{t}_2]$ and then consider them separately. Recall that t_1^* is the time when the orbit passes the cylinder T of the last section.

We consider the interval $[t_1^*, \tilde{t}_2]$ first. By properly choosing the cylinder T (i.e., enlarging v^+ if necessary), we can make r_{12}^2/r_{13}^2 and r_{12}^2/r_{14}^2 arbitrarily small for all $t \in [t_1^*, \tilde{t}_2]$. Thus, by the above means, we may assume that

$$|\mathbf{f}r_{12}^2| \leq \frac{M_1 r_{12}^2}{r_{13}^2} + \frac{M_2 r_{12}^2}{r_{14}^2} \leq \frac{1}{10}.$$

On the other hand, for all $t \in [t_1^*, \tilde{t}_2]$, we have $|w_{12}| \leq 2w^+ \leq 2/10$. Therefore

$$|\mathbf{f}r_{12}^2| \leq 1/10 \leq (6/10)^{1/2} \leq (1 - 2w_{12})^{1/2}.$$

Thus inequality (5.3) holds. Therefore, for all $t \in [t_1^*, \tilde{t}_2]$, all zeroes of $\dot{r}_{12}(x, t)$ are nondegenerate critical points of $r_{12}(x, t)$.

For the interval $[\tilde{t}_1, t_1^*]$, since m_3 is now closely involved with the binary m_1 and m_2 , inequality (5.3) is harder to obtain. To do so, we use the properties of the triple-collision manifold. Recall that, for the orbit with an angular momentum of zero, all the critical points of $r_{12}(x, t)$ are nondegenerate; therefore, on N_0 , this same property holds. From continuity, for the orbits of N close enough to N_0 , this property also holds. Since T is defined in a small neighborhood of N_0 , and since T_0 is compact, we can always make T small enough that, for all x such that $x(t_1^*) \in T$, the zeroes of $\dot{r}_{12}(x, t)$ are nondegenerate for

all $t \in [\tilde{t}_1, t_1^*]$. Here we used the fact that there is a positive number $M_3 > 0$ such that $|\tau_1^* - \tilde{\tau}_1| < M_3$ for all x so that $x(t_1^*) \in T$, where $\tau_1^* = \tau(t_1^*)$ and $\tilde{\tau}_1 = \tau(\tilde{t}_1)$.

We have shown above that all the critical points of $r_{12}(x, t)$ are non-degenerate for all $x \in \Delta$ and $t \in [\tilde{t}_1, \tilde{t}_2]$. Consequently the only discontinuities of $N(x)$ and $n(x)$ are when \tilde{t}_1 or \tilde{t}_2 is a local minimum or maximum of $r_{12}(x, t)$. This proves Lemma 5.1. □

In the 2-body problem it is a well-known fact that, for an elliptic orbit with energy $h < 0$, the period of motion is $\sqrt{2}/(2|h|^{3/2})$. We will need a similar estimate on the maximum time between two consecutive maxima or minima for what we do next.

Again consider the perturbed 2-body problem

$$(5.4) \quad \ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} + \mathbf{f},$$

where $\mathbf{f} = \mathbf{f}(t)$ is a small perturbation term.

LEMMA 5.2. *Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $|h_0^{-2}\mathbf{f}| < \delta$, with $h_0 < 0$ and $|w_{12}(x)| \leq 1/4$, then*

$$(5.5) \quad \left| \frac{\tau_{12}}{\sqrt{2}/(2|h_0|^{3/2})} - 1 \right| \leq \epsilon,$$

where τ_{12} is the time interval between the first two consecutive maxima or minima for $t > 0$.

First let us remark that the important feature of the lemma is that we have a *uniform* δ such that inequality (5.5) is satisfied for all $h_0 \leq 0$.

Proof. Consider the initial conditions on the energy hypersurface $h_0 = -1$ with $|w_{12}(x)| \leq 1/4$. The period for the unperturbed 2-body problem is $\sqrt{2}/2$. To each point x in this set, the proof of Lemma 5.1 leads to the zeroes of $r(x(t))$ being continuous functions of x , provided that $|\mathbf{f}r^2| < (1 - 2w_{12})^{1/2}$. Note that $r \leq 1/|h|$, $h_0(x) = -1$, and $|w_{12}(x)| \leq 1/4$, $h(x, t)$ and $w_{12}(x, t)$ depend continuously on \mathbf{f} . Therefore there is a $\delta(x)$ depending on x such that if $|\mathbf{f}| \leq \delta(x)$, then

$$(5.6) \quad |\tau_{12}(x)\sqrt{2} - 1| < \epsilon.$$

By the compactness of the set $\{x \mid h_0 = -1, |w_{12}(x)| \leq 1/4\}$ (in the regularized 2-body problem) there is a uniform δ independent of x such that if $|\mathbf{f}| \leq \delta$, then inequality (5.6) is satisfied.

We showed above that Lemma 5.1 is true for the energy hypersurface $h_0 = -1$. Now we want to show that the lemma is true for an initial value

with h_0 standing for any value of $h < 0$. In the 3-body problem, by using McGehee's coordinates, one may drop the r variable from the equations of motion and reduce the dimensions of the system by one. Thus the complete solutions of the original system can be obtained from the solutions of the reduced system and integration of the linear equation for r . In other words, if the only difference between two initial values is in r for the McGehee coordinates, then the solutions only differ in r with a constant factor. The rest of the proof of the lemma is motivated by the above fact; instead of going full scale through McGehee's coordinate change, we use a simplified version of it.

Consider a system of differential equations

$$(5.7) \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \nabla U(\mathbf{x}) + \mathbf{f}, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{v} \in \mathbb{R}^n,$$

where $U(x)$ is a homogeneous potential function of degree -1 and \mathbf{f} is a perturbation term. Let

$$\underline{x} = |h_0|\mathbf{x}, \quad \underline{v} = |h_0|^{-1/2}\mathbf{v}, \quad \underline{t} = |h_0|^{3/2}t,$$

where h_0 is an arbitrary constant. Then

$$d\underline{x}/d\underline{t} = \underline{v}, \quad d\underline{v}/d\underline{t} = \nabla U(\underline{x}) + |h_0|^{-2}\mathbf{f}.$$

Compare this system to the original (5.7). We have exactly the same set of equations, except that the perturbation term now has a factor of $|h_0|^{-2}$. Apply this to the 2-body problem and let h_0 be the initial energy of the system. Then the initial energy for the new system is exactly $h_0/|h_0| = -1$ for all $h_0 < 0$. Note that the time has been rescaled by a factor of $|h_0|^{3/2}$. Thus the lemma follows from what we have proved on the energy surface $h_0 = -1$. \square

The next result shows the limit behavior of $N(x)$ and $n(x)$ as x approaches x^* .

LEMMA 5.3. $N(x) \rightarrow \infty$ and $n(x) \rightarrow \infty$ as $x \rightarrow x^*$, $x \in \Delta$.

Proof. As $x \rightarrow x^*$, it follows from Theorem 4.4 that $\dot{z}_3(x, \bar{t}_1) \rightarrow -\infty$. Thus there is a positive number $c_1 > 0$ such that the kinetic energy for the subsystem m_1, m_2 and m_3 is greater than $c_1 \dot{z}_3^2(x, \bar{t}_1)$ for x sufficiently close to x^* and $t \in [\bar{t}_1, \bar{t}_2]$. From conservation of energy there is a $c_2 > 0$ such that

$$|h_{12}| \geq c_2 \dot{z}_3^2(x, \bar{t}_1)$$

for x sufficiently close to x^* and $t \in [\bar{t}_1, \bar{t}_2]$.

One sees from Lemma 5.2 that there is a $c_3 > 0$ such that

$$\tau_{12}(x) \leq c_3 \sqrt{2} |h_0|^{-3/2} / 2 \leq c_4 \dot{z}_3^{-3}(x, \bar{t}_1)$$

for some $c_4 > 0$, x sufficiently close to x^* and $t \in (\bar{t}_1, \bar{t}_2)$.

On the other hand, there is a $c_5 > 0$ such that

$$(\tilde{t}_2 - \tilde{t}_1) \geq c_5 B / |\dot{z}_3(x, \tilde{t}_1)|.$$

Thus $N(x) > c_6 \dot{z}_3^{-2}(x, \tilde{t}_1)$ and $n(x) > c_6 \dot{z}_3^{-2}(x, \tilde{t}_1)$; therefore, as $x \in x^*$,

$$N(x) \rightarrow \infty \quad \text{and} \quad n(x) \rightarrow \infty,$$

which proves the lemma. □

Observe from Lemma 5.2 that $N(x)$ and $n(x)$ change values only when $r_{12}(x, t)$ reaches a local minimum or maximum at \tilde{t}_1 or \tilde{t}_2 . As there is only a finite number of minima and maxima for $r_{12}(x, t)$ at $t = \tilde{t}_1$, for any curve in Δ that ends at x^* there is an infinite number of points on the curve such that $r_{12}(x, \tilde{t}_2)$ is a local minimum. And similarly there is an infinite number of points on the curve such that $r_{12}(x, \tilde{t}_2)$ is a local maximum.

So far we have centered our discussion around the 2-dimensional wedge given by Theorem 4.4. In order to have more flexibility and a clearer global picture we introduce the 3-dimensional wedge. The main reason we extend the scope of this section by considering a 3-dimensional section is that the noncollision singularity we construct here is in the neighborhood of the set of orbits that lead to simultaneous binary and triple collisions, which is a codimension-3 smooth manifold.

Just as in the triple-collision case we can also use McGehee’s coordinates to blow up a simultaneous binary and triple collision. This time, however, we blow up the set given by $R^2 = r_{123}^2 + \alpha^2/4 r_{45}^2 = 0$, where $\alpha = (4/m_4)^{1/3}$. We use the coordinates $r_{123} = R \sin \beta$ and $r_{45} = 2\alpha R \cos \beta$. In this way we also obtain a collision manifold. The flow on this collision manifold is more or less the product flow of the one over the binary-collision manifold and the one over the triple-collision manifold (with an appropriate rescaling of the time variable). However the next step is a partial blowup of the simultaneous binary and triple collisions, in which we keep the time rescaling used for the triple-collision manifold. In this way we may treat the binary collision and triple collision differently.

Let us introduce the following McGehee-type of change variables for the binary m_4 and m_5 :

$$\begin{aligned} (\alpha x_4, \alpha y_4) &= (\underline{r} \cos \underline{\theta}, \underline{r} \sin \underline{\theta}), \\ \beta &= \tan^{-1} \left(\frac{r}{\underline{r}} \right), \\ \underline{y} &= \begin{pmatrix} \cos \underline{\theta} \\ \sin \underline{\theta} \end{pmatrix} \cdot \begin{pmatrix} \alpha \dot{x}_4 \\ \alpha \dot{y}_4 \end{pmatrix}, \\ \underline{x} &= \begin{pmatrix} \alpha \dot{x}_4 \\ \alpha \dot{y}_4 \end{pmatrix} - \underline{y} \begin{pmatrix} \cos \underline{\theta} \\ \sin \underline{\theta} \end{pmatrix}, \end{aligned}$$

$$\underline{u} = r^{1/2} \underline{x} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},$$

$$\underline{v} = r^{1/2} \underline{y}.$$

In the above coordinates the equations of motion (4.5) become

$$(5.8) \quad \begin{aligned} r' &= (\mathbf{s} \cdot \mathbf{z})r, \\ \mathbf{s}' &= \mathbf{z} - (\mathbf{s} \cdot \mathbf{z})\mathbf{s}, \\ \mathbf{z}' &= \nabla U_1(\mathbf{s}) + \frac{1}{2}(\mathbf{s} \cdot \mathbf{z})\mathbf{z} + r^2 \frac{\partial U_2(r\mathbf{s}, \mathbf{q}_4)}{\partial(r\mathbf{s})}, \\ z'_4 &= r^{3/2} H_{p_{z_4}}(r, \mathbf{s}, \mathbf{z}, \mathbf{p}_4, \mathbf{q}_4), \\ p'_{z_4} &= -r^{3/2} H_{z_4}(r, \mathbf{s}, \mathbf{z}, \mathbf{p}_4, \mathbf{q}_4), \\ \beta' &= v \sin \beta \cos \beta - \underline{v} \sin^{5/2} \beta / \cos^{1/2} \beta, \\ \underline{v}' &= \tan^{3/2} \beta \left(\frac{1}{2} \underline{v}^2 + \underline{u}^2 - 1 \right) + O(r^{3/2}), \\ \underline{\theta}' &= \tan^{3/2} \beta \underline{u} + O(r^{3/2}), \\ \underline{u}' &= \tan^{3/2} \beta \left(-\frac{1}{2} \underline{u} \underline{v} \right) + O(r^{3/2}), \end{aligned}$$

where U_1 , as defined before, contains the terms of U , which contains only \mathbf{q}_1 and \mathbf{q}_3 . That is,

$$U_1(\mathbf{s}) = \frac{1}{\sqrt{2}} m_1^{3/2} m_3 \left(m_1 (s_1^2 + s_2^2)^{1/2} / m_3 + 4 (s_1^2 + s_2^2 + (1 + 2m_1/m_3) s_3^2)^{-1/2} \right).$$

Again, ' denotes differentiation with respect to τ , and $d\tau/dt = r^{-3/2}$. Note that the equations for m_1 , m_2 and m_3 are exactly the same as before in (4.5).

In equations (5.8), $r = 0$ is an invariant manifold. It is the collision manifold for the simultaneous triple collision of m_1 , m_2 and m_3 and the binary collision of m_4 and m_5 . We shall refer to it as the simultaneous collision manifold. Note that the single triple-collision and single binary collision correspond to $\beta = 0$ and $\beta = \pi/2$, respectively. In formula (5.8) there is still a singularity at $\beta = \pi/2$ that corresponds to the single binary collision of m_4 and m_5 . Standard methods can be used to regularize this singularity. Probably one could use the Levi-Civita coordinates for m_4 and m_5 and rescale the time to remove the singularity. However we note that one cannot use the usual time-rescaling factor r_{45} , since here the time variable τ is already rescaled by $r^{3/2}$. One would have to use the factor $\cos \beta$. Rescaling any more than $\cos \beta$ may put the binary collisions into rest points for the new time variable.

Here we choose Easton’s block regularization to regularize the single binary collision. The method is to join the collision orbit with an ejection orbit. It turns out that, after joining the two orbits, we find the resulting extended flow is smooth globally, after passing the collisions. Note the well-known fact that, for all $r > 0$ small, the singularity $\beta = \pi/2$ can be regularized as a block. For $r = 0$, the motion of m_4 and m_5 is a 2-body problem, without any perturbation from other particles. Therefore the singularity $\beta = \pi/2$ is also removable from the simultaneous collision manifold.

Here and after we shall speak of the flow as the regularized one.

We remark that, intentionally, we do not restrict our simultaneous collision manifold to the invariant sets given by the energy integral or energy relations. However various interesting fixed points lie in these invariant sets.

Now let us concentrate on the flow on the collision manifold. For $r = 0$, the equation concerning the triple m_1, m_2 and m_3 is exactly the same as discussed before. One easily sees that the rest points of the simultaneous collision manifold are either of the type

$$\tilde{P} = \{P\} \times \left\{ \underline{u} = 0, \underline{v} = \pm\sqrt{2}, \beta = \tan^{-1}((v_P/\sqrt{2})^2), \theta \in (0, \pi/2) \right\}$$

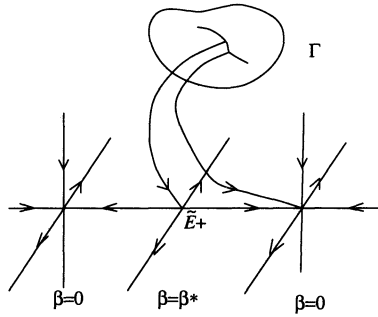
or

$$\{P\} \times \{\beta = 0\},$$

where P is one of the rest points in the triple-collision manifold, v_P is the value of v at P and the \pm sign is the same as that of v_P .

We are particularly interested in the rest points with $P = E_+$, where E_+ is the rest point on the triple-collision manifold. We can easily compute that for the rest point \tilde{E}_+ , besides the two positive eigenvalues from the triple-collision manifold, there are two additional positive eigenvalues. However, since we are restricted to the invariant subspace with $c_{12} + c_{45} = 0$, we can use this relation to eliminate the variable \underline{u} . Therefore the stable manifold of \tilde{E}_+ has codimension 3. Similar arguments hold for \tilde{E}_- . We conclude that the set of points, whose orbits lead to simultaneous triple and double collisions, forms a smooth manifold with codimension 3.

The 2-dimensional unstable manifold of \tilde{E}_+ on the triple-collision manifold has been studied in the previous sections. Now we discuss the additional branch of unstable manifolds arising from the simultaneous collisions. The new unstable eigenvector is in the $\delta\beta$ direction. To see how the unstable manifold goes in this direction we fix $\underline{u} = 0, v = v_{E_+}$ and $\underline{v} = -\sqrt{2}$. Then the equation for β only depends on β itself; i.e., this gives an invariant set for the flow. If the initial value of β is less than $\beta^* = \tan^{-1}(v_{E_+}^2/2)$, one sees that $d\beta/d\tau < 0$ for all $\tau > 0$; therefore $\beta \rightarrow 0$ as $\tau \rightarrow \infty$. However, if the initial value of β is larger than β^* , one easily sees that $d\beta/d\tau > 0$ and

FIGURE 6. \tilde{E}_+ on the simultaneous collision manifold

$\beta \rightarrow \pi/2$ as τ approaches some finite value; i.e., the orbit experiences a binary collision between m_4 and m_5 . By the regularization procedure, one joins this binary-collision orbit with a binary-ejection orbit, which is exactly the orbit with $v = \sqrt{2}$ and with θ incremented by π . After the orbit is rejected from the binary collision, we have $d\beta/d\tau < 0$. Again, $\beta \rightarrow 0$ as $\tau \rightarrow \infty$; i.e., this branch of unstable manifolds of \tilde{E}_+ also ends up in the triple collision of m_1 , m_2 and m_3 .

The procedure of joining collision orbits with ejection orbits may seem artificial and unsmooth; however, we point out that if the nearby solutions are considered, this indeed regularizes the flow. See [4].

Note that the rest points on the simultaneous collision manifold with $\beta = 0$ and $v < 0$ are attracting in the $\delta\beta$ direction. Figure 6 illustrates various stable and unstable manifolds and their relative positions.

Let x^* be a point in the stable manifold of \tilde{E}_+ , i.e., the orbit starting at x^* that ends up in the simultaneous triple collision of m_1 , m_2 and m_3 and double collision of m_4 and m_5 . Let t^* be the time when the solution of x^* ends. Thus $\mathbf{q}_4(x^*, t^*) = \mathbf{q}_5(x^*, t^*)$. The above blowup technique shows that the unstable manifold $\text{Un}(\tilde{E}_+)$ of \tilde{E}_+ forms a codimension-3 smooth submanifold. Let Π be a 3-dimensional section that intersects transversally $\text{Un}(\tilde{E}_+)$ at x^* . It is easily seen that Π intersects the set Σ_1 in a small curve near x^* (see Figure 6). Let ι be this curve; thus $\iota \in \Sigma_1$. We remark that ι may not be smooth at x^* , due to the simultaneous collision.

Let us assume additionally that Π intersects the set $\{x_4(x, \tilde{t}_1) = 0, y_4(x, \tilde{t}_1) = 0\}$ transversally at x^* and that the curve of the intersection of this set and Π crosses the curve ι at x^* .

From now on we shall focus on the solutions starting from Π . We remark that the only conditions imposed on Π are the transversality conditions. It is obvious that such a hypersurface Π exists.

Once again we point out that, on the simultaneous collision manifold, the variables associated with the triple collision are independent of other variables.

Therefore the analysis of Theorem 4.4 extends to the point x^* on the stable manifold of \tilde{E}_+ .

From Theorem 4.4 and the transversality condition on Π we know that there is a 3-dimensional wedge W with vertex at ι such that the following are satisfied:

(1) For all $x \in W$, $\tilde{t}_1(x)$, $t_1(x)$, $\bar{t}_1(x)$ and $\tilde{t}_2(x)$ are well-defined and continuous functions on x and $\dot{z}_3(t_1) \rightarrow \infty$, $\dot{z}_3(\bar{t}_1) \rightarrow \infty$ as $x \rightarrow \iota$.

(2) Let c^+ and c^- be two pieces of the boundary of W that have ι as their common boundary; then for all $x \in c^+$,

$$\frac{1}{4}w^+ \leq w_{12}(x, \tilde{t}_2) \leq 2w^+,$$

and for all $x \in c^-$,

(5.9)
$$\frac{1}{4}w^+ \leq -w_{12}(x, \tilde{t}_2) \leq 2w^+.$$

(3) There is a $K > 1$, where K depends only on the masses of m_1 , m_2 and m_3 such that, for any $\epsilon > 0$, if W is made small enough, then for all $x \in \Delta$

$$\begin{aligned} z_1(x, \tilde{t}_2) &\geq Kz_1(x, \tilde{t}_1) > 0 \\ |z_4(x, \tilde{t}_2) - z_4(x, \tilde{t}_1)| &< \epsilon. \end{aligned}$$

(4) The sets ι , c_1 and c_2 do not intersect the set $\{x_4(x, \tilde{t}_1) = 0, y_4(x, \tilde{t}_1) = 0\}$.

The next two lemmas are essential for justifying our iteration process and the use of symbolic dynamics.

LEMMA 5.4. *For the wedge W defined above, $W \subset \Pi$. There is a non-empty closed set $\iota_1 \in W$ containing x^* such that $r_{45}(x, \tilde{t}_2) = 0$ for all $x \in \iota_1$. That is, the orbits starting from ι_1 end at the triple collision of m_3 , m_4 and m_5 at \tilde{t}_2 . Furthermore the set ι_1 is connected and contains more than one point.*

Proof. Let c be a small closed curve in the boundary of W around x^* . By the transversality condition imposed on the section Π , c does not intersect the set $\{x_4(x, \tilde{t}_1) = 0, y_4(x, \tilde{t}_1) = 0\}$. Moreover the following map h_1 from c to S^1 is well defined and topologically nontrivial:

$$h_1(x) = \left(\frac{x_4(x, \tilde{t}_1)}{(x_4^2(x, \tilde{t}_1) + y_4^2(x, \tilde{t}_1))^{1/2}}, \frac{y_4(x, \tilde{t}_1)}{(x_4^2(x, \tilde{t}_1) + y_4^2(x, \tilde{t}_1))^{1/2}} \right) \text{ for all } x \in c.$$

The map h_1 is a degree-one map. Now we define the following map h_2 , also from c to S^1 :

$$h_2(x) = \left(\frac{x_4(x, \tilde{t}_2)}{(x_4^2(x, \tilde{t}_2) + y_4^2(x, \tilde{t}_2))^{1/2}}, \frac{y_4(x, \tilde{t}_2)}{(x_4^2(x, \tilde{t}_2) + y_4^2(x, \tilde{t}_2))^{1/2}} \right) \text{ for all } x \in c.$$

By the properties of the wedge W , we see that h_2 is also a well-defined, topologically nontrivial, degree-one map.

Now the lemma follows from a simple topological argument. Let S be a surface in W with c as its boundary. Then S must contain at least one point such that $x_4(x, \tilde{t}_2) = 0$ and $y_4(x, \tilde{t}_2) = 0$; otherwise the map h_2 can be continuously extended to the set S , which is impossible. Let ι'_1 be the set of all points in W such that $x_4(x, \tilde{t}_2) = 0$ and $y_4(x, \tilde{t}_2) = 0$. Then ι'_1 is a nonempty closed set and $x^* \in \iota'_1$. Let ι_1 be the connected component of ι'_1 containing x^* . It is obvious that ι_1 contains points other than x^* , which proves the lemma. \square

The next result concerns the solutions starting from ι_1 .

LEMMA 5.5. *Let $\iota_1 \in \Sigma_4$ be the curve of Lemma 5.4. Then there are infinitely many points $x_1, x_2, x_3, \dots, x_i \in \iota_1, i \in \mathbb{N}, x_i \rightarrow x^*$ as $i \rightarrow \infty$ such that $r_{12}(x_i, \tilde{t}_2) = 0$. That is, for the orbits starting from x_i the solution ends at simultaneous double and triple collisions.*

Proof. It follows from Lemma 5.3 that $n(x) \rightarrow \infty$ as $x \rightarrow x^*, x \in \iota_1$. Recall that $n(x)$ is a step function that changes values only when $r_{12}(x, t)$ reaches a local minimum at \tilde{t}_1 or \tilde{t}_2 . As there is no local minimum of $r_{12}(x, t)$ for $x \in \iota_1$ and $t = \tilde{t}_1$ if W is chosen small enough, the discontinuity of the function $n(x)$ occurs only if $r_{12}(x, t)$ has a local minimum at $t = \tilde{t}_2(x)$. Since ι_1 is connected, for any positive integer i large enough, there exists a point $x_i \in \iota_1$ such that $n(x)$ is discontinuous at x_i and $n(x_i) = i$. (Otherwise the set ι_1 would be disconnected and consist of two disjoint nonempty components with $n(x) \geq i$ and $n(x) < i$, respectively.) Renumber these points as x_i 's and let $x_i, i = 1, 2, 3, \dots$ be these points such that $r_{12}(x_i, t)$ is the local minimum. Then $\dot{r}_{12}(x_i, \tilde{t}_2) = 0$ or $r_{12}(x_i, \tilde{t}_2) = 0$. It follows from conservation of angular momentum, $c_{12} + c_{45} = 0$, that $c_{12}(x_i, \tilde{t}_2) = -c_{45}(x_i, \tilde{t}_2) = 0$, since $c_{45}(x_i, \tilde{t}_2) = 0$. Therefore $\ddot{r}_{12}(x_i, \tilde{t}_2) < 0$, and $\dot{r}_{12}(x_i, \tilde{t}_2) = 0$ can only give the local maximum. From this we conclude that $r_{12}(x_i, \tilde{t}_2) = 0$, proving the lemma. \square

Now we summarize the results obtained thus far. Let $x^* \in \Sigma_1$ be a point such that the orbit of x^* ends up in a simultaneous triple collision of m_1, m_2, m_3 and a binary collision of m_4, m_5 ; let Π be a small, generic, 3-dimensional hypersurface in the phase space. The section Π intersects Σ_1 in a curve ι having x^* in its interior. Then there is a wedge W with vertex at ι such that, for all the orbits starting from the wedge W , m_3 shoots out from m_1 and m_2 and catches up with m_4 and m_5 in a very short time. If we make W smaller, the time for m_3 to catch up with m_4 and m_5 can be made arbitrarily small. Furthermore, inside the wedge W , there are infinitely many points $x_1, x_2, \dots, x_i \rightarrow x^*$ such that, for any $i \in \mathbb{N}$, the orbit starting from x_i ends up

in a simultaneous triple collision of m_3, m_4, m_5 and a binary collision of m_1, m_2 .

Now we are ready to prove one of our major results, Theorem 1.1.

Proof of Theorem 1.1. We begin by repeating the above process of constructing the wedge W and finding the points x_1, x_2, \dots . This time, in place of x^* , we use $x_i, i \in \mathbb{Z}$ and ι_1 ; and in place of the triple collision of m_1, m_2, m_3 , we use the triple collision of m_3, m_4, m_5 . For each x_i we take a small neighborhood of $x_i, \Pi_i \subset W \subset \Pi$, and consider solutions starting from the hypersurface Π_i . We may assume that Π_i satisfies the transversality condition similar to that imposed on Π . This is true because transversality conditions are generic conditions; so we may make an arbitrary small change to Π such that all the transversality conditions are satisfied. It follows from Theorem 4.4 and Lemmas 5.4 and 5.5 that, again, a wedge W_i can be found, and W_i has its vertex at a subarc of ι_1 having x_i in its interior. Note that W_i has a property similar to that of W : for the orbits starting from W_i, m_3 escapes from m_4 and m_5 with a large velocity and catches up with m_1 and m_2 in a very short time. Let $\tilde{t}_3(x), x \in W_i$ be the time when m_3 reaches midpoint between m_1 and m_2 . The difference $\tilde{t}_3 - \tilde{t}_2$ can be made arbitrarily small by the restriction of x to a smaller wedge in W_i .

Again, for each i , we can find infinitely many points in the wedge $W_i, x_{ij}, j = 1, 2, 3, \dots, x_{ij} \rightarrow x_i$ such that the orbit of x_{ij} ends in the simultaneous triple collisions of m_1, m_2, m_3 and binary collisions of m_4, m_5 . For all $j \in \mathbb{Z}, x_{ij}$ has similar properties to those of x_i . Continuing in this way, and given any infinite sequence of positive integers, $i_1 i_2 i_3 \dots$, we can find a sequence of wedges:

$$\dots \subset W_{i_1 i_2 i_3} \subset W_{i_1 i_2} \subset W_{i_1} \subset W$$

and define a sequence of time:

$$\tilde{t}_1(x) < \tilde{t}_2(x) < \tilde{t}_3(x) < \dots$$

for all points of

$$W \cap W_{i_1} \cap W_{i_1 i_2} \cap W_{i_1 i_2 i_3} \cap \dots$$

In addition $\tilde{t}_{i+1}(x) - \tilde{t}_i(x)$ can be arbitrarily small for any $i \in \mathbb{Z}$. Thus we may assume that

$$\tilde{t}_{i+1}(x) - \tilde{t}_i(x) \leq 2^{-i}$$

whenever $\tilde{t}_{i+1}(x)$ is defined.

We conclude the proof of Theorem 1.1 by using symbolic dynamics. For the infinite sequence of positive integers $i_1 i_2 i_3 \dots$, by the above construction we have a corresponding infinite sequence of wedges

$$\dots \subset W_{i_1 i_2 i_3} \subset \overline{W}_{i_1 i_2 i_3} \subset W_{i_1 i_2} \subset \overline{W}_{i_1 i_2} \subset W_{i_1} \subset \overline{W}_{i_1} \subset W,$$

where \bar{V} is the closure of the set V . This nested sequence has a nonempty intersection. Let \mathbf{q}^* be a point in this intersection. Then

$$\mathbf{q}^* \in W_{i_1 i_2 i_3 \dots}$$

and $\tilde{t}_i(\mathbf{q}^*)$ is defined for all $i = 1, 2, 3, \dots$,

$$\tilde{t}_1(\mathbf{q}^*) < \tilde{t}_2(\mathbf{q}^*) < \tilde{t}_3(\mathbf{q}^*) < \dots$$

Since $\tilde{t}_{i+1}(\mathbf{q}^*) - \tilde{t}_i(\mathbf{q}^*) \leq 2^{-i}$, the sequence has a limit t_∞ :

$$t_\infty = \lim_{i \rightarrow \infty} \tilde{t}_i(\mathbf{q}^*) \leq \tilde{t}_1 + \sum_{i=1}^{\infty} 2^{-i} < \infty.$$

Therefore, for the orbit starting from \mathbf{q}^* , the solution is defined for all t , $0 \leq t < t_\infty < \infty$. To prove that

$$(5.10) \quad z_1(t) = z_2(t) \rightarrow \infty \quad \text{and} \quad z_4(t) = z_5(t) \rightarrow -\infty$$

as $t \rightarrow t_\infty$, for the solution starting from \mathbf{q}^* we could use von Zeipel’s theorem, given in the Introduction. Since now $t_\infty < \infty$ is a singularity and, from the construction, t_∞ is not a collision singularity, therefore $\lim I \rightarrow \infty$ as $t \rightarrow t_\infty$, where I is the moment of inertia of the total system. (We must point out, however, that as the binary collisions have been regularized in our problem, von Zeipel’s theorem cannot be directly applied here. But it is not hard to extend von Zeipel’s theorem to allow a regularization of binary collisions and to treat solutions with binary collisions as regular solutions.) Here we choose to use some estimates that were derived previously to prove equation (5.10). From the third property of Theorem 4.4, for any sequence of positive numbers $\epsilon_1, \epsilon_2, \dots$, we may make $W, W_i, W_{i_1 i_2}, \dots$ small enough such that the following are satisfied:

$$\begin{aligned} z_1(\mathbf{q}^*, \tilde{t}_2) &\geq K z_1(\mathbf{q}^*, \tilde{t}_1), \\ z_1(\mathbf{q}^*, \tilde{t}_3) &\geq z_1(\mathbf{q}^*, \tilde{t}_2) - \epsilon_1, \\ z_1(\mathbf{q}^*, \tilde{t}_4) &\geq K z_1(\mathbf{q}^*, \tilde{t}_3), \\ z_1(\mathbf{q}^*, \tilde{t}_5) &\geq z_1(\mathbf{q}^*, \tilde{t}_4) - \epsilon_2, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

where $K > 1$ is a constant. For any $1 < K_0 < K$, one easily sees that if $\epsilon_1, \epsilon_2, \dots$ are small enough, then

$$z_1(\mathbf{q}^*, \tilde{t}_{2n+2}) \geq K_0 z_1(\mathbf{q}^*, \tilde{t}_{2n}) \geq K_0^n z_1(\mathbf{q}^*, \tilde{t}_2)$$

for all $n > 0$. Therefore

$$z_1(\mathbf{q}^*, t) = z_2(\mathbf{q}^*, t) \rightarrow \infty \quad \text{as } t \rightarrow t_\infty.$$

And likewise

$$z_4(\mathbf{q}^*, t) = z_5(\mathbf{q}^*, t) \rightarrow -\infty \quad \text{as } t \rightarrow t_\infty.$$

Let Λ be the collection of all of such points; i.e., for any $\mathbf{q}^* \in \Lambda$ there is a corresponding infinite sequence of positive integers, i_1, i_2, \dots , such that

$$\mathbf{q}^* \in \bigcap_{n=1}^{\infty} W_{i_1 i_2 \dots i_n}.$$

The orbits starting from Λ are unbounded solutions in finite time.

It is obvious that Λ is uncountable. This completes the proof of Theorem 1.1. □

For later use, we state some of the properties of the unbounded solutions in finite time constructed here. Since we can always choose some smaller wedges in the construction of Λ if necessary, we may assume that the following properties hold for all $x \in \Lambda$:

- (1) $|\bar{t}_{n+1} - \bar{t}_n| \leq 2^{-2n}$ for all $n > 0$;
- (2) $|\dot{z}(\bar{t}_n)/\dot{z}_3(\bar{t}_{n+1})| \leq 2^{-n}$ for all $n > 0$;
- (3) $h_{123}(t) \gg 0$ whenever $z_3 > 0$, and $h_{345}(t) \gg 0$ whenever $z_3 < 0$.

Therefore the gradient-like property holds when we consider the subsystem near triple collision.

We finish the proof of Theorem 1.2 in the next section.

6. Noncollision singularities

In the last section we constructed uncountably many unbounded solutions in finite time on the 3-dimensional hypersurface Π . As we noticed, the binary collisions between m_1 and m_2 and m_4 and m_5 are all regularized. Therefore it is possible that our solutions involve binary collisions. The main purpose of this section is to show that, for some of the solutions constructed in the last section, there is indeed no binary collision involved.

We prove our second main result, Theorem 1.2, by proving a series of lemmas. Lemma 6.1 states for the unbounded solution constructed in the last section that the eccentricities of both the binaries m_1 and m_2 and m_4 and m_5 approach one. In other words, let $\mathbf{q}^* \in \Lambda$ and let t_∞ be the time when the solution ends. Then $w_{12}(\mathbf{q}^*(t)) \rightarrow 0$ as $t \rightarrow t_\infty$. This means that the solutions of the binaries are closer to a collinear problem. Lemma 6.2 asserts that the elliptic axes of binaries m_1, m_2 and m_4, m_5 also have limits as $t \rightarrow t_\infty$. Finally in Lemma 6.3 we prove that if these two limit axes form an angle other than $0, \pi/2, 3\pi/2$ or π , then for all t sufficiently close to t_∞ , the angular momentum for one pair, say m_1 and m_2 , is positive and the angular momentum for the other pair, m_4 and m_5 , is negative. That is, for t sufficiently close to t_∞ , the

angular momentum assigned to any pair does not change sign. This guarantees that no binary collision is possible for t sufficiently close to t_∞ . Lemma 6.3 is the most important lemma of this section. Its proof can be intuited as follows: as $t \rightarrow t_\infty$, move m_1, m_2 and m_4, m_5 closer and closer to the respective limit axes, and this is true for a large percentage of the time. Thus the change in the angular momentum is mostly in one direction, either decreasing or increasing. So, in some sense, the angular momentum for either binary behaves more and more like a monotonic function so that it remains away from zero when t is sufficiently close to t_∞ . From these lemmas we will see that if we carefully choose our initial conditions, we will have the noncollision singularity.

LEMMA 6.1. *Let $\mathbf{q}^*(t)$ be an unbounded solution in finite time, as constructed in the last section. Then $w_{12}(\mathbf{q}^*(t)) \rightarrow 0$ and $w_{45}(\mathbf{q}^*(t)) \rightarrow 0$ as $t \rightarrow t_\infty$, where $t_\infty < \infty$ is the time when solution $\mathbf{q}^*(t)$ ends.*

Proof. We first prove that $w_{12}(\mathbf{q}^*(\bar{t}_n)) \rightarrow 0$ as $n \rightarrow \infty$, where \bar{t}_n is as defined in the last section. For later use, we will also show that

$$|w_{12}(\mathbf{q}^*(\bar{t}_n))| \leq 2^{-n} M_1$$

for some positive constant M_1 , for all $n = 1, 2, \dots$

By construction of $\mathbf{q}^*(t)$,

$$(6.1) \quad |w_{45}(\mathbf{q}^*(\bar{t}_{2n}))| < 2w^+$$

and

$$|w_{45}(\mathbf{q}^*(\bar{t}_{2n}))| = |c_{45}^2(\bar{t}_{2n})h_{45}(\bar{t}_{45})| \geq c_1 c_{45}^2 |\dot{z}_3(\bar{t}_{2n})|$$

for some $c_1 > 0$. Hence

$$(6.2) \quad c_{45}^2(\bar{t}_{2n}) \leq \frac{2w^+}{|c_1 \dot{z}_3^2(\bar{t}_{2n})|}.$$

On the other hand, there is a $c_2 > 0$ such that

$$\begin{aligned} |w_{12}(\mathbf{q}^*(\bar{t}_{2n}))| &= |c_{12}(\bar{t}_{2n})h_{12}(\bar{t}_{2n})| \\ &\leq c_2 c_{12}^2(\bar{t}_{2n}) |\dot{z}_3^2(\bar{t}_{2n-1})| \leq 2w^+ \frac{c_2}{c_1} \frac{|\dot{z}_3^2(\bar{t}_{2n-1})|}{|\dot{z}_3^2(\bar{t}_{2n})|} \leq 2^{-2n} M_1. \end{aligned}$$

Here we use the fact that $c_{12} = -c_{45}$. Therefore

$$|w_{12}(\mathbf{q}^*(\bar{t}_{2n}))| \leq 2^{-2n} M_2$$

for some $M_2 > 0$.

To show that

$$|w_{12}(\mathbf{q}^*(\bar{t}_{2n-1}))| \leq 2^{-2n+1} M_3$$

for some $M_3 > 0$, notice that $\dot{w}_{12} \leq c_3$ for some $c_3 > 0$ and for all $t \in [\bar{t}_{2n-1}, \bar{t}_{2n}]$. We may assume that $|\bar{t}_{2n} - \bar{t}_{2n-1}| < 2^{-2n}c_4$ for some $c_4 > 0$. Therefore

$$(6.3) \quad |w_{12}(\mathbf{q}^*(\bar{t}_n))| < 2^{-n}M_1$$

for some $M_1 > 0$ and for all $n > 0$. A similar result holds for the pair m_4 and m_5 :

$$(6.4) \quad |w_{45}(\mathbf{q}^*(\bar{t}_n))| \leq 2^{-n}M_4$$

for some $M_4 > 0$ and for all $n > 0$.

Instead of continuing to prove Lemma 6.1, we will use these inequalities to prove our next lemma and then return to Lemma 6.1. A complete proof of Lemma 6.1 will be given later.

Let us consider a 2-body problem, m_1 and m_2 moving in \mathbb{R}^2 under Newton’s law, and let \mathbf{r}_{12} be their mutual distance. The equation of motion is

$$\frac{d^2\mathbf{r}_{12}}{dt^2} = -\frac{m_1 + m_2}{r_{12}^3}\mathbf{r}_{12}, \quad \mathbf{r}_{12} \in \mathbb{R}^2.$$

Besides the usual energy and angular-momentum integral, this equation also admits another integral. One easily checks that

$$(6.5) \quad \begin{aligned} \mathbf{p}_{12} &= \frac{\mathbf{r}_{12}}{r_{12}} + \frac{1}{m_1 + m_2}(\mathbf{c}_{12} \times \mathbf{r}_{12}) \\ &= \frac{\mathbf{r}_{12}}{r_{12}} + \frac{1}{m_1 + m_2} \left(\mathbf{r}_{12} \times \frac{d\mathbf{r}_{12}}{dt} \right) \times \mathbf{r}_{12} \end{aligned}$$

is a constant of motion. The value of \mathbf{p} has some important physical meaning: if m_1 and m_2 move in some elliptic orbits, then the vector \mathbf{p} points toward the perigee of elliptic orbit, and the magnitude of \mathbf{p} is exactly the eccentricity of the ellipse. See Pollard [13] for a more detailed discussion.

For the solutions constructed in the last section we note that, for most of the time, m_1, m_2 are far away from the rest of the particles m_3, m_4 and m_5 . This suggests that we treat the motion of m_1 and m_2 as a perturbation problem. Motivated by the integral \mathbf{p} in the 2-body problem, we define a vector \mathbf{p}_{12} similarly to \mathbf{p} above. Let

$$(6.6) \quad \mathbf{p}_{12} = \frac{(x_1, y_1, 0)}{(x_1^2 + y_1^2)^{1/2}} + \frac{4}{m_1}(\mathbf{c}_{12} \times (\dot{x}_1, \dot{y}_1, 0)),$$

where

$$\mathbf{c}_{12} = (0, 0, x_1\dot{y}_1 - y_1\dot{x}_1).$$

At first glance, our definition for \mathbf{p}_{12} depends on r . However, if we write \mathbf{p}_{12} in McGehee’s coordinates, then the r factor cancels out ($r_{12} \sim O(r)$),

$\dot{r}_{12} \sim O(r^{-1/2})$). Therefore \mathbf{p}_{12} is well defined even on the triple-collision manifold.

LEMMA 6.2. *Let $\mathbf{q}^*(t)$ be an unbounded solution in finite time, as constructed in the last section, and let t_∞ be the time when the solution $\mathbf{q}^*(t)$ ends. Then the limits of $\mathbf{p}_{12}(\mathbf{q}^*(t))$ and $\mathbf{p}_{12}(\dot{\mathbf{q}}^*(t))$ exist and, moreover, if*

$$\lim(\mathbf{p}_{12}(\mathbf{q}^*(t))) = \mathbf{p}_{12}(\mathbf{q}^*) \text{ and } \lim(\mathbf{p}_{12}(\dot{\mathbf{q}}^*(t))) = \mathbf{p}_{12}(\dot{\mathbf{q}}^*) \text{ as } t \rightarrow t_\infty,$$

then $|\mathbf{p}_{12}(\mathbf{q}^*)| = |\mathbf{p}_{12}(\dot{\mathbf{q}}^*)| = 1$.

Proof. An easy computation shows that

$$(6.7) \quad \dot{\mathbf{p}}_{12} = -\mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) + O(|z_4|^{-2}).$$

What we want to do next is estimate the change of \mathbf{p}_{12} for the unbounded solutions constructed in the last section. Let $\Delta \mathbf{p}_{12} = \mathbf{p}_{12}(t_\infty) - \mathbf{p}_{12}(0)$. Then

$$\Delta \mathbf{p}_{12} = - \int_0^{t_\infty} \mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) dt + \int_0^{t_\infty} O(|z_4|^{-2}) dt.$$

The second integral on the right side converges, and the first integral is an improper integral, because $\inf(r_{13}) \rightarrow 0$ as $t \rightarrow t_\infty$. What we prove next is that this improper integral also converges:

$$\begin{aligned} \int_0^{t_\infty} \mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) dt &= \int_0^{t_1} + \int_{t_1}^{\bar{t}_2} + \int_{\bar{t}_2}^{t_3} \\ &\quad + \dots \left(\mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) \right) dt \\ &= \sum_{n=0}^{\infty} \int_{\bar{t}_{2n}}^{t_{2n+1}} \left(\mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) \right) dt \\ &\quad + \sum_{n=0}^{\infty} \int_{t_{2n+1}}^{\bar{t}_{2n+2}} \left(\mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) \right) dt. \end{aligned}$$

The second summation is convergent, since the integrand is bounded (note that $r \geq A$). Therefore we only need to show that $\sum_{n=0}^{\infty} a_n$ is a convergent series, where

$$(6.8) \quad a_n = \int_{\bar{t}_{2n}}^{t_{2n+1}} \left(\mathbf{c}_{12} \times \mathbf{r}_{12} \left(\frac{m_1 m_3}{4} r_{13}^{-3} \right) \right) dt.$$

Using the substitution of variables $\tau = \tau(t)$ defined by $dt = r^{3/2} d\tau$, and letting $\bar{\tau}_n = \tau(\bar{t}_n)$ and $\tau_n = \tau(t_n)$, we have

$$a_n = \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}} \left(\frac{m_1 m_3}{4} \frac{(0, 0, u) \times (\cos \phi \cos \theta, \sin \phi \sin \theta, 0)}{(1 + 2\alpha \sin^2 \phi)^{3/2}} \right) d\tau.$$

Therefore

$$(6.9) \quad |a_n| \leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |u \cos \phi|}{4} d\tau.$$

Let $\tau_{2n+1}^0 \in (\bar{\tau}_{2n}, \tau_{2n+1})$ be the time such that $v(\tau_{2n+1}^0) = -v_0 < 0$, where v_0 is any positive number such that $-v_0 > v(E_+)$. Recall that E_+ is the rest point in the triple-collision manifold. It follows from the gradient-like property of N_+ that τ_{2n+1} is uniquely defined. Thus

$$(6.10) \quad |a_n| \leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |u \cos \phi|}{4} d\tau + \int_{\tau_{2n+1}^0}^{\tau_{2n+1}} \frac{m_1 m_3 |u \cos \phi|}{4} d\tau.$$

We will consider the above two integrals separately. For the first integral,

$$(6.11) \quad \begin{aligned} \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |u \cos \phi|}{4} d\tau &\leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |u|}{4} d\tau \\ &= \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12}|}{4r^{1/2}} d\tau \\ &= \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12}^0 + c_{12} - c_{12}^0|}{4r^{1/2}} d\tau \\ &\leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12}^0|}{4r^{1/2}} d\tau \\ &\quad + \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12} - c_{12}^0|}{4r^{1/2}} d\tau, \end{aligned}$$

where $c_{12}^0 = c_{12}(t_{2n+1}^0)$.

Following from $\dot{r} = r^{-1/2}v$, $v \leq -v_0$ for all $t \in [\bar{t}_{2n}, t_{2n+1}^0]$, we have

$$\dot{r} \leq -r^{-1/2}v_0.$$

Therefore

$$r(t) \geq (r_0^{3/2} + v_0(t_{2n+1}^0 - t))^{2/3}$$

for all $t \in [\bar{t}_{2n}, t_{2n+1}^0]$.

From $|\dot{c}_{12}| \leq c_1 r_{12} r_{45} r_{14}^{-2}$ for some $c_1 > 0$ we may assume that

$$|c_{12} - c_{12}^0| < 2^{-2n} M_1 (t_{2n+1}^0 - t)$$

for some $M_1 \geq 0$, $t \in [\bar{t}_{2n}, t_{2n+1}^0]$; therefore

$$\begin{aligned} \frac{|c_{12} - c_{12}^0|}{r^2} &\leq \frac{2^{-2n} M_1 (t_{2n+1}^0 - t)}{(r_0^{3/2} + v_0(t_{2n+1}^0 - t))^{4/3}} \\ &\leq 2^{-2n} M_1 (t_{2n+1}^0 - t)^{-1/3} \end{aligned}$$

and

$$(6.12) \quad \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12} - c_{12}^0|}{4r^{1/2}} d\tau \leq \int_{\bar{t}_{2n}}^{t_{2n+1}^0} \frac{m_1 m_3 2^{-2n} M_1 (t_{2n+1}^0 - t)^{-1/3}}{4} dt \leq M_2 2^{-2n}$$

for some $M_2 > 0$ and for all n .

For the first integral of (6.11) note that $r' \leq -rv_0$ for all $\tau \in [\bar{\tau}_{2n}, \tau_{2n+1}^0]$. We have

$$r \geq r_0 \exp(v_0(\tau_{2n+1}^0 - \tau)).$$

Therefore

$$(6.13) \quad \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |c_{12}^0|}{4r^{1/2}} d\tau \leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3 |u_0| r_0^{1/2}}{4r^{1/2}} d\tau \leq \int_{\bar{\tau}_{2n}}^{\tau_{2n+1}^0} \frac{m_1 m_3}{4} |u_0| \exp\left(-\frac{1}{2}v_0(\tau_{2n+1}^0 - \tau)\right) d\tau \leq M_3 |u_0|$$

for some $M_3 > 0$. Later on, we will show that $|u_0| \leq 2^{-n} M_9$ for some $M_9 > 0$.

We now consider the second integral in (6.10). For this we need an estimate concerning u ,

$$u' = \frac{1}{2}vu + rf(t),$$

where $f(t)$ is a function of t such that

$$|f(t)| = |\dot{c}_{12}| \leq c_1 r_{12} r_{45} r_{15}^{-2}.$$

Therefore

$$u(\tau) = \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) \left(u_0 + \int_{\tau_{2n+1}^0}^{\tau} f(t)r \exp\left(\int_{\tau_{2n+1}^0}^{\tau} \frac{1}{2}v(\tau)d\tau\right) d\tau\right),$$

where $u_0 = u(\tau_{2n+1}^0)$. Change variable τ to t in one of the above integrals with $f(t)$,

$$u(\tau) = \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) \times \left(u_0 + \int_{t_{2n+1}^0}^t f(t)r^{-1/2} \exp\left(\int_{\tau_{2n+1}^0}^{\tau} \frac{1}{2}v(\tau)d\tau\right) dt\right).$$

From that $r' = rv$ it follows that

$$r = r_0 \exp\left(\int_{\tau_{2n+1}^0}^{\tau} v(\tau)d\tau\right);$$

therefore

$$\exp\left(\int_{\tau_{2n+1}^0}^{\tau} \pm \frac{1}{2}v(\tau)d\tau\right) = \left(\frac{r}{r_0}\right)^{\pm \frac{1}{2}}.$$

Using the above equation, we obtain

$$(6.14) \quad \left| \int_{t_{2n+1}^0}^t f(t)r^{-1/2} \exp\left(\int_{\tau_{2n+1}^0}^{\tau} \frac{1}{2}v(\tau)d\tau\right) dt \right| = \left| \int_{t_{2n+1}^0}^t f(t)r^{-1/2} \frac{r^{1/2}}{r_0^{1/2}} dt \right| \leq \max |f(t)|r_0^{-1/2}|t - t_0|.$$

It follows from the proof of Lemma 4.2 that there is a constant d_5 such that

$$r_0^{-1/2}|t_{2n+1} - t_{2n+1}^0| \leq d_5$$

for all $n > 0$. On the other hand, for all $t \in [t_{2n+1}^0, t_{2n+1}]$ we may assume that

$$\max |f(t)| \leq c_2r_{12}r_{45}r_{15}^{-1} \leq 2^{-2n}M_4$$

for some $M_4 > 0$ and all $n > 0$. Therefore

$$\int_{t_{2n+1}^0}^{t_{2n+1}} \frac{|f(t)|}{r_0^{1/2}} dt \leq 2^{-2n}M_5$$

for some $M_5 > 0$ and all n . Thus we have the second integral of formula (6.10)

$$(6.15) \quad \begin{aligned} & \int_{\tau_{2n+1}^0}^{\tau_{2n+1}} \frac{m_1m_3|u \cos \phi|}{4} d\tau \\ & \leq \int_{\tau_{2n+1}^0}^{\tau_{2n+1}} \frac{m_1m_3|u|}{4} d\tau \\ & \leq \int_{\tau_{2n+1}^0}^{\tau_{2n+1}} (|u_0| + 2^{-2n}M_5) \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) d\tau \\ & \leq (|u_0| + 2^{-2n}M_5) \int_{\tau_{2n+1}^0}^{\tau_{2n+1}^*} \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) d\tau \\ & \quad + (|u_0| + 2^{-2n}M_5) \int_{\tau_{2n+1}^*}^{\tau_{2n+1}} \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) d\tau, \end{aligned}$$

where $\tau_{2n+1}^*(x)$ is the value of τ when the orbit of x reaches the solid cylinder T .

We want to show that both of the above integrals are bounded. Similar to the argument used in proving Lemma 4.2, we see that

$$(6.16) \quad |\tau_{2n+1}^* - \tau_{2n+1}^0| \leq M_6,$$

$$(6.17) \quad \left| \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) \right| \leq M_7$$

for all $\tau \in [\tau_{2n+1}^0, \tau_{2n+1}^*]$, where M_6 and M_7 are positive constants.

For the second integral in (6.15) we use the same argument as that used to obtain inequality (6.13). Note that

$$(6.18) \quad \exp\left(\int_{\tau_{2n+1}^0}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) = \exp\left(\int_{\tau_{2n+1}^0}^{\tau_{2n+1}^*} -\frac{1}{2}v(\tau)d\tau\right) \exp\left(\int_{\tau_{2n+1}^*}^{\tau} -\frac{1}{2}v(\tau)d\tau\right).$$

The first term above is bounded. For the second term, following from that $v(\tau) \geq v^+$ for $\tau \geq \tau_{2n+1}^*$, we have

$$\exp\left(\int_{\tau_{2n+1}^*}^{\tau} -\frac{1}{2}v(\tau)d\tau\right) \leq \exp\left(-\frac{1}{2}v^+(\tau - \tau_{2n+1}^*)\right).$$

Together with inequalities (6.15), (6.16) and (6.17) this yields

$$(6.19) \quad \int_{\tau_{2n+1}^0}^{\tau_{2n+1}^*} \frac{m_1 m_3 |u \cos \phi|}{4} d\tau \leq (|u_0| + 2^{-2n} M_5) M_8$$

for some $M_8 > 0$ and all $n > 0$. Therefore it follows from inequalities (6.10), (6.11), (6.12) and (6.13) that

$$(6.20) \quad \begin{aligned} |a_n| &\leq (|u_0| + 2^{-2n} M_5) M_8 + 2^{-2n} M_2 + M_3 |u_0| \\ &= (M_3 + M_8) |u_0| + 2^{-2n} (M_2 + M_5 M_8). \end{aligned}$$

In order to show that $\sum |a_n|$ is a convergent series, we only have to show that

$$(6.21) \quad |u_0| \leq 2^{-2n} M_9$$

for some $M_9 > 0$. For this we use inequality (6.3): $|w_{12}(\bar{t}_n)| \leq 2^{-n} M_1$. Recall from the definition that $|w_{12}| = |h_{12} c_{12}^2|$; therefore

$$(6.22) \quad |c_{12}^2(\bar{t}_n) h_{12}(\bar{t}_n)| \leq 2^{-n} M_1.$$

Next we derive some estimates for $c_{12}(\bar{t}_n)$ in terms of u_0 . Again let t_{2n+1}^* , $t_{2n+1}^* \in [\bar{t}_{2n}, t_{2n+1}]$ be the time such that the orbit reaches T , and let $r(t_{2n+1}^*) = r_*$. It follows from equation (6.14) that

$$(6.23) \quad \begin{aligned} |c_{12}(\bar{t}_{2n})| &= |r^{1/2}(\bar{t}_{2n}) u(\bar{t}_{2n})| \\ &= r^{1/2}(\tau_{2n+1}^0) |u_0 + s(\bar{t}_{2n})| \\ &\geq M_{10} r_*^{1/2} |u_0 + s(\bar{t}_{2n})|, \end{aligned}$$

where

$$s(t) = \int_{t_{2n+1}^0}^t \exp\left(\int_{\tau_{2n+1}^0}^{\tau} \frac{1}{2}v(\tau)d\tau\right) f(\tau) r^{-1/2} dt.$$

The inequality in (6.23) comes from

$$\frac{r_*}{r(\tau_{2n+1}^0)} = \exp \int_{\tau_{2n+1}^0}^{\tau_{2n+1}^*} \frac{1}{2} v(\tau) d\tau \leq M_{10}^{-2}$$

for all x such that $x(t_{2n+1}^*) \in T$ and some $M_{10} > 0$. Here we used inequality (6.16). Note that it follows from equation (6.14) that

$$|s(t)| \leq 2^{-2n} M_5 \quad \text{for all } t \in [t_{2n+1}^0, t_{2n+1}^*].$$

On the other hand, as in the proofs of Lemma 4.2 and 4.3, one can easily see that there are $d_7 > 0$ and $d_8 > 0$ such that, for r_* sufficiently small,

$$(6.24) \quad d_7 \leq \frac{|h_{12}(\bar{t}_{2n})|}{|h_{12}(t_{2n+1}^*)|} \leq d_8$$

for all x such that $x(t_{2n+1}^*) \in T$. Combining formulas (6.22), (6.23) and (6.24), we have

$$2^{-(2n+1)} M_1 \geq |w_{12}(\bar{t}_{2n})| \geq d_7 |h_{12}(t_{2n+1}^*)| M_{10}^2 r_* |u_0 + s(\bar{t}_{2n})|.$$

Therefore

$$|u_0 + s(\bar{t}_{2n})|^2 \leq \frac{2^{-(2n+1)} M_1 d_7^{-1} M_0^{-2}}{|r_* h_{12}(t_{2n+1}^*)|}.$$

To prove inequality (6.21), it follows from $|s(\bar{t}_{2n})| \leq 2^{-2n} M_5$ that we only need to show that

$$|r_* h_{12}(t_{2n+1}^*)| > M_{11}$$

for some $M_{11} > 0$ and for all x such that $r(t_{2n+1}^*) \in T$. Since T_0 is a compact set, it suffices to show that, for any $x \in T$,

$$(6.25) \quad |rh_{12}(x)| \neq 0.$$

Notice that $rh_{12}(x)$ does not depend on the value of r and, in McGehee's coordinates, that

$$|rh_{12}(x)| = \left| g - \frac{1}{2} v^2 \sin^2 \phi - w^2 + 2vw \sin \phi - m_1^{3/2} m_3 (1 + 2\alpha \sin \phi)^{-1/2} \right|.$$

For $x \in M$, where M is the triple-collision manifold, $g(x) = 0$. Similar to the argument used in proving Lemma 2.1, we see that $rh_{12}(x) \neq 0$ for all $x \in M$. By the continuity of $rh_{12}(x)$ and the compactness of $T_0 \cap M$, statement (6.25) is true provided that g is sufficiently small. Therefore $|rh_{12}(x)| \neq 0$, if T_0 is made small enough. Since we can always choose T_0 as small as desired, we may assume that this is true. Therefore inequality (6.21) holds; i.e., $|u_0| \leq 2^{-2n} M_9$ for some $M_9 > 0$. Thus $\sum a_n$ is a convergent series and, therefore, the limit of $\mathbf{p}_{12}(\mathbf{q}^*(t))$ exists as $t \rightarrow t_\infty$.

From $1/2w_{12} = (1 - |\mathbf{p}_{12}|^2)^{1/2}$ and $w_{12}(t_n) \rightarrow 0$ as $n \rightarrow \infty$, we have $|\mathbf{p}_{12}(\mathbf{q}^*(t_n))| \rightarrow 1$ as $n \rightarrow \infty$. Because the limit of $\mathbf{p}_{12}(\mathbf{q}^*(t))$ exists as $t \rightarrow t_\infty$, then $|\mathbf{p}_{12}(\mathbf{q}^*(t))| \rightarrow 1$ as $t \rightarrow t_\infty$. This proves Lemma 6.2. \square

Observe that

$$w_{12} = 2(1 - |\mathbf{p}_{12}|^2)^{1/2} \rightarrow 0.$$

Thus $w_{12}(\mathbf{q}^*(t)) \rightarrow 0$ as $t \rightarrow t_\infty$, which proves Lemma 6.1, as promised. \square

Now we can state the following important lemma, which concludes proving the existence of the noncollision singularity.

LEMMA 6.3. *Let $\mathbf{q}^*(t)$ be an unbounded solution in finite time, as constructed in the last section. Also let $\mathbf{p}_{12}^*, \mathbf{p}_{45}^*$ be the same as those of Lemma 6.2. If $\mathbf{p}_{12}^* \cdot \mathbf{p}_{45}^* \neq 0$ and $\mathbf{p}_{12}^* \neq \pm \mathbf{p}_{45}^*$, i.e., if \mathbf{p}_{12}^* and \mathbf{p}_{45}^* are neither parallel nor perpendicular to each other, then there exists a $t^a < t_\infty$ such that, for all $t \in (t^a, t_\infty)$, $c_{12} \neq 0$. In other words, there is no binary collision between m_1, m_2 and m_4, m_5 in the time interval (t^a, t_∞) .*

Proof. The proof of this lemma is easier than one might expect. First we compute that

$$\dot{c}_{12} = 2m_1m_4(x_1y_4 - x_4y_1)(r_{14}^{-3} - r_{15}^{-3}).$$

Let θ be the angle between the two vectors \mathbf{r}_{12} and \mathbf{r}_{45} ,

$$\cos \theta = \frac{x_1x_4 + y_1y_4}{(x_1 + y_1)^{1/2}(x_4 + y_4)^{1/2}}.$$

Then

$$\begin{aligned} \dot{c}_{12} &= \frac{1}{2}m_1m_4r_{12}r_{45} \sin \theta \frac{3}{2}r_{12}r_{45} \cos \theta \left((z_1 - z_4)^{-6} \right. \\ (6.26) \quad &\quad \left. + O((z_1 - z_4)^{-9}) \right) \\ &= \frac{3}{8}m_1m_4r_{12}^2r_{45}^2 \sin 2\theta \left((z_1 - z_4)^{-6} + O((z_1 - z_4)^{-9}) \right). \end{aligned}$$

Let $\mathbf{q}^*(t)$ be an unbounded solution, as constructed in the last section. It follows from Lemma 6.2 that $\mathbf{p}_{12}(\mathbf{q}^*(t))$ and $\mathbf{p}_{45}(\mathbf{q}^*(t))$ have limits \mathbf{p}_{12}^* and \mathbf{p}_{45}^* as $t \rightarrow t_\infty$, where t_∞ is the time when the solution $\mathbf{q}^*(t)$ ends. Let θ^* be the angle between \mathbf{p}_{12}^* and \mathbf{p}_{45}^* . By the assumption of the lemma,

$$\sin 2\theta \neq 0.$$

Assuming that $\sin 2\theta > 0$, we find that the case with $\sin 2\theta < 0$ can be treated equally. Let ω be the angle between $\mathbf{p}_{12}(\mathbf{q}^*(t))$ and $\mathbf{p}_{45}(\mathbf{q}^*(t))$. Then

$$\omega(t) \rightarrow \theta^* \quad \text{as } t \rightarrow t_\infty.$$

Let $t^b < t_\infty$ be the time such that, for all $t \in (t^b, t_\infty)$,

$$\sin 2\omega(t) > 0$$

and also

$$(z_1 - z_4)^{-6} + O((z_1 - z_4)^{-9}) > 0,$$

where the high-order term $O((z_1 - z_4)^{-9})$ is from that of equation (6.26). This is possible since $(z_1 - z_4) \rightarrow \infty$ as $t \rightarrow t_\infty$.

Now we have two cases:

(a) If $c_{12} \neq 0$ for all $t \in (t^b, t_\infty)$, then we take $t^b = t^a$, thus proving the lemma.

(b) If for some $t = t^a$, $t^a \in (t^b, t_\infty)$, there is $c_{12}(t^a) = 0$, then we shall show for all $t \in (t^a, t_\infty)$ that $c_{12}(t) \neq 0$, and again the lemma is proved.

However note that, by definition,

$$\mathbf{p}_{12} = \frac{\mathbf{r}_{12}(t^a)}{r_{12}(t^a)} + \frac{4(\mathbf{c}_{12} \times (\dot{x}_1, \dot{y}_1, 0))}{m_1}.$$

Since $\mathbf{c}_{12}(t^a) = 0$, we have $\mathbf{p}_{12}(t^a) = \mathbf{r}_{12}(t^a)r_{12}^{-1}(t^a)$; therefore

$$\theta(t^a) = \omega(t^a).$$

Hence

$$\dot{c}_{12}(t^a) = \frac{3}{8}m_1m_4r_{12}^2r_{45}^2 \sin(2\omega(t^a))((z_1 - z_4)^{-6} + h.o.t) > 0,$$

provided that $r_{12}(t^a) \neq 0$ and $r_{45}(t^a) \neq 0$. Therefore, for $r_{12}(t^a) \neq 0$ and $r_{45}(t^a) \neq 0$, we have the following:

- (1) $c_{12}(t) < 0$ for all $t < t^a$, $(t^a - t)$ sufficiently small; and
- (2) $c_{12}(t) > 0$ for all $t > t^a$, $(t - t^a)$ sufficiently small.

The above also holds for the case $r_{12}(t^a) = 0$ and/or $r_{45}(t^a) = 0$. Now we will prove for all $t \in (t^a, t_\infty)$ that $c_{12}(t) > 0$.

We do this by contradiction. Suppose that this is not true. Then there exists $t^c \in (t^a, t_\infty)$ such that $c_{12}(t) > 0$ for all $t \in (t^a, t^c)$ and $c_{12}(t^c) = 0$. Since $c_{12}(t^c) = 0$, it follows from the above discussion (with t^a replaced by t^c), that $c_{12}(t) < 0$ for $t < t^c$ and $t^c - t$ sufficiently small, which is a contradiction to our assumption.

This proves Lemma 6.3. □

Finally, we are in a position to prove our main result: the existence of noncollision singularities.

Proof of Theorem 1.2. From Theorem 1.1 and Lemma 6.3 we know that we only need to show that there is an uncountable subset Λ_0 of Λ such that,

for all $x \in \Lambda_0$, $\mathbf{p}_{12}^*(x)$ and $\mathbf{p}_{45}^*(x)$ are neither parallel nor perpendicular to each other.

Let x^* be the point in Theorem 1.1 such that

$$\sin(2\theta(x^*, t^*)) \neq 0,$$

where θ is the angle between $\mathbf{p}_{12}(x^*, t^*)$ and $\mathbf{p}_{45}(x^*, t^*)$. The existence of such a point is obvious. The proof of Lemma 6.2 leads to the idea that, for any $\epsilon > 0$, we may select the wedges $W, W_i, W_{ij}, W_{ijk}, \dots$ small enough such that, for all $x \in \Lambda_0 = W \cap W_i \cap W_{ij} \cap W_{ijk} \cap \dots$,

$$|\mathbf{p}_{12}(x^*, t^*) - \mathbf{p}_{12}(x, t)| < \epsilon$$

and

$$|\mathbf{p}_{45}(x^*, t^*) - \mathbf{p}_{45}(x, t)| < \epsilon$$

for all t such that $\tilde{t}_2 \leq t \leq t_\infty$. Therefore we can make $\epsilon > 0$ small enough that

$$\sin 2\theta(x, t_\infty) \neq 0$$

for all $x \in \Lambda_0$.

Theorem 1.2 now follows from Lemma 6.3. □

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(Received April 11, 1989)

(Revised November, 1991)