Convinced of this essential independence of the two types of force, the one acting on the mass of the conductor in the presence of currents, and the other on what we call "electricity" itself, Maxwell regularly distinguishes them by name as, respectively, the *mechanical* and the *electromotive* forces, both exerted by the electromagnetic field. One well-known consequence of this insistence of Maxwell's was that Oliver Lodge was, by his own account, discouraged from discovery of the Hall effect.[29] A note on the Hall effect (discovered in 1880, the year after Maxwell's death), was added at this point to the third edition of the *Treatise* by J. J. Thomson. In the Hall effect, a potential is produced in a solid conductor precisely by the action which Maxwell denied, that is, by the action of the forces discussed in this chapter on the current itself rather than on the mass of the conductor.

A further discussion of these two supposed types of forces will arise in connection with our discussion of the general dynamical theory of electromagnetism, in Chapter 6, below.

AMPÈRE'S THEORY IN THE *TREATISE*

Having given us an account of the mechanical force between currents, shaped to Faraday's concepts, Maxwell now, in the second chapter of Part iv, gives an entirely different theory of the same thing. It is more than likely that he felt obliged to include Ampère's theory in the *Treatise*; it had been a standard Cambridge topic since the publication of Murphy's treatise for Cambridge students in 1833.[30] Beyond that, it was a complete and elegant theory, virtually universally accepted, and capable of accounting for all of the known phenomena. It had furnished the foundation for such extensions as Neumann's and Weber's, through which Faraday's discovery of electromagnetic induction and the phenomena of diamagnetism had been successively integrated into a single connected account. Maxwell no doubt felt a responsibility to insure the literacy of Cambridge students in this action-at-a-distance theory. It is significant, at the same time, that he placed the account in Faraday's terms first, to introduce the ideas of field theory firmly in the reader's mind at the outset.

My own impression is that Maxwell would have included his chapter on Ampère's theory even if he had not been required by circumstances to do so. As I have suggested, he was eager to place alternative types of theory before the reader in order to make the significance of Faraday's method clearer, and no doubt also to give the student freedom of choice in an area

29. Whittaker, *op. cit.*, p. 289.

30. Robert Murphy, *Elementary Principles of Electricity, Heat, and Molecular Actions: Part I, On Electricity* (Cambridge: 1833). Maxwell began his studies with "a little antipathy to Murphy's Electricity" (Larmor, *Origins*, p. 3).

of science whose future was very much in doubt. Furthermore, he was in possession of a joke which he probably could not resist sharing. It is at once a joke on Ampère, and a telling consequence of the use of excessively formal methods in mathematical physics. Ampère had claimed, in the very title of the treatise which constitutes the ultimate statement of his theory, a special *methodological* victory. Not only had he achieved a completely successful mathematical theory of electrodynamics, but he proudly claimed to have *deduced* it "uniquely" [*uniquement*] from the phenomena.[31] In this, as in other respects, Ampère is following the strict tradition of the *Principia* as closely as possible: Newton likewise claims, not to *induce*, but to *deduce* the law of gravity from the phenomena. Given the laws of force and the propositions which follow from them in Book I of Newton's *Principia*, together with the experimental evidence concerning the lines of apsides of the planetary orbits summarized briefly at the beginning of "The System of the World," the inverse-square law of gravity indeed follows as the necessary consequence of a deductive argument. In the same way, Ampère was confident that his law of electrodynamics followed inevitably from an elegant set of experimental results, together with certain undeniable principles.

Maxwell's presentation of Ampère's theory in the *Treatise* is not the same as Ampère's, and the difference reveals the joke: not one unique law, but an infinite variety of possible force laws emerge rather ridiculously from Ampère's argument as Maxwell reconstructs it. Maxwell by no means feels that he has introduced this difficulty through his reformulation, but rather that he has revealed a problem that Ampère had not acknowledged. Maxwell certainly takes satisfaction in this perplexity, which results, he is convinced, from the artificiality of Ampère's methods.

In order to follow Maxwell properly, it will be necessary to review the theory very briefly as Ampère himself presents it, and then to outline the argument which Maxwell gives. To proceed as economically as possible, I shall put Ampère's argument into the terms of Maxwell's figures and symbols from the start, and I shall give the argument only in schematic outline.[32]

As we have noted earlier (page 118n, above), an issue was drawn between Ampère and others, among them Faraday, on the question of electromagnetic rotations. Ampère's theory, true to its Newtonian paradigm, allows only forces which act in direct lines between the elements of circuits; the circular form of the Oersted phenomenon, on the other hand, did not appear to be reducible to such linear forces. Ampère soon proved himself able to derive these circularly-directed forces from his own

31. See page 18n, above.

32. For a more complete outline of Ampère's theory, see Tricker, *Early Electrodynamics*, pp. 42 ff.

principles, but the question of rotational effects nonetheless took on special interest partly because of the challenge it seemed to offer to a theory which took linear forces as fundamental. Faraday's first triumph in the science of electromagnetism was in the demonstration of continuous electromagnetic rotations, and for him, as for Maxwell, they are highly important as clues to the circular configuration of the field. Maxwell's analysis of the Faraday rotator, in terms of the concepts of Chapter I which we have just reviewed, is particularly interesting. Therefore, after we have derived Ampère's force law both by his own methods and by Maxwell's, we will turn to this question of electromagnetic rotations, reviewing its history briefly, and then examining the account of it which Maxwell gave in Chapter I of Part iv of the *Treatise*.

We have said that the Oersted phenomenon seemed to resist inclusion in the framework of Newton's *Principia*. But there was no other mathematical physics than that of Newton, as perfected by Laplace. The issue for Ampère was thus very simple: either to find some way to include Oersted's discovery within the program of Newton and Laplace, or to abandon hope of dealing with it through strict mathematical reasoning. The extension of Newton's work to include the phenomena of electricity known up to this time had already been carried out brilliantly by Coulomb and Poisson; it fell to Ampère to carry out this seemingly impossible task for the new electromagnetic phenomenon. In the opening paragraph of his *Théorie mathématique*, Ampère places his effort squarely in relation to Newton:

> L'époque que les travaux de Newton ont marquée dans l'histoire des sciences n'est pas seulement celle de la plus importante des découvertes que l'homme ait faites sur les causes des grands phénomènes de la nature, c'est aussi l'époque où l'esprit humain s'est ouvert une nouvelle route dans les sciences qui ont pour objet l'étude de des phénomènes.[33]

This is the "road" which Ampère is resolved to follow. Specifically, this is the demand upon the Newtonian theorist:

> Newton nous a appris que cette sorte de mouvement doit, comme tous ceux que nous offre la nature, être ramenée par le calcul à des forces agissant toujours entre deux particules matérielles suivant la droite qui les joint, de manière que l'action exercée par l'une d'elles sur l'autre soit égale et opposée à celle que cette derniere exerce en même temps sur la première. ... Mais il ne suffisait pas de s'être élevé à cette haute conception, il faillait trouver suivant quelle loi ces forces varient avec la situation ... en exprimer la valeur par une formule.[34]

33. Ampère, *Théorie mathématique*, p. 1.

34. *Ibid.*, pp. 1–2.

Oersted, looking at the action of his wire upon the magnetic needle, was full of visions of vortices in a medium.[35] Ampère, speaking of the time before Newton, and hence literally referring to Descartes and the Cartesians, says:

> [P]artout où l'on voyait un mouvement révolutif, on imaginait un tourbillon dans le même sens.[36]

The remark applies, however, to Oersted as well as to Descartes. Newton had written the *Principia* to refute the Cartesian hypothesis of vortices; Ampère now sees himself called upon to perform the same task with respect to electricity and magnetism: once again to perform the Newtonian magic, to show that a rotational motion can be reduced to the operations of a rectilinear force—to dispel the vortices, and thereby to preserve mathematical intelligibility of nature.

The first step, which Ampère performed with legendary swiftness after hearing Oersted's experiment reported to the French academy, was to produce a further phenomenon.[37] In terms of a "current" (a concept which was not clear in Oersted's account), what Ampère showed was that one current exerts a force on another, quite independently of any magnets or poles. The program which was to dispel vortices was this: first, to show that the forces with which currents act on currents can in every case by analyzed as the sum of forces acting between differential elements of the currents, each of these elementary forces being strictly Newtonian in the sense Ampère had specified in the quotation above, and second, to show that magnets are equivalent to aggregates of currents flowing in small closed paths, so that the Oersted phenomenon in which currents act on magnets, as well as all other magnetic forces, is reducible to these Newtonian actions of the first type.

The interaction of currents was at first shown by Ampère with movable frames carrying currents, an apparatus that has become familiar in school laboratories ever since. But when putting his argument into rigorous form in the *Théorie mathématique*, he chose to base it rather on the smallest possible number of independent experiments of a somewhat different sort, and, he insists, on no other empirical evidence. Ampère, in listing five principal accomplishments upon his application for appointment to the Collège de France, states as the fifth:

35. Oersted, *op. cit.*, pp. 116–17. What is translated "circular motion" in this reference was rendered "*mouvements tourbillonnaires*" in the French translation of 1820; Jules Joubert, ed., *Collection de memoires relatifs a la physique vols. 2 & 3: Memoires sur l'electrodynamique* (Paris: 1885, 1887), 2, p. 6. This is a very useful collection of papers of Oersted, Ampère, Biot and Savart, et al., with valuable notes by Joubert.

36. Ampère, *op. cit.*, p. 1.

37. Ampère (1820), "The Mutual Action of Two Electric Currents," in Tricker, *op. cit.*, pp. 140 ff.

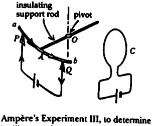


Figure 10. Ampère's Experiment III, to determine whether segment *ab* will move along its own length under the influence of current in coil C.

5. La marche qui m'a conduit à cette formule sera toujours un modèle de celle qu'on doit suivre pour arriver à de telles formules par l'expérience seulement et sans aucune supposition.[38]

Interestingly, one of those who held up Ampère's work as a model of scientific method was James David Forbes, Maxwell's teacher at the University of Edinburgh.[39]

Ampere's four basic experiments were these:[40]

Experiment I: If a wire is doubled on itself, so that the same current flows in opposite directions in adjacent conductors, the net force on a nearby current-carrying conductor is zero.[41]

Experiment II: If in the above experiment, one of the wires is given a series of tight twists or bends, which do not however take it far at any point from the original straight line, the results are as before.[42]

Experiment III: A current-carrying conductor experiences no force along its length as the result of the action of a second conductor carrying current in a closed loop (Figure 10).[43]

38. Launay, *Le grand Ampère: d'après des documents inédits* (Paris: 1925), p. 211.

39. James David Forbes, *Dissertation Sixth ... of the Progress of Mathematical and Physical Science*] (Edinburgh: [1856?]), p. 975: "He is at least as well entitled as any other philosopher who has yet appeared to be called 'the Newton of Electricity'." Forbes compares Ampère's reduction of magnets to circulating currents, as an artifice, with Newton's "fits of easy reflection and transmission" of light. Maxwell echoes Forbes' characterization of Ampère as the "Newton of electricity" at (*Tr* ii/175).

40. The four experiments are diagrammed in the *Treatise* in very poor, sometimes indeed unintelligible, reproductions of Ampère's figures from Plate I of the *Théorie mathématique*. They are adequately reproduced in Tricker, *op. cit.*, pp. 164, 168, and 171.

41. Ampère, *op. cit.*, pp. 9–10; (*Tr* ii/160).

42. *Ibid.*, pp. 10–14; (*Tr* ii/160–61).

43. *Ibid.*, pp. 14–17; (*Tr* ii/161–62).

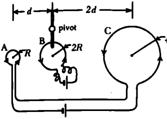


Figure 11. Ampère's Experiment IV

Experiment IV: If two current-carrying conductors lie in the same plane, the force between them will be the same whatever the scale of the apparatus, provided the currents are held constant as the scale is changed, and the geometry is kept similar to itself during the scale change (Figure 11).[44]

The null results of these four experiments are so elegant and have such an inherent plausibility that it may hardly seem necessary to perform the experiments at all; indeed, although Ampère makes no such admission, the *Théorie mathématique* closes with what must be one of the great anticlimaxes of scientific history:

> Je crois devoir observer, en finissant ce Mémoire, que je n'ai pas encore eu le temps de faire construire les instruments répresentés dans la figure [that of Experiment IV]. Les expériences auxquelles ils sont destinés n'ont donc pas encore été faites....[45]

He explains that the result has been assured by other means.

No less an experimenter than Weber, whom Maxwell very much admired, later criticized Ampère's experiments in detail and rather severely, pointing out that a null experiment is nonetheless a measurement which yields the value zero only to within certain limits of accuracy which Ampère ought to have reported; on the whole, it seems likely that Weber was right in regarding Ampère as a thinker rather than an experimenter.[46] Nevertheless, there

44. *Ibid.*, pp. 17–18; (*Tr* ii/162–63).

45. Ampère, *op. cit.*, p. 151. Tricker (*op. cit.*, pp. 46–48) calls attention to this, and describes his own version of the fourth experiment.

46. Weber terms Ampère "mehr Theoretiker als Experimentator" (*Wilhelm Weber's Werke* (6 vols.; Berlin: 1892–94), *3*, p. 213). He makes a number of comments to the same effect in the *Elektrodynamische Maasbestimmungen* of 1846. Carl Neumann, citing Weber's criticism, recommends regarding "the results of Ampère's so-called fundamental experiments not as experimental facts, but as hypotheses" and he offers a list of six such hypotheses. Carl Neumann, ed., *Franz Neumann's Gesammelte Werke* (3 vols.; Leipzig: 1912), *3*, pp. 340–41.

can be no objection to null-experiments in principle (provided they are performed!), and Ampère's experiments have as precedent the Cavendish determination of the force law for electricity, an experiment which was repeated with great precision at the Cavendish Laboratory under Maxwell.[47] Newton's Proposition 70 of Book I of the *Principia*, on which the Cavendish experiment was based, is the ultimate prototype for Ampère's reasoning in interpreting his own Experiment IV.[48] Whether it is in fact owing to experimental care on Ampère's part, or to an insight into the ways in which nature *ought* to work, the four null results are empirically quite correct, as verified by a much later repetition.[49]

Experiments I and II yield postulates of Ampère's theory directly. From I we conclude that currents can be added algebraically, and from II, that the force exerted by one current on another is equal to the vector sum of components into which the first may be resolved in any way. But Experiments III and IV are best interpreted in the context of the theory as it develops.

Ampère assumes initially that the elementary force must be proportional to the length of the element of the circuit through which the current is flowing, and that, with other factors constant, the magnitude of the force may be assumed proportional to the quantity of the current as well:

$$df \propto ii'ds\,ds',$$

where i and i' are the currents in the two elements of conductors **ds** and **ds**$'$.[50] The distance between ds and ds' is taken as unity in the above expression (thereby implying a definition of unit current).[51] Ampère takes a

47. See Maxwell's note as editor of Cavendish's electrical papers: Henry Cavendish, *The Scientific Paper of the Honourable Henry Cavendish, R.R.S.: Volume I, The Electrical Researches*, ed. James Clerk Maxwell (revised by Joseph Larmor) (Cambridge: Cambridge University Press, 1921), pp. 404–9; cf. (*Tr* i/81–86).

48. Maxwell points out that Laplace had improved upon Newton's demonstration of the basic theorem (*Tr* i/85n); it was Laplace who deduced an inverse-square law from Biot's electromagnetic experiments and it was this demonstration in turn which suggested to Ampère the direct argument from his Experiment IV which he appends to the *Théorie mathématique*, pp. 152 ff., and which Maxwell in effect repeats (*Tr* ii/162–63).

49. Von Ettinghausen, "Ueber Ampère's elektrodynamische Fundamentalversuche," *Königliche Akademie der Wissenschaften, Wien: Mathematische-Naturwissenschaftliche Klasse, Sitzungsberichte* (11) 77 (1878), p. 109. Cf. J. J. Thomson, "Report on Electrical Theories," *British Association Reports, 1885*, p. 98.

50. Ampère, *op. cit.*, p. 19. Maxwell follows Ampère unquestioningly in this same assumption (*Tr* ii/166).

51. Ampère's definition falls in naturally with his force law, but it differs from the definition of unit current in the cgs-emu system which Maxwell and the subsequent tradition adopted. Maxwell points out that "the unit current adopted in electromagnetic measure is greater than that adopted in electrodynamic measure [Ampère's] in the ratio of $\sqrt{2}$ to 1" (*Tr* ii/173).

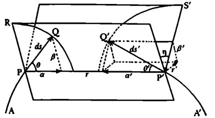


Figure 12. The geometry of Ampère's theory

positive sign as denoting attraction. It must be emphasized that this first equation is not the result of an experiment, and cannot in principle be confirmed by any experimental test at Ampère's command. Electrodynamic analysis of a circuit, perhaps unlike chemical analysis of a compound, is a purely intellectual act. One is certainly impressed by the extent to which the *Théorie mathématique* is, from this first step, *made* by Ampère, and not found in experiment.

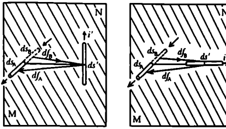
There is a major difficulty lurking in this first assertion of an elementary force: in what direction does it act? There is certainly no doubt in Ampère's mind on this point, and yet it is precisely the question which Maxwell insists on raising. Ampère asserts:

> [O]n ne peut pas concevoir cette force autrement que comme une tendance de ces deux points à se rapprocher ou à s'éloigner l'un de l'autre suivant la droite qui les joint....[52]

But there is no direct evidence in Ampère's favor; indeed, in electromagnetic experiments, we meet forces which act in very strange directions, as we have already seen.

Ampère's next step is to consider the effect of the *positions* of the current elements. For Newton, dealing with planets, this was only a question of their separation r, but here Ampère must consider as well their spatial attitudes with respect to each other—hence the far greater complexity of Ampère's mathematical problem. In Figure 12 (corresponding to Maxwell's Figure 29 (*Tr* ii/164)), the two elements **ds** and **ds′** are resolved into components in the following manner: pass the plane RP' through **ds′** and **r**. Let α and β denote the components of **ds** parallel to **r**, and perpendicular to **r**, respectively, in this plane. Then resolve **ds′** into components of α', β', and γ', respectively in the direction of **r**, perpendicular to **r** in the plane RP', and perpendicular to that plane. Finally, pass a second plane PS'

52. Ampère, *op. cit.*, p. 86. Contrast Maxwell: "[W]e shall not at first assume that their mutual action is necessarily in the line joining them" (*Tr* ii/165).



(a) (b)

Figure 13. Two instances of Ampère's crossed current elements.

through **ds'** and **r**, and let η denote the angle between the two planes; let $\theta = \measuredangle(\mathbf{ds},\mathbf{r})$ and $\theta' = \measuredangle(\mathbf{ds}',\mathbf{r})$, as shown in Figure 12.

If we assume, as both Ampère and Maxwell do without question, that the dependence on distance of separation **r** and the dependence on the spatial attitude characterized by θ, θ', and η are *independent*, then the force equation will be separable. If we represent the dependence on *attitude* by the unknown function $\rho(\theta,\theta',\eta)$, and if we assume, in the manner of Newton, that the force will be inversely proportional to some power *n* of the distance, we may write with Ampère: [53]

$$df = \frac{\rho(\theta,\theta',\eta)}{r^n} i\,i'ds\,ds'.$$

Ampère's next step is to use the right afforded by Experiment II to consider the interactions of the components of **ds** and **ds'** separately. In general, then, each component of **ds** will interact with each component of **ds'**, but Ampère argues that not all of these interactions could in fact occur in nature. He asserts:

> [U]ne portion infiniment petite de courant électrique n'exerce aucune action sur une autre portion infiniment petite d'un courant situé dans un plan qui passe par son mileur et que est perpendiculaire a sa direction.[54]

Ampère's "proof" is this (see Figure 13): let **ds'** lie in the plane perpendicular to **ds** through the latter's center. Then current *i* flowing through $\mathbf{ds}_B$ and through $\mathbf{ds}_A$ must exert opposite effects on **ds'**, one being attractive and the other repulsive, *since one flows toward the plane MN and the other flows away from it*. In the limit, the two components cancel; hence there can be no net force of this kind in nature.

53. Ampère, *op. cit.*, p. 19.
54. *Ibid.*, p. 20.

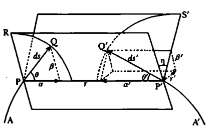


Figure 12 (repeated)

This theorem is crucial, since it very much simplifies the force equation by allowing us to disregard all possible terms due to such crossed currents. What Ampère is assuming is in effect that the world must behave the same way as its mirror-image. Suppose the current in $\mathbf{ds}_A$ is as shown in Figure 13a and that the force on it is repulsive. Now suppose that the plane *MN* is a mirror; the mirror-image of $\mathbf{ds}_A$ with its current away from the mirror would be an element $\mathbf{ds}_B$ similarly carrying current *away from the mirror.* Since $\mathbf{ds}_A$ experiences a repulsion, so would the image in the mirror. Therefore if (disregarding mirrors) we set up an experiment with a current element $\mathbf{ds}_B$ carrying current away from the plane *MN*, it (like the mirror image of $\mathbf{ds}_A$) would necessarily be observed to be repelled. If the current in it were then reversed, it would be attracted, as Ampère affirms. Maxwell endorses Ampère's assumption; he says:

> The only action possible between elements so related is a couple whose axis is parallel to *r*. (*Tr* ii/166)

The "couple" is evidently that formed by $\mathbf{df}_A$ and $\mathbf{df}_B$ as assumed by Ampère and drawn in Figure 13; Maxwell apparently allows the possibility that both forces might act, yielding a couple but no net force. But his theory does not take couples into account, so he, like Ampère, omits this term. J. J. Thomson, commenting as editor of the third edition of the *Treatise*, questions the assumption by proposing an alternative in terms of a vector cross-product relation, and then answers his own question in this way:

> The reason for assuming that such a force does not exist, is that the direction of the force would be determined merely by the direction of the currents, and not by their relative position. (*Tr* ii/166n)

By "relative position" he evidently means, their position relative to such a reference plane as *MN* in Figure 13. It was Carl Neumann who, in listing

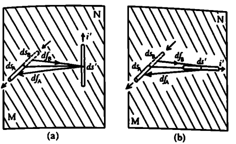


Figure 13 (repeated)

Ampère's axioms, made the parity-assumption explicit.[55]

Only two pairs of components now remain which might exert forces on one another: the collinear components α and α', and the parallel components β and β' (Figure 12). Let us denote these respectively df_1 (due to α, α'), and df_2 (due to β, β'). We have no information at all about their relative strengths, or even whether both in fact exist, although we know they both *might* act. This information is the rabbit in the hat in the third experiment.

For the present, Ampère merely introduces a constant k to measure the ratio of the force df_1 to the force df_2. He defines unit current in terms of parallel current elements, so that he writes first

$$\text{PARALLEL ELEMENTS:} \quad df_2 = \frac{ii'\beta\beta'}{r^n},$$

then

$$\text{COLLINEAR ELEMENTS:} \quad df_1 = k\frac{ii'\alpha\alpha'}{r^n}.$$

55. Carl Neumann, ed., *Franz Neumann's Werke, 3*, pp. 340–41. Note that the development of mathematical representations of spatial relations suggested alternative force laws, and was closely related to the problems of electrodynamics. This is true of Hermann Grassmann's "Ausdehnungslehre" (see Grassmann, "Neue Theorie der Elektrodynamik," *Annalen der Physik, 64* (1845), pp. 1 ff.); Grassmann remarks with some surprise that he had invented his new mathematics "... zwar ehe ich von dieser neuen Theorie eine Ahnung hatte," but it proved very appropriate (*Ibid.*, p. 11). P. G. Tait took electrodynamics as a problem upon which to exercise the art of quaternions: "Quaternion Investigations Connected with Electrodynamics and Magnetism," *Quarterly Journal of Pure and Applied Mathematics, 3* (1860), pp. 331 ff.

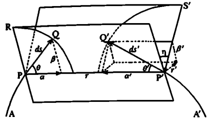


Figure 12 (repeated)

Since both forces act along the direction of **r**, they add algebraically, and we might write for the total force:

$$\mathbf{df} = \left(df_1 + df_2\right)\hat{\mathbf{e}}_r\,.$$

Substituting, and expressing the elements in terms of the angles of Figure 12 in a way which is readily confirmed:

$$\mathbf{df} = \frac{ii'}{r^2}\left(\sin\theta\cos\theta'\cos\eta - \frac{1}{2}\cos\theta\cos\theta'\right)ds\,ds'\,\mathbf{e}_r$$

Note that k and n are both unknown; these two elements of the force equation are to be determined from the information of Experiments III and IV.

Let us look only at the result of the argument by which Ampère brings Experiment III to bear on the evaluation of k and n. He transforms the force equation into an expression in terms of partial derivatives (substituting these derivatives for the corresponding trigonometric functions of the angles), and he then uses the following strategy: he evaluates the force *along the length* of a current element ds' due to a complete circuit of elements ds (let us call this df'). He gets this expression:[56]

$$df' = -\frac{1}{2}ii'ds'(1 - n - 2k)\oint\frac{\cos^2\theta'}{r^n}ds.$$

The analysis has been cut to fit the experiment, for in Experiment II a length (ds') of a conductor is pivoted so as to be free to move only along its length, and it is submitted to the action of a current in an adjacent loop. The fact that the experimental result is nil permits Ampère to set the above expression equal to zero, and since the integral is not in general zero (in the integrand, r and θ' are independent), he obtains this relation between the two coefficients:

56. Ampère, *op. cit.*, p. 26.

$$k = \frac{1-n}{2},$$

leaving only one further relation to be found.

Through an argument analogous (as has been mentioned) to that of Proposition 70, Book I, of the *Principia*, Experiment IV yields the exponent $n = 2$ in the law of force.[57] This in turn tells us that $k = -\frac{1}{2}$. We recall that k determines the relative direction and magnitude of the forces due to parallel and collinear components: the analysis has therefore revealed that collinear components contribute half as much as parallel components of the same length and carrying the same current, and collinear components repel whereas parallel attract (compare pages 32 ff., above). The force law may now be written explicitly:[58]

$$\mathbf{df} = \frac{ii'}{r^2}\left(\sin\theta\cos\theta'\cos\eta - \frac{1}{2}\cos\theta\cos\theta'\right)ds\,ds'\,\mathbf{e}_r .$$

It is convenient to use a relationship from spherical trigonometry which helps us to condense and interpret the above. If ε denotes the angle between $\mathbf{ds}$ and $\mathbf{ds}'$, (compare Figure 12), then:[59]

$$\cos\varepsilon = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\eta,$$

and finally

$$\mathbf{df} = \frac{ii'}{r^2}\left(\cos\varepsilon - \tfrac{3}{2}\cos\theta\cos\theta'\right)ds\,ds'\,\mathbf{e}_r .$$

The fact that $k < 0$ shows that Ampère's analysis has reported a force of repulsion between two collinear elements, α and α'. Does this force really exist in nature? Our earlier discussion indicates that experiment cannot yield an answer: we cannot experiment with two current elements. But strangely, in this case Ampère and de la Rive felt that they had verified this conclusion empirically.[60] This could only appear to be a verification of the theory if we forget that, as Ampère is usually the first to point out, it is not the interaction of two current elements which we observe in an experiment,

57. Note that in the body of Ampère's text, a long argument intervenes before the consequence of Experiment IV can be drawn. The simplified argument is appended (pp. 152 ff. of Ampère's text).

58. Ampère gives his result first in another form (*op. cit.*, p. 44; but see p. 116, where the result above is given explicitly).

59. For any spherical triangle, if a, b, and c are the sides and A the angle opposite side a, then $\cos a = \cos b \cos c + \sin b \sin c \cos A$ (the spherical "law of cosines"). Here if $\mathbf{ds}$, $\mathbf{ds}'$, and $\mathbf{r}$ are laid out from a single point along three radii of a sphere, the arcs $(\pi-\theta')$, θ, and $(\pi-\varepsilon)$ constitute three sides of a spherical triangle, with $(\pi-\eta)$ as the angle opposite $(\pi-\varepsilon)$, and the theorem applies.

60. Ampère, *op. cit.*, p. 28; Joubert, *op. cit.*, pp. 33–34.

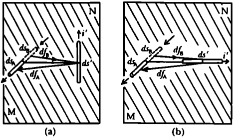


Figure 13 (repeated)

but the effect of *one whole circuit* on an element. It is difficult to understand how Ampère allowed this misunderstanding to arise. I suspect the error arises through the uncritical inclusion of an earlier paper in a later work.

The fact is that no experiments in which currents flow in closed circuits can confirm this fundamental law directly.[61] It must stand or fall by the strength of Ampère's argument, and Maxwell was not the first to question this. Hermann Grassmann (1809–1877) pointed to a peculiarity in the result when the law is applied to parallel elements which make an angle θ with the line joining them: he showed (as is easily seen) that when that angle takes the value 35°30′, the force implausibly goes to zero, and for smaller angles changes from attraction to repulsion. He remarked:

> Already the tangled form of this equation must raise a doubt about it. This doubt must be enhanced when we try to apply it.[62]

For the sake of obtaining a law with less arbitrary behavior, Grassmann was willing to abandon Ampère's first requirement of equal and opposite forces acting along the line **r**.

Ampère's law, we should note in passing, is not the same as the "law" usually given in modern texts, which is in open contradiction to Newton as well as to Ampère.[63] The modern law, which we may write

$$\mathbf{df} = ii'\left(\mathbf{ds} \times \frac{\hat{\mathbf{e}}_r \times \mathbf{ds}'}{r^2}\right),$$

61. The possibility of experimenting with the electrodynamic effects of unclosed circuits, and thereby obtaining empirical evidence which would decide between alternative theories, interested several investigators, among them Helmholtz. "Ueber die Bewegungsgleichungen der Elektrodynamik" (1870), in Hermann Helmholtz, *Wissenschaftliche Abhandlungen* (3 vols.; Leipzig: 1882–95), *I*, pp. 547, 563. Cf. J. J. Thomson, *op. cit.*, pp. 144 ff., where a number of contributions are reviewed.

62. Grassmann, *op. cit.*, p. 4 (compare page 151n, above).

63. Tricker, *op. cit.*, p. 43.

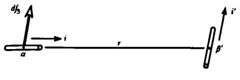


Figure 14. Maxwell's force $\mathbf{df}_3$

appears fleetingly in Ampère's text, but by no means as a law of nature; it is *Reynard's law*, apparently first enunciated in 1870.[64]

Maxwell's analysis in Chapter II of Part iv of the *Treatise* agrees with Ampère's with respect to the forces we have called $\mathbf{df}_1$ and $\mathbf{df}_2$; he reformulates the expressions for them slightly (*Tr* ii/166):

$$\mathbf{df}_1 = A\alpha\alpha' ii' \hat{\mathbf{e}}_r$$

$$\mathbf{df}_2 = B\beta\beta' ii' \hat{\mathbf{e}}_r .$$

The only respect in which Maxwell diverges from Ampère's argument is in the admission of one form of crossed-element force. Like Ampère, he denies a force due to elements perpendicular both to each other and to the line joining them (the case of Figure 13a, discussed earlier), but he includes a force due to elements perpendicular to each other, one of which however lies along the direction of $\mathbf{r}$ (Figure 13b). Why is this not subject to the stricture against the first case? It would be, if it were supposed to take the form of an attraction or a repulsion. Maxwell, though, rejects this possibility, assuming instead that, if it exists, it must be a force $\mathbf{df}_3$, parallel to the current element which is perpendicular to $\mathbf{r}$ (Figure 14).

Three different pairs of components can give rise to such a force, namely $\alpha\beta'$, $\beta\alpha'$, and $\alpha\gamma'$. A typical expression for the force will be

$$\mathbf{df}_3 = C\alpha\beta' ii' \hat{\mathbf{e}}_{\beta'} .$$

Arguing for this choice of direction, Maxwell says:

> The sign of this expression is reversed if we reverse the direction in which we measure β'. It must therefore represent either a force in the direction of β' or a couple. (*Tr* ii/166)

In saying this Maxwell rejects the possibility of a cross-product relation such as $\alpha \times \beta'$ determining the direction of the force; such a relation would formally reverse $\mathbf{df}_3$ with a reversal of either α or β'—as it should—but it would assume that "handedness" could be distinguished in a fundamental law of nature, an assumption that Maxwell tacitly rejects.

In admitting a force which acts perpendicularly to the line joining the elements $\mathbf{ds}$ and $\mathbf{ds}'$, Maxwell has denied Ampère's interpretation of the

64. Joubert, *op. cit.*, *3*, p. 123; F. Reynard, "Nouvelle théorie des actions électrodynamiques," *Annales de chimie et physique*, *19* (1870), p. 272.

Newtonian program, according to which "one cannot conceive" (on ne peut pas concevoir) the force except as acting along the line of centers (page 148, above). Maxwell does say, at the end of the chapter, when surveying the array of possibilities which his new analysis has produced,

> [T]hat of Ampère is undoubtedly the best, since it is the only one which makes the forces on the two elements not only equal and opposite but in the straight line which joins them. (*Tr* ii/174)

But by putting the question this way, Maxwell has made it a matter of taste rather than a scientific necessity. Throughout the analysis which follows, he continues to deal with a component of the force which does not obey Ampère's principle; Maxwell must therefore regard it as conceivable.

Having introduced components of the force in directions other than along **r** itself, Maxwell has to carry out his analysis as a vector problem. Taking coordinates in the directions of α', β', and γ', we have now:

$$\mathbf{df}_{\alpha'} = A\alpha\alpha' + B\beta\beta'$$
$$\mathbf{df}_{\beta'} = C(\alpha\beta' - \alpha'\beta)$$
$$\mathbf{df}_{\gamma'} = C\alpha\gamma'.$$

To avoid carrying the currents i and i' through the equation, Maxwell takes **df** as an intensive measure of the force between circuits carrying unit currents (force per unit current per unit current).

I shall abridge the complexities of Maxwell's analysis as much as possible, in order to focus upon the relation of his results to Ampère's. Of a number of quantities which he introduces, we will need to use the following (*Tr* ii/168, 170):

$$P = \int_r^\infty \frac{A+B}{r^2}dr$$
$$Q = \int_r^\infty C\,dr$$
$$\rho = \tfrac{1}{2}\int_r^\infty (B-C)dr$$

It will be useful to define two further quantities as well which are not separately identified by Maxwell, namely potentials of the forces $\mathbf{df}_1$ and $\mathbf{df}_2$:

$$W = \int_r^\infty A\,dr$$
$$V = \int_r^\infty B\,dr.$$

Maxwell's quantity Q presents a peculiar problem of interpretation. The force $\mathbf{df}_3$, of which C is the measure, is not a central force, but acts in every case normally to **r**. If we carried the element **ds'** along **r** at right angles to the current element **ds** from $\mathbf{r}_0$ to infinity, the component $\mathbf{df}_3$ would contribute nothing to the work done. Q then seems to be a fictitious potential: the potential which $\mathbf{df}_3$ *would* have if it were central. Since $C = C(r)$, this works out perfectly well mathematically. Since Q enters into the other apparent potential expressions in Maxwell's chapter, they all share this fictitious character. For example, ρ may be written

$$\rho = \frac{1}{2}\left(\int B dr - \int C dr\right) = \frac{V-Q}{2},$$

and later Maxwell will work with $(Q+\rho)$, which is

$$(Q+\rho) = \frac{V-Q}{2} + Q = \frac{V+Q}{2}.$$

It should be made clear that Maxwell does not apply the term "potential" to these; but he is arranging his theory to yield the true scalar and vector potentials M and $\mathbf{A}$ later in the argument, as we shall see.

The general force expression with A, B, and C undetermined in relation to one another is too complex to be of interest, but it simplifies in the case of *that component* of the force of $\mathbf{ds}'$ on $\mathbf{ds}$ which acts in the direction of $\mathbf{ds}$—that is, the component which is pertinent to Ampère's Experiment III. Writing this expression and integrating around a closed circuit of elements $\mathbf{ds}'$, Maxwell obtains for the net force along the length of $\mathbf{ds}$:

$$df_s = -\oint (2Pr - B - C) l' \lambda ds'.$$

Here l' is $\cos \varepsilon$, and λ is $\cos(x, r)$. From the independence of these two quantities, we conclude that (*Tr* ii/169)

$$(2Pr - B - C) = 0,$$

corresponding to Ampère's evaluation of k in terms of n. That is, Experiment III has this time told us that

$$P = \frac{1}{2r}(B + C).$$

Using this result to eliminate P, and with it any reference to force $\mathbf{df}_1$, Maxwell now obtains a general expression for the total force of a closed circuit upon a current element $\mathbf{ds}$. Notice that we are now approaching a point of convergence with the Faraday theory of Maxwell's preceding chapter. Maxwell defines an auxiliary quantity which we may write

$$\mathbf{D} = \oint \frac{B - C}{2} \hat{\mathbf{e}}_r \times \mathbf{ds}' \quad [\mathbf{D} \leftrightarrow \alpha', \beta', \gamma'],$$ [65]

where the integration is around the circuit of the elements $\mathbf{ds}'$. In terms of this quantity, the force $\mathbf{df}$ on $\mathbf{ds}$ is (*Tr* ii/169)

65. The symbol **D** is not related to Maxwell's displacement vector **D**, to be discussed in Chapter 6. It is in fact the symbol Ampère uses to denote his "directrix" (Ampère, *op. cit.*, p. 31).

$$d\mathbf{f} = (d\mathbf{s} \times \mathbf{D})ii'.$$

In a passage in the *Théorie mathématique* which we did not cite, Ampère derived a result that will be seen to agree exactly with the above expression once B and C have been determined.[66] Maxwell's enlargement of Ampère's theory does not affect any testable case such as this.

Maxwell says of the quantity we have written **D**, but which he writes in terms of the components (α', β', γ'—no relation to his earlier quantities of the same name!):

> The quantities α', β', γ' are sometimes called the determinants of the circuit s' referred to the point P. Their resultant is called by Ampère the directrix of the electrodynamic action. (*Tr* ii/169)

The reader has only to recall a result from Maxwell's preceding chapter to recognize what has happened: Faraday's line of magnetic force has emerged *out of the formalism of analysis*. Maxwell goes on to add:

> Since we already know that the directrix is the same thing as the magnetic force due to a unit current in the circuit s', we shall henceforth speak of the directrix as the magnetic force due to the circuit. (*Tr* ii/169–70)

Strictly, as Maxwell has defined the quantity here, it is equal to the magnetic force per unit current in the source loop. This coincidence of the formal theory and Faraday's view was certainly unknown to Ampère and Faraday; I do not know that it was remarked by anyone before Maxwell pointed it out.[67]

Maxwell postpones, seemingly until the last moment, the evaluation of the dependence of the force on the distance r; he accomplishes this very simply, interpreting Experiment IV essentially as Ampère did. The factor $(B-C)/2$ in the equation previously written for the directrix is now shown to be $1/r^2$ (*Tr* ii/173), so that

$$\mathbf{D} = \oint \frac{\hat{\mathbf{e}}_r \times d\mathbf{s}}{r^2}.$$

To compare Maxwell's result with Ampère's, let us set aside for a moment a section of Maxwell's chapter which has no counterpart in Ampère (*Tr* ii/170–71), and go directly to his final statement of the force law (*Tr* ii/173). He formulates it in terms of a set of oblique coordinates which we may represent by the non-orthogonal unit vectors $\hat{\mathbf{e}}_s$, $\hat{\mathbf{e}}_{s'}$, and $\hat{\mathbf{e}}_r$,

66. Ampère, *op. cit.*, p. 31.

67. Tait, for example, in discussing it, does not point out any relation to Faraday: "This vector ... which is of great importance in the whole theory of the effects of closed or indefinitely extended circuits, corresponds to the line which is called by Ampère 'directrice de l'action électrodynamique'. It has a definite value at each point of space..." (*op. cit.*, p. 334); cf. p. 151n, above.

parallel to **ds**, **ds'**, and **r**.[68] Then

$$\mathbf{df} = \left(R\hat{\mathbf{e}}_r + S\hat{\mathbf{e}}_s + S'\hat{\mathbf{e}}_{s'}\right)\mathbf{ds}\,\mathbf{ds}'.$$

Maxwell finds that[69]

$$R = \frac{1}{r^2}\left(\cos\varepsilon - \frac{3}{2}\cos\theta\cos\theta'\right) + r\frac{\partial^2 Q}{\partial s\partial s'}$$

$$S = -\frac{\partial Q}{\partial s'}$$

$$S' = \frac{\partial Q}{\partial s}.$$

We have traced schematically how Ampère's four experiments serve to eliminate the unknown coefficients in the force equation, P having been eliminated by Experiment III, and $(B-C)/2$ by Experiment IV. What about Q, an unknown function of r, which remains? The answer is that Q, appearing always as a perfect differential, is immune to experiment; in integration around the closed loop of the source circuit, each of the terms involving Q goes to zero, so Q is left to our free choice:

> Since the form and value of Q have no effect on any of the experiments hitherto made, in which the active current at least is always a closed one, we may, if we please, adopt any value of Q which appears to us to simplify the formulae. (*Tr* ii/174)

Maxwell mentions three interesting options in addition to Ampère's; this last arises if $Q = 0$, thereby suppressing the components other than $\hat{\mathbf{e}}_r$. The others are (1) that there be no force between collinear elements (that is, $A = 0$); (2) that the attraction R be proportional to $\cos\varepsilon$; (3) that the attraction and the oblique force depend on θ and θ' only. Each yields a new law of force—but others might be invented without limit.[70]

Maxwell's opinion of the process is perhaps reflected in this comment in a Royal Institution lecture:

> The formula of Ampère, however, is of extreme complexity, as compared with Newton's law of gravitation, and many attempts have been made to resolve it into something of greater apparent simplicity.

68. Maxwell had before him a paradigm of the use of oblique coordinates, in Newton's analysis of the three-body problem (*Principia*, Book I, Proposition 66.) He taught the *Principia* and knew it well, though I do not know that he had this example in mind here.

69. (*Tr* ii/173). I have transcribed Maxwell's result by substituting the geometrical equivalents of the partial derivatives as given at (*Tr* ii/165), taking into account the fact that Maxwell's θ' is the supplement of the angle θ' shown in Figure 12 (page 148 above).

70. On the generalization of Ampère's law of force—by others as well as by Maxwell—see J. J. Thomson, *op. cit.*, p. 115; Whittaker, *History*, pp. 233–36; O'Rahilly, *Electromagnetic Theory*, pp. 102–123

> I have no wish to lead you into a discussion of any of these *attempts to improve a mathematical formula*. Let us turn to the independent method of investigation employed by Faraday in those researches ... which have made this Institution one of the most venerable shrines of science. (*SP* ii/318; emphasis added)

This suggests, I think, the point of the chapter for Maxwell: there is a moral in the elusive function Q. This kind of investigation with which Ampère has endowed science is an attempt "to improve a mathematical formula"; that it is not an investigation of nature is proven by the fact that the mathematics proves absolutely deaf to anything nature might say. Maxwell is impatient to return to Faraday, and no doubt hopes that the reader shares this feeling.

Before returning to Faraday, however, we must consider the section of Maxwell's Ampère chapter referred to above, a section that does not correspond to anything in Ampère. This is a section (*Tr* ii/170–71) in which the result for the force on a current element is reformulated in terms of potentials. To turn immediately to the result, Maxwell shows that the force on $\mathbf{ds}$ due to $\mathbf{ds}'$ can be written in this way (*Tr* ii/171):

$$\mathbf{df} = ii'\frac{\partial}{\partial s \partial s'}\left(\nabla M - \nabla L + \mathbf{A} - \mathbf{A}'\right),$$

in which

$$M = \int_0^{s}\int_0^{s'}\frac{\mathbf{ds}\cdot\mathbf{ds}'}{r},$$

$$L = \int_0^{r}(Qr+1)dr,$$

$$\mathbf{A} = \int_0^{s}\frac{\mathbf{ds}}{r}, \qquad \mathbf{A}' = \int_0^{s'}\frac{\mathbf{ds}'}{r}.$$

We see that here, Maxwell is looking at Ampère's "elements" from the point of view that was naturally adapted, in Maxwell's Chapter I of Part iv, to Faraday's concepts involving a view of systems as wholes. This is perhaps a little like looking the wrong way through a telescope. Maxwell has rewritten Ampère's force law in terms of an incremental mutual potential, an incremental vector potential, and one quantity, L, that will disappear on the first integration.

The force between two finite currents, neither of which is closed, would be

$$\mathbf{f} = ii'\left(\nabla M - \nabla L + \Delta\mathbf{A} - \Delta\mathbf{A}'\right),$$

where $\Delta\mathbf{A}$ is the increment in the vector-potential of conductor s between the end-points of conductor s', *etc.*, and M and L are as given above. Since Maxwell has denied the existence of such open currents, this is for him purely an exercise of thought, but not necessarily for that reason altogether idle, as it may lead to a clearer understanding both of the vector potential and of the significance of the earlier insistence on closed currents. We see that **A** would have a role in the force between currents, if currents did not always form closed circuits.

Maxwell investigates whether the force between unclosed currents would have a potential, and shows that because of the vector-potential terms, it would not. Forming the differential of work **f**·d**l**, the terms $\Delta\mathbf{A}\cdot d\mathbf{l}$ and $\Delta\mathbf{A}'\cdot d\mathbf{l}$ are not perfect differentials.71

If we integrate around one closed loop only, say of circuit s', then $\Delta\mathbf{A}$ goes out, and we may form an empirically meaningful expression for the force on an element $d\mathbf{s}$. This force will be $d\mathbf{F}$, having components such as:

$$dF_x = \frac{\partial}{\partial x}M + d\mathbf{s}\cdot\nabla A_x\,.$$

This is an alternative form for the law of force on a current element in a magnetic field.72 Again, this force does not have a potential, since $\Delta\mathbf{A}\cdot d\mathbf{l}$ is not a perfect differential. We see that so long as either circuit is unclosed, we do not have to do with the mutual potential alone; instead it is modified, and the force is made nonconservative, by a contribution from the vector potential.

It is only when we deal with two complete circuits that the remaining term in the vector potential disappears, and the force is conservative:

$$\mathbf{F} = \nabla M\,.$$

If Maxwell feels that Faraday's method involves the interactions of "systems of power" which should be expressible as potentials, this analysis has shown that Faraday's insights are relevant only to whole circuits; we must have displacement currents if we are to have a potential theory of electromagnetism. We might say Maxwell has found that incomplete circuits do not have sphondyloids.

To review for a moment, we have followed two derivations of a law of force between current elements, one Ampère's, and the other what is almost Maxwell's parody of Ampère's. Both start from the same set of four null experiments, and use the evidence of the experiments in essentially the same way. They differ in that Ampère denies from the outset the possibility of a force which acts otherwise than along the line connecting the two elements, while Maxwell is willing to admit a force perpendicular to this line; as a result, Maxwell is able to include a force arising from one case of crossed elements, while Ampère rejects all such cases. Maxwell is thus led to develop a much more complex theory of a force in which three vector

71. O'Rahilly similarly demonstrates that it is not in fact possible to sustain a theory of the mutual potential of two current elements, or of a current element and a complete circuit; O'Rahilly, *op. cit.*, p. 117.

72. The equation can be obtained from the vector triple-product for $d\mathbf{f}$ of our contemporary texts (page 154, above), by expanding the triple product as the sum of two terms.

components must be considered, instead of Ampère's one, along the line of centers.

Maxwell's analysis employs functions analogous to potentials, but is on the whole parallel to Ampère's. One unknown function $Q(r)$ runs through Maxwell's analysis and remains, untouched by any possible evidence based on experiments with circuits, in the final result; it represents the irremovable uncertainty about the forces introduced by the crossed current elements. The two results are these:

AMPÈRE:

$$\mathbf{df} = \frac{ii'}{r^2}\left(\cos\varepsilon - \frac{3}{2}\cos\theta\cos\theta'\right)ds\,ds'\,\hat{\mathbf{e}},$$

MAXWELL:

$$\mathbf{df} = df_r\hat{\mathbf{e}}_r + df_s\hat{\mathbf{e}}_s + df_{s'}\hat{\mathbf{e}}_{s'}$$

$$df_r = ii'\left[\frac{1}{r^2}\left(\cos\varepsilon - \frac{3}{2}\cos\theta\cos\theta'\right) + r\frac{\partial^2 Q}{\partial s\partial s'}\right]ds\,ds'$$

$$df_s = -ii'\frac{\partial Q}{\partial s'}ds\,ds'$$

$$df_{s'} = -ii'\frac{\partial Q}{\partial s}ds\,ds'$$

Finally, Maxwell throws his force equation into the form of an incipient equation of potential, as it were. His effort here, I believe, is to find an intelligible relation between Ampère's view and Faraday's:[73]

$$\mathbf{df} = ii'\frac{\partial^2}{\partial s\partial s'}\left(\nabla M - \nabla L + \mathbf{A} - \mathbf{A}'\right).$$

ELECTROMAGNETIC ROTATIONS

Let us turn now to the question of electromagnetic rotations. As has been mentioned, the possibility of producing a continuous rotation, either of a current about a magnet, or a magnet about a current, led to a confrontation of two modes of analysis—Ampère's, in which the rotation was a complex phenomenon arising out of couples produced by simultaneous attractions and repulsions; and Faraday's, in which the rotation followed naturally from a circular configuration of the field. The rotations were

73. Compare this early remark in a letter to William Thomson (September 13, 1855): "[Y]ou are acquainted with Faraday's theory of lines of force & with Ampère's laws of currents and of course you must have wished to understand Ampère in Faraday's sense." (Larmor, *Origins*, p. 18)