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A METRIC FOR AN EVANS-VIGIER FIELD

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In this paper we present an example of a specific metric which geometrizes explicitly a light-like four-vector potential field (Evans-Vigier field). We define the concepts of 'semilocal' and 'complete' geometrization and show that a light-like vector field has the same geometrical structure as a gravitational Kerr field. With this background in mind we discuss a theoretical proposition that a rotating body generates, besides a special gravitational field, a magnetic-type gauge field which might be identified with a geometrized Evans-Vigier field. We finally present a discussion which inform us that a classical Evans-Vigier field represents a novel type of field because we cannot identify it with any of the known electromagnetic fields.

Key words: light-like vector potential, force-free field, complete geometrization.

1. INTRODUCTION

In this contribution, we construct a metric which appears appropriate for a geometrization, within the framework of a Riemannian spacetime, of a light-like 4-vector potential field which can be assigned to an electromagnetic-type field. Such an exotic field with a 4-vector potential A_α satisfies the relation

$$A_\alpha A^\alpha = 0 \tag{1}$$

and is denoted by us as an *Evans-Vigier field* since in accordance with our information it emerged for the first time in the work of these authors in connection with the hypothesis of the existence of a special kind of magnetic field (see, for instance [1]).

The starting point is the well known approach to the geometrization of physical fields involving the construction of spacetime geometries (the so called *force-free geometries*) within which the geodesic equation proves to be identical to the equation of motion of a particle when interacting with such (nongravitational) fields. This method derives in fact from the generalized Einstein's equivalence principle which asserts that "any trajectory is a geodesic of some geometry" [2]. Furthermore, the laws of motion, in the case of interacting particles, are given by the differential equations of the geodesics for the metric in question at the instantaneous position of each particle [3].

Pursuing this subject, we observe that for the formulation of the geodesic equations also in the presence of nongravitational forces, some efforts have been directed towards applying changes to the metric (see, for instance, [4, 5]) and other efforts to modifications of the connection [6, 7], in a Riemann or a Riemann-Cartan spacetime. There appeared also papers which consider the possibility of applying a Finsler [8] or a Randers geometry [9] or a fractal spacetime geometry [10-12] in order to establish unitary theories of gravitation and electromagnetism in conjunction with a probabilistic interpretation of the geometry of the background spacetime.

However, all these alternative interpretations of force-free geometries have not yet reached the same level of elaboration and experimental verification as is the case for Einstein's general theory of relativity the formal structure of which has continuously invited the development of gauge theories. These are the reasons that why we maintain in the present work the framework of a Riemannian spacetime which helps us to geometrize a vectorial field. We propose a geometrization of a vectorial field in the sense that the associated physical quantity (*e.g.*, the four-vector potential A_α) enters directly into the metric which may be interpreted, alternatively, as an 'interior' ($T_{\alpha\beta} \neq 0$) or 'exterior' ($T_{\alpha\beta} = 0$) solution of Einstein's equations. However, from an Einsteinian point of view, the field defined by A_α is *completely (truly) geometrized* (like the gravitational field itself) if it leads to a determination of the geometry of the (curved) vacuum spacetime in which no other (non-geometrized) matter manifests its presence in conjunction with a non-zero energy-momentum tensor. We emphasize that the physical quantities (*e.g.*, density, pressure, electromagnetic field tensor etc) which generally appear on the right hand side of Einstein's equations represent non-geometrized quantities, *i.e.*, the source of the (geometrized) gravitational field.

In the present paper we adhere to the Einstein's general relativity and thus the energy and momentum of the geometrized Evans-Vigier field are encapsulated solely in the pseudotensor $t_{\alpha\beta}$ on the same geometrical footing as any gravitational field. We recollect that the general relativity is a very special non-Abelian gauge theory and thus it is possible that a truly spacetime geometrization can be applied also to a non-Abelian analogue of the electromagnetic field. The Yang-Mills field may serve as such a field.

We parenthetically note that in a generalisation of Einstein's gravity theory which assumes a non-zero stress energy of the gravitational field (see, *e.g.*, [13] and its criticism in [14]) we can assert that there exists a third alternative geometrization, but we will not refer to this aspect here.

Attempts have also been made to mix directly the standard symmetric Riemannian metric tensor with an antisymmetric (electromagnetic) field tensor, but the new nonsymmetric metric cannot achieve a real geometrization of the electromagnetic field [15].

Even if, for the time being, we cannot propose a firm experimental program to detect or generate an Evans-Vigier field, the latter retains the merit of enhancing the search of exotic forms of gauge fields in Abelian and non-Abelian gauge theories. Moreover, a possible existence of a light-like 4-vector electromagnetic field would be a proof that the most important metrics of general relativity, Schwarzschild and Kerr solutions (which in Eddington coordinates are described also by light-like four vectors) have an electromagnetic analogue. Thus, the Kerr metric, which represents the gravitational field exterior to a spinning source which 'drags' space around with it, has the *same geometrical structure* as a geometrized Evans-Vigier field. This calls for a possible general relativistic physical explanation of the mutual relation between a magnetic dipole (or, generally, another 'gauge dipole' described by a light-like non-abelian vector potential) and the angular momentum, as observed already in the case of astrophysical bodies (see the *Wilson-Blackett-Ahluwalia-Wu relation* [16]).

On a microscopic level, the Evans' *optical (light) magnet* [17] produced by a circularly polarised light beam appears as a natural and physically possible hypothesis. A search for cyclically symmetric equations, similar to spin angular momentum relations but now refering to a magnetic-type field, seems also tempting from a geometrical point of view. Of course, as for gravitation or perhaps for the entire field of physics we do not yet know the physical intrinsic mechanism of such a *magneto-rotation induction*: 'rotation generates magnetic-type field and magnetic field generates rotation', and yet we attempt to model and describe it here.

A simple experimental proposal for the verification of these hypotheses may be the detection of an Aharonov-Bohm effect as arising, for example, in the usual two-slit electron diffraction experiment in which the solenoid is replaced by a rotating body. (For a recent overview of microturbines see [18], and for a two-slit electron diffraction experiment with a rotating superconductor see [19].) These explorations might be extended to an astrophysical scale in which a natural cosmic gravitational Aharonov-Bohm situation may arise [20]. Indeed, the gravitational field of a rotating astrophysical lens object plays the role of both a double slit (by its electriclike and curvature inducing effects by gravity) and an 'external' field (with a magneticlike contribution of the gravitation). A proposal for a laboratory experiment for an observation of a gravitational Aharonov-Bohm effect in conjunction with photons is described in [21].

The section 'Physical Content of the Evans-Vigier Condition' tries to construe an answer to the much disputed question: What is the Evans-Vigier $\mathbf{B}^{(3)}$ field?

In the final section we present a discussion on the possibility of identifying an Evans-Vigier field within the set of known electromagnetic fields.

2. A SPECIAL METRIC AND BASIC RELATIONS

Let us consider a null-like four-vector with components

$$A_\alpha(x^\beta) := (A_0, A_1, A_2, A_3) = (A_0, \mathbf{A}). \tag{2}$$

We denote by

$$A^2 = \eta^{\alpha\beta} A_\alpha A_\beta = 0 \tag{3}$$

its Minkowskian module, in which

$$\eta_{\alpha\beta} = [+1, -1, -1, -1] \tag{4}$$

is the Minkowski (flat) diagonal metric. We should mention that $A_\alpha(x^\beta)$ is here a standard spacetime vector which may represent the vector potential of an electromagnetic-type gauge field. For the moment, we cannot foresee if A_α may be associated with a massive or a zero-mass field or if we must include the subject of a gauge invariance. Consequently, all the calculations are given in the *tangent bundle* of spacetime.

We propose to study under which conditions a metric $g_{\alpha\beta}$ having the special form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + K A_\alpha A_\beta, \tag{5}$$

where $K \neq 0$ is a constant still to be determined, can define an Evans-Vigier field.

The determinant of the metric tensor $g_{\alpha\beta}$ is given by

$$\det(g_{\alpha\beta}) \equiv g = -\left(1 + KA^2\right) = -1, \tag{6}$$

and thus the inverse (contravariant) metric is

$$g^{\alpha\beta} = \eta^{\alpha\beta} - KA^\alpha A^\beta. \tag{7}$$

The metric (5) is similar to the one which describes a weak gravitational field, *i.e.*,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{8}$$

However, for the time being we do not impose yet any condition on the value or the strength of the term $KA_\alpha A_\beta$. There follows that

$$A^\alpha = g^{\alpha\beta}A_\beta = \eta^{\alpha\beta}A_\beta, \quad \eta^{\alpha\beta}A_\alpha A_\beta = g^{\alpha\beta}A_\alpha A_\beta = 0, \tag{9}$$

and thus the indices of A^α may be raised and lowered with either the metric $g_{\alpha\beta}$ or the Lorentz metric $\eta_{\alpha\beta}$. It is easy to show that

$$A_\alpha A^\alpha{}_{;\beta} = A_\alpha A^\alpha{}_{,\beta} = 0, \tag{10}$$

where the ordinary partial derivatives are denoted by commas (or alternatively by ∂_α and $\partial/\partial x^\alpha$), and covariant derivatives by semicolons. The Christoffel symbols are

$$\begin{aligned}
\Gamma^\alpha_{\beta\gamma} &= g^{\alpha\sigma}[\beta\gamma,\sigma] \\
&= \frac{1}{2}Kg^{\alpha\sigma}\Big[(A_\sigma A_\beta)_{,\gamma} + (A_\sigma A_\gamma)_{,\beta} - (A_\beta A_\gamma)_{,\sigma}\Big] \\
&= \frac{1}{2}Kg^{\alpha\sigma}\Big[A_\beta B_{\gamma\sigma} + A_\gamma B_{\beta\sigma} + A_\sigma(A_{\beta,\gamma} + A_{\gamma,\beta})\Big] \\
&= \frac{1}{2}K\eta^{\alpha\sigma}\Big[A_\beta B_{\gamma\sigma} + A_\gamma B_{\beta\sigma} + A_\sigma(A_{\beta,\gamma} + A_{\gamma,\beta})\Big] \\
&- \frac{1}{2}K^2A^\alpha A^\sigma(A_\beta B_{\gamma\sigma} + A_\gamma B_{\beta\sigma}),
\end{aligned} \tag{11}$$

where

$$B_{\nu\beta} = A_{\beta,\nu} - A_{\nu,\beta}, \tag{12}$$

and $[\beta\gamma,\sigma]$ is the Christoffel symbol of the first kind.

Because $g = \text{constant} = -1$, it follows that

$$\Gamma^{\alpha}_{\beta\alpha} = 0, \tag{13}$$

and thus the Ricci tensor is given by

$$\begin{aligned} R_{\beta\gamma} &= -\eta^{\alpha\sigma}[\beta\gamma,\sigma]_{,\alpha} + K\Big[A^{\alpha}A^{\sigma}[\beta\gamma,\sigma]_{,\alpha} \\ &+ \left(A^{\alpha}_{,\alpha}A^{\sigma} + A^{\alpha}A^{\sigma}_{,\alpha}\right)[\beta\gamma,\sigma]\Big] + \eta^{\alpha\mu}\eta^{\sigma\nu}[\beta\sigma,\mu][\gamma\alpha,\nu] \\ &- K\left(\eta^{\alpha\mu}A^{\sigma}A^{\nu} + \eta^{\sigma\nu}A^{\alpha}A^{\mu}\right)[\beta\sigma,\mu][\gamma\alpha,\nu] \\ &+ K^{2}A^{\alpha}A^{\mu}A^{\sigma}A^{\nu}[\beta\sigma,\mu][\gamma\alpha,\nu] \\ &=: KR_1 + K^2R_2 + K^3R_3 + K^4R_4. \end{aligned} \tag{14}$$

3. FORCE-FREE FIELD AND A SEMILOCAL GEOMETRIZATION

Introducing the parameter s defined by

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}, \tag{15}$$

the equations of geodesics,

$$\frac{du^{\alpha}}{ds} + \Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma} = 0, u^{\mu} = \frac{dx^{\mu}}{ds}, \tag{16}$$

become

$$\frac{du^{\alpha}}{ds} + KA^{\alpha}\frac{dC}{ds} = KC\eta^{\alpha\sigma}B_{\sigma\gamma}u^{\gamma}, \tag{17}$$

where

$$C = A_{\beta}u^{\beta} \neq 0. \tag{18}$$

At this point it is easy to see that an Evans-Vigier field described by the metric (5) becomes a 'force-free field' with respect to the motion of a charged test particle having the characteristic parameter q/m_0, and subject to the constraint

$$KC = \text{constant} = \frac{q}{m_0c^2}. \tag{19}$$

With this constraint, the geodesic equation (17) reduces formally to the Lorentz equation

$$\frac{du^\alpha}{ds} = \frac{q}{m_0c^2}\eta^{\beta\alpha}B_{\beta\nu}u^\nu. \tag{20}$$

We may identify $B_{\beta\nu}$ given by (12) with an electromagnetic field tensor if the 4-vector potential A_μ is related to an electromagnetic potential $\mathcal{A}_\mu$ by a gauge transformation of the second kind

$$A_\mu = \mathcal{A}_\mu + \frac{\partial f}{\partial x^\mu}. \tag{21}$$

Since it is possible to demonstrate that constraints such as (19) and (21) are consistent and in fact do not contradict each other along the trajectory of the test-particle (see,*e.g.*, [22]), we can assert that we have achieved a local or a *semilocal geometrization* (*i.e.*, one along a curve) of the Evans-Vigier field.

The final conclusion of this section is that any field described by a metric of the form (5) may act on a test particle with a Lorentz-type force (20). In such geometrical terms, a Lorentz-type force was known until now only for a weak gravitational field (see, *e.g.*, [23]).

4. A COMPLETE GEOMETRIZATION OF EVANS-VIGIER FIELD

Bearing in mind that the metric tensor is given in our account by equations (5) and (7), we need only derive the $R_{\alpha\beta}$, R, and also the Einstein's tensor from the $g_{\alpha\beta}$ and establish in this way the components of the matter tensor $T_{\alpha\beta}$. If this energy-momentum tensor coincides with one which is known for a given (physical, phenomenological) material scheme, we say that (5) represents a solution of Einstein's equations for such a scheme. If we do not possess such a coincidence, we say that we face an *exotic matter* which might determine the desired properties of the spacetime (*e.g.*, 'traversable wormhole' [24] or 'warp drive' [25]). From this point of view the general theory of relativity is not a closed theory, and sometimes the Einstein's equations seem to form a mathematical identity if a suitable metric is chosen:

$$G_{\alpha\beta} := \kappa\frac{G_{\alpha\beta}}{\kappa} =: \kappa T_{\alpha\beta}. \tag{22}$$

In other words, in this case Einstein's equations are used merely for a definition of an energy-momentum tensor which generates a given gravitational field.

In the following we will not use this *identity aspect* of the Einstein's equations since we intend to geometrize the Evans-Vigier field A_α which may be considered as a gravitational perturbation of a vacuum spacetime. Then the field equations correspond to an 'exterior case' and are given by

$$R_{\beta\gamma} = 0, \tag{23}$$

where $R_{\beta\gamma}$ is given by Eq. (14). In a way, the constant K may be called a 'coupling constant' because it characterizes the strength of the perturbation of the vacuum spacetime generated by an Evans-Vigier field. We assume that the form of the metric (5) retains its independence from the value of K. In other words, the metric $g_{\alpha\beta}$ given by (5) remains a solution for any arbitrary value of K. Thus in the expression (14) of $R_{\beta\gamma}$, each coeficient of K and of its powers must be cancelled separately. In this way, we obtain four equations:

$$R_1 = 0 = -\eta^{\alpha\sigma}[\beta\gamma,\sigma]_{,\alpha}, \tag{24}$$

$$\begin{aligned} R_2 = 0 = K\Big[&A^\alpha A^\sigma[\beta\gamma,\sigma]_{,\alpha} + \left(A^\alpha_{,\alpha}A^\sigma + A^\alpha A^\sigma_{,\alpha}\right)[\beta\gamma,\sigma]\Big] \\ &+ \eta^{\alpha\mu}\eta^{\sigma\nu}[\beta\sigma,\mu][\gamma\alpha,\nu], \end{aligned} \tag{25}$$

$$R_3 = 0 = -K\left(\eta^{\alpha\mu}A^\sigma A^\nu + \eta^{\sigma\nu}A^\alpha A^\mu\right)[\beta\sigma,\mu][\gamma\alpha,\nu], \tag{26}$$

$$R_4 = 0 = +K^2 A^\alpha A^\mu A^\sigma A^\nu[\beta\sigma,\mu][\gamma\alpha,\nu]. \tag{27}$$

We note that, in accordance with Eq. (26) the potential A_α generates a new light-like vector a_α which, by analogy with the kinematics of a timelike congruence of curves, may be called an 'acceleration-potential vector' and has the following properties:

$$a^\alpha = A^\alpha{}_{;\beta}A^\beta = A^\alpha{}_{,\beta}A^\beta = -b(x^\gamma)A^\alpha, \tag{28}$$

$$a^\alpha = g^{\alpha\beta}a_\beta = \eta^{\alpha\beta}a_\beta, \qquad \eta^{\alpha\beta}a_\alpha a_\beta = g^{\alpha\beta}a_\alpha a_\beta = 0, \tag{29}$$

$$a_\alpha a^\alpha{}_{;\beta} = a_\alpha a^\alpha{}_{,\beta} = a_\alpha A^\alpha{}_{;\beta} = a_\alpha A^\alpha{}_{,\beta} = 0, \tag{30}$$

$$a^\alpha A_\alpha = 0. \tag{31}$$

We define also an 'expansion', $\mathcal{E}(x^\beta)$, of the light-like potential congruence in the form of

$$\mathcal{E}(x^\beta) = -A^\alpha{}_{;\alpha} = -A^\alpha{}_{,\alpha}.$$

We notice that Eqs. (25) and (27) are satisfied identically, and that Eq. (26) is reduced to the definition of the acceleration potential (28). Thus the Einstein field equations (24)-(27) become

$$\Box^2(A_\beta A_\gamma) - \Big[(\mathcal{E}+b)A_\beta\Big]_{,\gamma} + \Big[(\mathcal{E}+b)A_\gamma\Big]_{,\beta} = 0. \tag{33}$$

For the stationary case, $\square^2 \to -\nabla^2$, there arise two remarkable type (2,2) D solutions of Eq. (33), namely, the Schwarzschild-type solution [see, *e.g.*, [26], p. 111, Eq. (9.7)]

$$\begin{aligned} A^{\mathrm{S}}{}_{\alpha} := (A^{\mathrm{S}}{}_0, A^{\mathrm{S}}{}_1,\ A^{\mathrm{S}}{}_2, A^{\mathrm{S}}{}_3) &= \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \\ &= \left[\frac{1}{\sqrt{r}}, \nabla\,(2\sqrt{r})\right] \end{aligned} \tag{34}$$

and the Kerr-Schild type metric (see, *e.g.*, [27], p. 146)

$$A^{\mathrm{KS}}{}_{\alpha} = \sqrt{\frac{\rho^3}{\rho^4 + a^2 z^2}}\left(1, \frac{\rho x + a y}{a^2 + \rho^2}, \frac{\rho y + a x}{a^2 + \rho^2}, \frac{z}{\rho}\right), \tag{35}$$

where

$$r^2 = x^2 + y^2 + z^2, \tag{36}$$

$$\rho^2 = \frac{1}{2}(r^2 - a^2) + \left[\frac{1}{4}(r^2 - a^2)^2 + a^2 z^2\right]^{1/2}. \tag{37}$$

Here a is a parameter related to the angular velocity and, thus, to the angular momentum of the source. We remind the reader that the Kerr metric represents a vacuum field exterior to a spinning source. Hence, an Evans-Vigier field and a type (2,2) gravitational field have the same topological properties. It is important to stress that, for the Schwarzschild-type solution (34),

$$\nabla \times \mathbf{A}^{\mathrm{S}} = \mathbf{0} \quad (\text{no magnetic} - \text{type field}) \tag{38}$$

and, for the Kerr-Schild type metric (35),

$$\nabla \times \mathbf{A}^{\mathrm{KS}} \neq \mathbf{0} \quad (\text{magnetic} - \text{type field}). \tag{39}$$

(The physical interpretation and other formal details of these equations will be discussed in another paper which is now under study [28].) An immediate consequence of these results is that rotating bodies generate, besides a special kind of gravitational field, also some magnetic-type gauge fields defined by light-like vector potentials (see Secs. 1 and 5). For the time being all experimental tests of general relativity (*e.g.*, advance of the perihelion of Mercury, bending of light, gravitational red shift etc) are expressed only as functions of the mass of the central gravitating body. In order to evaluate the physical implications of the Evans-Vigier field we must evaluate all these effects in terms of the light-like vector potential A_α. This is not a simple task because we have to apply Cartesian-type coordinates to spherically or axially symmetric solutions (see [28]).

5. PHYSICAL CONTENT OF THE EVANS-VIGIER CONDITION

5.1. Four Independent Electromagnetic Invariants

In Classical electrodynamics there exist only four independent electromagnetic (EM) field invariants [29, 30], namely (in units with $c = 1$),

$$I_0 = A_\alpha A^\alpha, \tag{40}$$

$$I_1 = \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} = |\mathbf{E}|^2 - |\mathbf{B}|^2, \tag{41}$$

$$I_2 = -\frac{1}{2} F_{\alpha\beta} \mathcal{F}^{\alpha\beta} = 2\mathbf{E} \cdot \mathbf{B}, \tag{42}$$

$$I_3 = -2A_\alpha T^{\alpha\beta} A_\beta, \tag{43}$$

where $F_{\alpha\beta}$ is the EM field tensor, $\mathcal{F}^{\alpha\beta}$ is the dual EM field tensor and $T^{\alpha\beta}$ is the Maxwell stress-energy tensor. Salingaros [30] used these invariants to announce the proposition: Plane monochromatic EM (transverse) waves are characterized by vanishing invariants $I_1 = I_2 = I_3 = 0$ in the Lorentz gauge. As we mentioned, the Evans-Vigier field is defined by a vanishing invariant $I_0 = 0$.

5.2. Is the Evans-Vigier Condition a Lorentz-Covariant Gauge Condition or a Constraint Defining an Exotic Electromagnetic Field?

In this section we consider the question if $A_\alpha A^\alpha = 0$ can be proposed as a particular (nonlinear) Lorentz-covariant gauge condition for Abelian or non-Abelian gauge theories. We mention that a similar nonlinear condition in which, however,

$$A_\alpha A^\alpha = \text{constant} \neq 0 \tag{44}$$

was proposed by Dirac [31]. This is equivalent to a proportionality between A^α and the particle four-velocity u^α [32, 33].

Generally, there are two constraints which arise in connection with any gauge condition: attainability and completeness (or uniqueness) [34]. Attainability means that given an arbitrary 4-vector potential B_α not satisfying the Evans-Vigier condition, one can find a gauge transformation U such that the gauge-transformed B_α,

$$B_\alpha \xrightarrow{U} B^U{}_\alpha = U^{-1} B_\alpha U - \frac{\mathrm{i}}{G} U^{-1} \partial_\alpha U, \tag{45}$$

satisfies the gauge condition. Imposing the condition (3) on $B^U{}_\alpha$, we find a nonlinear differential equation for U

$$B^{U\alpha} \partial_\alpha U = -\mathrm{i} G B_\alpha U B^{U\alpha}, \tag{46}$$

where

$$B^{U\alpha} = U^{-1} B^\alpha U - \frac{\mathrm{i}}{G} U^{-1} \eta^{\alpha\beta} \partial_\beta U. \tag{47}$$

Concerning the problem of uniqueness we assume that there exists a potential B_α satisfying Eq. (3)and also a gauge transformed $B^U{}_\alpha$ of that potential which also satisfies the Evans-Vigier condition, *i.e.*,

$$B_\alpha B^\alpha = B^U{}_\alpha B^{U\alpha} = 0. \tag{48}$$

Applying Eq. (45) to Eqs. (46) and (48), we obtain nonlinear differential equations for U which for a simply connected spacetime and for regular vector potentials can, in principle, be solved. Thus, the uniqueness is not fulfilled and the Evans-Vigier condition (3) is rather a constraint defining an exotic electromagnetic-type field than a Lorentz-covariant gauge condition.

5.3. Evans-Vigier Field and Non-Abelian Fields

It is well known that actually we cannot speak about a non-Abelian SU(2) electrodynamics but merely about a non-Abelian (nonlinear) analogue of Maxwell's equations or about a non-Abelian analogue of electric and magnetic fields which do not appertain to what we ordinarily associate with electromagnetism. In contrast to electrodynamics, the Lagrangian of the Yang-Mills field in vacuum contains, in addition to the second-order terms in such fields, higher-order terms. Thus, Yang-Mills fields possess a nontrivial self-interaction as in the case of a gravitational field. In other words, the mediating particles of the Yang-Mills field themselves possess charges.

On the other hand, these three classical theories, Maxwellian electrodynamics, Einsteinian general relativity, and Yang-Mills theory possess a common feature they all admit a Birkhoff theorem [35].

In a restricted sense, this means that if we consider [in each of these theories, *i.e.*, even for SU(2) and SU(3) simmetries)] the gauge field of a single point particle, for large r, it proves similar to the Coulomb type field.

For example, the non-Abelian SU(2) analogue of the electromagnetic field strength tensor is the curvature tensor of the SU(2) internal bundle

$$F_{\alpha\beta}^{\mathrm{NA}} = -\mathrm{i}[\mathrm{D}_\alpha,\ \mathrm{D}_\beta] = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \mathrm{i}G[A_\alpha,\ A_\beta], \tag{49}$$

where the matrices A_α are defined by

$$A_\alpha(x^\beta) = A_\alpha^a(x^\beta)T^a =: \mathbf{A}_\alpha \cdot \mathbf{T}, \tag{50}$$

and T^a are the isospin matrices appropriate to the particular multiplet of wavefunctions on which the gauge-covariant derivative

$$\mathrm{D}_\alpha = \partial_\alpha + \mathrm{i}A_\alpha(x^\beta) \tag{51}$$

acts. The matrix form of equation (49) depends on the particular representation of the gauge group SU(2) to which the matrices T^a appertain. However, in every representation, these matrices satisfy the commutation relations of the Lie algebra

$$[T^a,\ T^b] = \mathrm{i}\varepsilon^{abc}T^c. \tag{52}$$

In an isospin space, $\mathbf{T}$ and $\mathbf{A}_\alpha$ are 3-dimensional vectors (a, b, c=1, 2, 3). Then

$$F_{\alpha\beta}^{\mathrm{NA}} = (\partial_\alpha \mathbf{A}_\beta - \partial_\beta \mathbf{A}_\alpha - G\mathbf{A}_\alpha \times \mathbf{A}_\beta) \cdot \mathbf{T}. \tag{53}$$

If we want to describe the interaction between the gauge fields and n species of spin-$\frac{1}{2}$ particles (fermions) with masses m_i, we can use the total action

$$S^{\mathrm{NA}} = \int \mathrm{d}^4x \times \left[-\frac{1}{4}F_{\alpha\beta}^a F^{a\alpha\beta} + \sum_{i=1}^{n} \bar{\psi}_i(x^\nu)(\mathrm{i}\bar{\partial} - G\bar{A}(x^\nu) - m_i)\psi_i(x^\nu) \right], \tag{54}$$

where the 'bar' notation means the 'contraction' with (Dirac) γ^μ matrices, *i.e.*, for instance,

$$\bar{A} = \gamma^\mu A_\mu^a T^{(i)a} =: \gamma^\mu \mathbf{A}_\mu \cdot \mathbf{T}^i, \tag{55}$$

and where $T^{(i)a}$ is the ath generator matrix in the isospin-$T^{(i)}$ representation. The sum is over multiplets of wavefunctions ψ_i, each having $(2T^{(i)}+1)$ entries in the case of a SU(2) isospin, and each member being itself a Dirac spinor.

The variation of the action (54) with respect to $\mathbf{A}_\mu$ yields Euler-Lagrange equations for the gauge fields $A^a_\alpha(x^\beta)$ which are the Yang-Mills equations, *i.e.*, the *non-Abelian analogue of Maxwell's equations*:

$$\mathrm{D}_\alpha F^{\mathrm{NA}\,\alpha\beta} = J^\beta \tag{56}$$

or

$$\partial_\alpha F^{a\alpha\beta} - G\varepsilon^{abc} A^b_\alpha F^{c\alpha\beta} = J^{a\beta}, \tag{57}$$

where the current is given by

$$J^{a\beta} = g\sum_i \bar{\psi}_i \gamma^\beta T^{(i)a}\psi_i. \tag{58}$$

For free fields, written in terms of the vector potential in the Lorentz gauge,

$$\frac{\partial A_\alpha}{\partial x^\alpha} = 0, \tag{59}$$

the Yang-Mills equations become

$$\Box \mathbf{A}_\alpha + G\mathbf{A}^\beta \times (\partial_\beta \mathbf{A}_\alpha - \partial_\alpha \mathbf{A}_\beta - G\mathbf{A}_\alpha \times \mathbf{A}_\beta) = 0. \tag{60}$$

In order to compare this with the geometrized Evans-Vigier potentials (34) and (35), we remind the reader that the spherically symmetric solutions of the source-free Yang-Mills equations for a single point particle do not differ fundamentally from the Coulomb field [36-37]. Indeed, the solution has the form

$$\mathbf{A}_0 = \mathrm{i}\varphi(r), \quad \mathbf{A}_k = \frac{x_k}{r}\mathbf{f}, \tag{61}$$

which even for a canonical case cannot apparently satisfy the Evans-Vigier condition. We cannot apparently also identify an Evans-Vigier field in the set of monopoles and instantons, that is within solutions corresponding to Yang-Mills equations for SO(4) and SO(3,1) gauge groups. The word 'apparently' is used here because solutions of the form (61) are very particular, and it is possible that Yang-Mills fields with other symmetries admit Evans-Vigier type exotic potentials as exact solutions which, when they are geometrized in spacetime, may be of the form (34) or (35).

5.4. Rotation and Evans-Vigier Field

Following our preceding account, we may now state that a geometrized Evans-Vigier field represents a classical but exotic electromagnetic - type field which possesses similar properties to gravitational fields defined by Schwarzschild and Kerr metrics. The process of geometrizing such an Evans-Vigier field, through association of the vector potential with part of the structure of spacetime, leads to the supposition that, possibly, there exists a fundamental relation between rotation and a magnetic-type field. It should be emphasized that in a sense our results demonstrate a generalisation of and the reciprocity to a well-known physical phenomenon. Thus, considering a free particle in an external electromagnetic field defined by the tensor $F_{\alpha\beta}$, we observe the generation of a vorticity

$$\omega_{\alpha\beta} = u_{\alpha;\beta} - u_{\beta;\alpha}, \tag{62}$$

which is related to the field tensor $F_{\alpha\beta}$ via the (London) equation of superconductivity [38, 39]:

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = \frac{mc}{e}\omega_{\alpha\beta}. \tag{63}$$

In a geometrized field, free noninteracting test particles placed in an external gravitational field define a *gravitational superconducting state* because they move on (force-free) geodesics meeting no 'resistance' [40, 41].

Equation (63) expresses that the four-vector potential A_α is tangent to the particle trajectories at all points [42] and thus the particle velocity is proportional to the vector potential as we have seen above. It is important to stress that it is the external vectorial field A_α which determines the motion of a test particle and not vice versa. Moreover, generally, the four-velocity u_α may be defined as the vector-potential of an *inertial-gravitational field* and may be assigned to each point of the spacetime independently of the fact whether or not a test particle resides at that point [43, 45]. Hence, if the vacuum spacetime is perturbed by the presence of the vectorial field A_α we can assert that the source of vorticity is precisely this field.

Our generalisation arises from the fact that not only does a normalized (Dirac) vector potential field [see Eq. (44)] generate a vorticity field, but yields also a relation between the angular momentum of a rotating body and a geometrized light-like vector potential. This result is clearly illustrated by Eqs. (35) and (39). It is not possible to write a simple relation similar to Eq. (63) because, in the present case, A_α is light-like and u_α is time-like and the relation between these quantities is more complicated than in (63) (see [28]).

However, it is not unreasonable to search for a relation between a vector potential A_α and the 'vorticity' of the wavevector of a null-free electromagnetic field (an incoherent fluid of photons) for which we quote the relations [46]:

$$\mathbf{E}^2\mathbf{B}^2 = 0, \tag{64}$$

$$\mathbf{E}\,\mathbf{B} = 0, \tag{65}$$

$$T_{\mathrm{em}\alpha\beta} = \xi_\alpha \xi_\beta \quad (\text{stress} - \text{energy tensor}), \tag{66}$$

$$\xi_\alpha \xi^\alpha = 0, \tag{67}$$

$$\xi_{\alpha;\beta}\xi^\beta = 0 \quad (\text{null geodesics}). \tag{68}$$

If A_α is proportional to the null wavevector ξ_α which now plays the role of a four-velocity vector, the existence of an Evans-Vigier magnetic-type field follows in a natural way: *An 'axial' magnetic-type field* $\mathbf{B}^{(3)}$ *is generated by the vorticity of the null wavevector which defines a null (i.e., transverse) electromagnetic field.* Characterizing the null electromagnetic field by its wavevector $\xi_\alpha(x^\beta)$, we can write

$$F^{EV}{}_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = \text{constant } \times \left(\frac{\partial \xi_\beta}{\partial x^\alpha} - \frac{\partial \xi_\alpha}{\partial x^\beta}\right). \tag{69}$$

On the other hand, let us consider as a simple example, a circularly polarized plane wave with a 3-vector potential

$$\mathbf{A} = (\hat{\mathbf{x}} \pm \mathrm{i}\hat{\mathbf{y}})\frac{E_0}{\omega}\exp\left(\mathrm{i}\omega t - \frac{\mathrm{i}\omega z}{c}\right), \tag{70}$$

where the $\pm$ signes correspond, respectively, to the complex vector potentials $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ which define the circularly polarized wave. Of course, we note that $\mathbf{B}^{(1)} = \nabla \times \mathbf{A}^{(1)}$ and $\mathbf{B}^{(2)} = \nabla \times \mathbf{A}^{(2)}$ represent two transverse magnetic fields of the electromagnetic wave. The Evans-Vigier hypothesis that vectorial fields of the form $\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ or $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ [the latter being inspired by the Yang-Mills relation (53)] could be associated with real electromagnetic fields must be verified by experiment.

Another idea proposed as a hypothesis by Evans and Vigier is that the fields $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)} \propto \mathbf{B}^{(1)} \times \mathbf{B}^{(2)}$ are related by the angular commutator theory. In fact, we recognize a similar idea in a recent paper [47] where it is shown that magnetic fields might be related to spatial rotations. Furthermore, we observe that the number of the non-null components of an electromagnetic field is dictated by the particular form of the parameters and generators of the Lorentz group.

We finally remark that a possible reciprocal magneto-rotation-induction effect: 'rotating (even neutral) particles generate a magnetic-type field' (see *section 1*) is supported also by an analogy between charge and spin in general relativity (see, *e.g.*, [48]) and thus, from a geometrical point of view, both charges and angular momenta are sources of electromagnetic-type fields.

6. DISCUSSION

In this section we detail some ideas related to the possibility that a photon becomes massive or acquires a mass by its interaction with other fields. Because our understanding of the Evans-Vigier field is still incomplete, we expect in this way to demonstrate the similarities and differences between the Proca field and the Evans-Vigier field.

6.1. Classical Proca Massive Photons

In the case of the classical electrodynamics of massive spin one particles in a flat (Minkowski) spacetime, the Proca Lagrangian density (in the absence of charges and currents) is

$$L_{\text{Proca}} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha, \tag{71}$$

where

$$\mu = \frac{m_{\text{ph}} c}{\hbar} = \frac{1}{\Lambda} \tag{72}$$

is the inverse Compton wavelength of the photon. We assert that even if $A_\alpha A^\alpha$ were zero we can still formally use the Lagrangian density (71) but have to substitute ultimately in the equations of motion $A_\alpha A^\alpha = 0$ if such an expression were possibly involved.

The Euler-Lagrange equations of motion with respect to A_α are precisely the Proca equations

$$\partial_\beta F^{\beta\alpha} + \mu^2 A^\alpha = \Box^2 A^\alpha - \partial^\alpha \left(\partial_\beta A^\beta\right) + \mu^2 A^\alpha = 0, \tag{73}$$

which describe the wave aspect of massive photons. If we apply the divergence (∂_α) to these equations, we find

$$\partial_\alpha A^\alpha = 0 \tag{74}$$

for $\mu \neq 0$. We see that the Lorentz condition is now a necessary condition and not a choice of gauge.

On the basis of a *minimal coupling procedure* ($\eta_{\alpha\beta} \to g_{\alpha\beta}$, and $\partial_\beta F^{\beta\alpha} \to F^{\beta\alpha}{}_{;\beta}$), we observe that the Proca equations in a curved spacetime, assuming that the electromagnetic field is weak and does not perturb the background metric, are given by

$$F^{\beta\alpha}{}_{;\beta} + \mu^2 A^\alpha = 0. \tag{75}$$

The Proca equations arise here as the only possible linear generalization of the Maxwell equations [49]. As a result of the presence of the coupling constant μ (the mass of the photon), the potentials become directly measurable (observable) quantities.

For the static case, the Proca equations *in vacuo* reduce to

$$\nabla^2 A^\alpha = \mu^2 A^\alpha. \tag{76}$$

For example, the electrostatic scalar potential of a point charge placed at the origin (Proca-Yukawa potential) becomes

$$\phi(r) = \frac{\text{const}}{r} \exp(-\mu r). \tag{76}$$

Thus, we observe that the assumption of a non-vanishing mass for photons leads to a deviation from the Coulomb law. Another consequence is that a free electromagnetic wave with massive photons,

$$\begin{aligned} A^\alpha = P^\alpha(p) \exp[-\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{r})] &\equiv P^\alpha(p) \exp\left(-\frac{\mathrm{i}}{\hbar} p_\alpha x^\alpha\right) \\ &\equiv P^\alpha(p) \exp(-\mathrm{i} k_\alpha x^\alpha), \end{aligned} \tag{78}$$

possesses a dispersion relation in a vacuum,

$$\omega^2 = (k^2 + \mu^2)c^2, \tag{79}$$

as well as a group velocity

$$v_\mathrm{g} = \frac{d\omega}{dk} = \frac{c}{\omega}(\omega^2 - \mu^2 c^2)^{1/2}, \tag{80}$$

where $P^\alpha(p) \equiv (P^0, \mathbf{P})$ is the polarization vector of the photon, $p^\mu \equiv (p^0, \mathbf{p})$ is the 4-momentum of the photon, and $k \equiv (k^0, \mathbf{k})$ is the wave vector. To each component of the quantized four vector potential, there corresponds a special type of a photon. In conjunction with the Lorentz condition (74) we obtain

$$p_\alpha P^\alpha = 0. \tag{81}$$

Since there exists no gauge invariance, the massive photon possesses three degrees of polarization (freedom), corresponding to helicities (spin projections along the direction of propagation) $\varepsilon = \pm 1$ and 0. For instance, a massive photon with momentum $\mathbf{p}$ along the z axis displays the polarizations:

$$P^{\mathrm{R,L}} := P^{(\varepsilon=\pm 1)} \qquad (82)$$
$$:= \mp\frac{1}{\sqrt{2}}\left(\hat{e}_x \pm \mathrm{i}\hat{e}_y\right) \quad (\text{circular polarizations}: \text{ R and L}),$$

$$P^{(\varepsilon=0)} = \frac{1}{\mu}\left(|\mathbf{p}|, 0, 0, \frac{\mathrm{E}}{\mathrm{c}}\right). \qquad (83)$$

These polarizations satisfy the relation of completeness [50]

$$\sum_{\varepsilon} P_{\alpha}^{(\varepsilon)*} P_{\beta}^{(\varepsilon)} = -\eta_{\alpha\beta} + \frac{p_\alpha p_\beta}{\mu^2}, \qquad (84)$$

which suggests a possible link with the projection tensor in an infinitesimal 3-space orthogonal to p_α. Since

$$\frac{E^2}{c^2} - \mathbf{p^2} = \mathbf{m_{\mathrm{ph}}^2 c^2} \neq 0 \qquad (\text{"mass} - \text{shell" condition}), \qquad (85)$$

we observe that the Proca longitudinal polarization (83) does not satisfy the Evans-Vigier hypothesis $A_\alpha A^\alpha = 0$. The Maxwell-Proca stress-tensor corresponding to the Lagrangian density (71)) is

$$\begin{aligned} T_{\alpha\beta} &= T_{\alpha\beta}^{\mathrm{Maxwell}} + T_{\alpha\beta}^{\mathrm{Proca}} \\ &= \frac{1}{4\pi}\left(\frac{1}{4}\eta_{\alpha\beta}F_{\gamma}{}^{\delta}F^{\gamma}{}_{\delta} - F_{\alpha}{}^{\gamma}F_{\beta\gamma}\right) \\ &+ \frac{\mu^2}{4\pi}\left(\frac{1}{2}\eta_{\alpha\beta}A_\gamma A^\gamma - A_\alpha A_\beta\right), \end{aligned} \qquad (86)$$

and this expression can be consistent with an Evans-Vigier field only if we assume the condition $A_\alpha A^\alpha = 0$ here but do not extend this to the original Lagrangian (71).

We conclude this subsection with a discussion on some properties of Maxwell, Proca and, possibly, also of the Evans-Vigier field.

1. The electromagnetic field entries are composed of two parts: the transverse part for which energy and momentum form a

four vector and behave like the energy and momentum of free particles as far as their transformation properties are concerned, and the longitudinal part for which this does not apply. In conventional quantum electrodynamics, only the first (transverse) part is subject to a quantification, giving rise to a photon. The second part remains usually unquantified. The latter part gives rise, in the presence of matter, to the Coulomb interaction between charged bodies in classical electrodynamics (lines of force obeying Gauss' law) and the quantum electrodynamic effects (exchange messenger, virtual particles).

2. The longitudinal photon does not transport energy and cannot be observed as a free particle. Thus, for instance, there does not exist any gravitational interaction between a longitudinal photon and a massive object, and this is why a Coulomb field can cross the event horizon of a black hole. At this point we can offer a non-local interpretation to this phenomenon. Indeed, the longitudinal photons do not travel in space as do ordinary transverse photons at a finite speed c. They simply happen as instantaneous (*i.e.*, non-local) extended events in which the longitudinal quanta (or 'connections' [52]) are spatially and non-temporally distributed and connected reproductions as it happens in an ordinary still photograph. Such a longitudinal photon may possibly be associated with an Evans-Vigier $\mathbf{B}^{(3)}$ field. It is evident that we can define in a similar way a *longitudinal graviton* corresponding to a gravitomagnetic $\mathbf{B}_g^{(3)}$ field.

As regards the estimates of the (rest) mass of a photon we quote the upper limit which is extracted from "Galactic Electrodynamics" [53]

$$m_{\mathrm{ph}} \lesssim 10^{-58}\mathrm{gm}. \tag{87}$$

It is very difficult to hope that the effects of such a minute mass may be detected in a laboratory.

We remind the reader that the condition $\mu \neq 0$ is not consistent with gauge invariance, and for this reason Proca's electrodynamics appears aesthetically defective to many theoretical physicists. However, the only certain assertions on the value of μ which can be made must be based on experiments [54], and observations we might add. Furthermore, it is important to stress that Hora [55], studying the motion of electrons in a laser field, noted the presence of an unexpected longitudinal component. At this point we mention that in an infinite plane wave, the $\mathbf{E}$ and $\mathbf{B}$ fields are everywhere perpendicular to the wave vector and the energy flow is everywhere parallel to the wave vector. However, in a wave of finite transverse extent (e.g. a circularly polarized wave propagating in the z direction, as, for instance, in the laser beam contained within an optical fiber with a radius of about a micron) the **E** and **B** fields have a (longitudinal) component parallel to the wave vector (the field lines are closed loops). This is a well known classical result (see, for instance, [56]).

6.2. Interacting Electromagnetic and Gravitational Fields

We consider a free electromagnetic field, in a vacuum spacetime, which creates its proper gravitational field $g_{\alpha\beta}$. Such an interacting system incorporating a gravitational field ($g_{\alpha\beta}$) and an electromagnetic field (A_α) may be specified by the Lagrangian density [57]

$$\begin{aligned} L = &- \frac{1}{16\pi}\sqrt{-g}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{\kappa}\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta} \\ &+ \chi\frac{1}{\kappa}\sqrt{-g}A_\gamma A^\gamma g^{\alpha\beta}R_{\alpha\beta}, \end{aligned} \tag{88}$$

where χ is the coupling constant. If we require that the action emanating from this Lagrangian is to be stationary with respect to variations in $g_{\alpha\beta}(x)$, one obtains the Euler-Lagrange equations. These are, in fact, the Einstein's equations, for the nonlinear interaction between electromagnetic and gravitational fields. Thus we find

$$\begin{aligned} G_{\alpha\beta} &= -\kappa T_{\alpha\beta} \\ &\equiv -\kappa\Big[T^{\text{classical}}_{\alpha\beta} + \frac{\chi}{\kappa}A_\gamma A^\gamma G_{\alpha\beta} + \frac{\chi}{\kappa}RA_\alpha A_\beta \\ &- \frac{\chi}{\kappa}(A_\gamma A^\gamma)_{;\delta}{}^{;\delta}g_{\alpha\beta} + \frac{\chi}{\kappa}(A_\gamma A^\gamma)_{;\alpha;\beta}\Big]. \end{aligned} \tag{89}$$

Contracting (89) with $g^{\alpha\beta}$, we find

$$R = -3\chi(A_\gamma A^\gamma)_{;\delta}{}^{;\delta}. \tag{90}$$

Now, the Euler-Lagrange equations deduced from (88) by a variation of A_α are

$$F^{\alpha\beta}{}_{;\beta} + \frac{\chi}{\kappa}RA^\alpha = 0, \tag{91}$$

or

$$F^{\alpha\beta}{}_{;\beta} - 3\frac{\chi^2}{\kappa}RA^\alpha(A_\gamma A^\gamma)_{;\delta}{}^{;\delta} = 0. \tag{92}$$

We stress that these are the equations of an electromagnetic field interacting with its proper gravitational field. Comparing also with Eqs. (75), we establish a relation for the mass of the photon:

$$\mu^2 = \frac{\chi}{\kappa} R := -3\frac{\chi^2}{\kappa}(A_\gamma A^\gamma)_{;\delta}{}^{;\delta}. \tag{93}$$

Thus, we may conclude that a photon acquires a mass as a result of its nonlinear interaction with its proper gravitational and (or) electromagnetic field. The mass of the photon is directly proportional to the magnitude of the vector potential A_α and (or) to the curvature scalar of spacetime.

We finally remark that in the present case the Proca massive field does not coincide with the Evans-Vigier field.

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