The Study of Partial Differential Equations of the First Order in the 18th and 19th Centuries

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Communicated by A. P. YOUSHKEVITCH

1. Introduction

I consider studies in the theory[1] of first-order partial differential equations. From my standpoint, these studies constitute a process of gradual understanding of the ideas which formed the intrinsic essence of the theory, the essence revealed

[1] Mathematicians use the word *theory* in two essentially different meanings. In a

by LIE in the 1870's. The development of the theory of partial differential equations took place in several stages, or periods, during each of which only some of the ideas were prominent. Moreover, as seen from the vantage-ground of subsequent history, the prominent ideas were often considered from a particular point of view. Note that even in the presence of such dominant ideas as characterized one or another period, other concepts might well have been discussed in the outlying districts of the region under study, thus paving the way for a later stage.

I single out four stages. *The first one* (§ 2) lasted to the end of the 1760s, or the beginning of the 1770s. It was all but exclusively connected with EULER and D'ALEMBERT, and it was primarily characterized by integration of equations with the aid of a number of specific versions of the method of multipliers. Accordingly, mathematicians extensively used expressions representing total differentials. Specific methods used for the solution of equations were devoid of any geometric interpretation, and for this reason I call the whole period *formal-analytic*. The new concept of a complete solution of first order partial equations that matured in LAGRANGE'S studies began a new stage.

The distinctive feature of *the second period* (§ 3) (from the beginning of the 1770s to the 1830s) is the development of LAGRANGE'S theory. Other main characters involved at this stage besides LAGRANGE were MONGE, who developed the geometric aspect of the theory and, also, PFAFF, CAUCHY and C. G. JACOBI. They largely completed the program of research inherent in the theory.

JACOBI'S study of his so-called 'second method' prompted by requirements of mechanics constituted the chief subject of *the next* (*the third*) *period* (§ 4), which lasted until the end of the 1860s. HAMILTON was the first to establish close ties between mechanics and partial differential equations while JACOBI followed in his steps.

In the beginning of the 1870s, LIE constructed his 'general theory', which made up the subject of *the fourth period* (§ 5). The premisses of LIE'S 'theory' took shape, during the former periods while general geometric ideas framed exactly at the same time (the beginning of the 1870s) served as its foundation.

Concepts which acted as hidden nerves of studies on first-order partial differential equations were completed in LIE'S 'theory'. In its context, methods and ideas developed by previous authors were treated as parts of a single whole rather than a collection of so many scarcely interconnected fragments.

Each of the four stages ended when their central ideas became sufficiently realized in corresponding 'theories'. It is common knowledge that new fundamental ideas whose application demands a transition to a subsequent stage are formulated and developed under the influence of both internal and external factors.

Internal factors alone can determine the development of a mathematical theory but in the case I study external agents (mechanics in the first place) turned out to be powerful motives for progress. Without them, I presume, the theory would not

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narrow sense, it denotes a complex structure based on definite ideas and methods and covering a certain range of studies (thus, the theory of GALOIS, or LAGRANGE'S theory of first-order partial differential equations). In a broad sense, the word *theory* designates a province of thought (*e.g.*, theory of numbers; of differential equations). To distinguish between the two cases, I use single quotation marks in the latter instance ('theory').

have advanced so intensively. As evidenced by the work of D'ALEMBERT (see my § 2.2) and JACOBI (§ 4.1), mechanics exerted a permanent and prevailing influence on mathematical studies from their very beginning.

2. Formal-analytic Period

2.1. Euler. Historians of science (M. CANTOR [1], H. WIELEITNER [1], V. ANTROPOVA in A. P. YOUSHKEVITCH, ed. [1]) assume that the theory of first order partial differential equations commenced in 1740 with one of EULER's works [1]. This tradition can be traced back to the end of the 18^{th} century, and, in particular, to COUSIN [1, p. xiv]. His opinion contradicted the then generally accepted view according to which the origination of the theory took place in D'ALEMBERT's works published in the 1740s.

For my part, I [2] think that EULER's memoir [1] constituted no more than the prehistory of the new branch of analysis. Concerning himself with geometric problems, he encountered expressions which we now interpret as partial differential equations. EULER intuitively sensed their importance and considered them in detail. Nevertheless, this part of his work, though not closely tied to geometric problems, did not acquire any independent significance. Evidently, even EULER himself, to say nothing about his contemporaries, could not predict its future role. Only later on, in successfully attempting to establish EULER's priority, did his followers recall this part of his work.

2.2. D'Alembert. During the 1740s, D'ALEMBERT arrived at a number of equations, or systems of equations, of mathematical physics:

(1) In 1743 [1] at the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial y}{\partial s} - (l - s)\frac{\partial^2 y}{\partial s^2}. \tag{2.2.1}$$

(2) In 1747 [2] at systems

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial s}, \quad \nu\frac{\partial \beta}{\partial u} = \varrho\frac{\partial \alpha}{\partial s} + \varphi(u,s) \tag{2.2.2}$$

and

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial s}, \quad \varrho\frac{\partial \alpha}{\partial s} + p\frac{\partial \beta}{\partial s} = \gamma\frac{\partial \beta}{\partial u} + m\frac{\partial \alpha}{\partial u} + \varphi(u,s).$$

(3) In 1749 [3] at an equation for the vibrations of the string

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}. \tag{2.2.3}$$

D'ALEMBERT wrote one of his equations (2.2.1) as

$$\frac{dp}{dt} = q - (l - s)\frac{dq}{ds}. \tag{2.2.1'}$$

Here, in his own notation, which differs but little from the one in current use,

$$p = \frac{dy}{dt}, \quad q = \frac{dy}{ds}.$$

As to other equations ((2.2.2) and (2.2.3)), he put them down as expressions in total differentials. Thus he wrote the latter in the form of a system

$$dp = \alpha\, dt + \nu\, ds, \quad dq = \nu\, st + \alpha\, ds \qquad (2.2.4)$$

where

$$p = \frac{\partial y}{\partial t}, \quad q = \frac{\partial y}{\partial s}.$$

The appearance of such systems was likely occasioned by the method which D'ALEMBERT employed for the integration of equations ((2.2.2) and (2.2.3)).[2] He developed this *method of multipliers* undoubtedly proceeding from various tricks due to EULER [1]. At least, D'ALEMBERT knew this memoir: elsewhere [2] he referred to EULER'S *De infinitis curvis etc*, originally published in the same volume of the *Commentarii Acad. Sci. Imp. Petrop.* as the *Additamentum* [1] to it.[3]

In § 2.3 I adduce examples of the use of D'ALEMBERT'S method of multipliers for the solution of first-order equations. For the time being, I shall only say that starting from differential relations involved in a given system of equations (such as system (2.2.4)) he formed their linear combination with suitable numerical or functional coefficients; he then transformed the linear combination into an integrable expression by substitutions of the independent variables and the function sought.

Exactly in the works which I mentioned above D'ALEMBERT treated partial differential equations as an object belonging to a new branch of analysis; he formulated the problem of their solution[4] and, finally, introduced the first methods for their integration.

The most eminent mathematicians of the time at once turned their attention to D'ALEMBERT'S works. EULER himself became interested in the new field of study, and he was compelled to continue his own research [1]. Just after D'ALEMBERT published his study of the vibrations of strings [3], EULER [2] offered a modification of D'ALEMBERT'S method of integrating the system (2.2.4) and expressed his views on the nature of the solution obtained. From this moment onwards EULER began his long study of the theory of partial differential equations, in which he strove for superiority to D'ALEMBERT. The work of these outstanding scholars was the essence of the first period.

2.3. Main Achievements. The new domain of analysis provided enough room for research while pertinent methods proved to be indispensable for the solution of a series of problems in mechanics, thus provoking widespread interest. For

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[2] In 1743 he was not yet able to integrate equation (2.2.1).

[3] Naturally enough, the *Additamentum* followed just after the main memoir, *De infinitis curvis.*

[4] To find a function which transforms the given equation into an identity.

all that, only a selected few were able to master the new branch, which up to the 19th century remained the most complicated field of analysis.

At first, efforts were mainly concentrated on equations of the second order: mechanics, the dominant science of the 18th century, led to just these equations. Even so, the authors of the new calculus soon turned great attention to equations of the first order. Naturally enough, a systematic development of the general theory of partial equations, a goal perceived in D'ALEMBERT'S and EULER'S works of the 1760s, first and foremost demanded the study of equations of the first order.

The main achievements in this direction are due to EULER [4][5] and D'ALEMBERT [5] (1764 and 1768 respectively), who reduced partial equations to equations in total differentials and solved these by the aid of one or another specific version of the method of multipliers. Consider for example a problem due to EULER [6, problem 21]. It is required to solve the equation

$$px + qy = 0 \qquad \left(p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right).$$

First, EULER arrived at an equation in total differentials

$$dz = p\,dx + q\,dy = p\,dx - \frac{px}{y}\,dy = p\left(dx - \frac{x}{y}\,dy\right).$$

Then he noticed that the expression

$$dx - \frac{x}{y}\,dy$$

possesses an integrating factor $1/y$, so

$$dz = py\left(\frac{dx}{y} - \frac{x}{y^2}\,dy\right) = py\,d\left(\frac{x}{y}\right).$$

Finally, he concluded that py is a function of x/y, so

$$py = f'\left(\frac{x}{y}\right), \quad z = f\left(\frac{x}{y}\right)$$

where $f(t)$ is an arbitrary function of its argument.

This example vividly illustrates the two features characteristic of the first period, *viz*:

(1) Reduction of partial equations to equations in total differentials. This mode of action, which retained the ties between the virgin tract and the cultivated area of ordinary differential equations, was caused by the use of the method of multipliers.

(2) A formal-analytic approach combined with the use of clever tricks for reducing differential expressions to integrable forms; lack of any geometric interpretation either of equations or their solutions. Note that exactly the formal nature of analytic tricks which he used for integrating some partial equations[6] enabled

[5] See also his later publication [6].

[6] For example, equations $pq = 1$, $q = f(p)$, $q = f_1(p)\,x + f_2(p)$.

EULER [6, problems 12, 14, 17] to accept partial derivatives p (or q) as independent variables. Thus, EULER was the first to apply contact transformations effectively to partial differential equations.

Regarding equations of the first order, essential progress was achieved during this first stage. Mathematicians integrated the following important equations:

1. $\frac{\partial z}{\partial x} + \lambda(x, y) \frac{\partial z}{\partial y} + \omega(x, y) = 0$

(EULER, 1764 [4]).

2. $\frac{\partial z}{\partial x} + \lambda(x, y) \frac{\partial z}{\partial y} + \xi(x, y)\, z + \nu(x, y) = 0$

(D'ALEMBERT, 1768 [5]).

3. $\frac{\partial z}{\partial x} + \lambda(x, y) \frac{\partial z}{\partial y} + f(x, y, z) = 0$

(LAPLACE, 1777 [1]).

Needless to say, not all of the achievements of the 1750s and 1760s fall within my rigid description of characteristic features of this period (see above). Following D'ALEMBERT (see my equation (2.2.1') in § 2.2), mathematicians gradually developed an inclination to consider equations irrespective of corresponding equations in total differentials. The inclination became standard practice under EULER's influence (EULER [3], TRUESDELL [1, p. 260]).

The first methods of integration adapted to the new manner of writing the equations were the methods of separation of variables and of characteristic coordinates. The former is due to D'ALEMBERT (see also TRUESDELL [1, p. 241]); in 1752 he [4] applied what might be called the kernel of the method, and he subsequently [1, 2nd edition] fully developed his idea. The latter method is due to EULER who introduced it in 1766 [5] for the solution of wave equations.

3. Lagrange's 'Theory'

3.1. Origin (Euler and Lagrange). According to EULER [6, §§ 37 and 249] (see also ENGELSMAN [1]) an integral of an n^{th} order partial differential equation is complete if it includes n arbitrary functions.[7] He understood the completeness of the solution thus defined in the sense that it contained a totality of particular solutions obtained by corresponding specialization of the arbitrary functions.

The origin of LAGRANGE's 'theory' is connected with his gradual approach to the new concept of a complete solution. Naturally enough, he commenced from EULER's understanding of the term. In 1774, applying methods extremely similar in spirit to those used by EULER, he [1] considered anew some problems from the *Institutiones*. Still adhering to EULER's terminology, LAGRANGE noticed that solu-

[7] Note that EULER [6, § 38] guided himself by an analogy with the solution of ordinary equations, replacing arbitrary constants by arbitrary functions.

tions with two arbitrary constants are also complete in the sense that their variation yields all solutions including those which are complete in EULER's sense.

This remark acquired a dominant significance in a later (1776) memoir [2] which contained a new concept of a complete solution and thus signified that LAGRANGE had progressed beyond EULERIAN ideas. Here LAGRANGE called a solution of equation

$$f(x, y, z, p, q) = 0 \quad \left(p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right) \tag{3.1.1}$$

complete if it depended on two arbitrary constants (a and b): $z = \varphi(x, y, a, b)$. Justifying the name of this term he showed that, varying the two constants, it was possible to determine all other solutions. Indeed, suppose $b = \psi(a)$ where ψ is an arbitrary function of its argument and exclude a from system

$$z = \varphi(x, y, a, \psi(a)), \quad \frac{\partial z}{\partial a} = 0.$$

Then the solution thus obtained which depends on an arbitrary function will be general (*intégrale générale*), or complete, according to EULER. Finally, eliminate both a and b from the system

$$z = \varphi(x, y, a, b), \quad \frac{\partial z}{\partial a} = 0, \quad \frac{\partial z}{\partial b} = 0$$

and the corresponding solution will be singular (*intégrale particulière*).

Thus LAGRANGE reduced the integration of equation (3.1.1) to the discovery of its complete solution.

LAGRANGE revealed the geometric sense of his terms. A *complete* solution defined a two-parameter family of solutions while a *general* solution corresponded to a totality of envelopes of an arbitrarily chosen one-parameter subfamily of the surfaces contained in the complete solution.[8] Finally, a *singular* solution determined the envelope of the entire two-parameter family of surfaces contained in the complete solution.

Still, LAGRANGE did not construct a consistent geometric 'theory' of first order equations. This noteworthy step was taken by MONGE who published his findings in a whole series of memoirs on the subject; I shall mention only two of them [1; 2], published in 1787 and 1807 respectively. In particular, MONGE introduced the notion of characteristics effectively considered even by EULER and LAGRANGE and showed how solutions could be constructed with their help.

3.2. Charpit. LAGRANGE's creative work led him [3; 4] (1781 and 1787 respectively) to a problem of integrating the equation

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, z)\frac{\partial z}{\partial x_i} = b(x_1, x_2, \ldots, x_n, z) \tag{3.2.1}$$

[8] The subfamily was defined by relation $b = \psi(a)$.

for arbitrary n by reducing it to a system of ordinary equations

$$\frac{dx_i}{a_i(x_1, x_2, \ldots, x_n, z)} = \frac{dz}{b(x_1, x_2, \ldots, x_n, z)}. \tag{3.2.2}$$

P. CHARPIT completed LAGRANGE's study of equation

$$f(x, y, z, p, q) = 0. \tag{3.2.3}$$

In 1784 CHARPIT submitted his work to the Paris Academy of Sciences, but he died prematurely (in 1785) and the manuscript remained unpublished. LACROIX [1, §§ 740–741] published some information about CHARPIT's findings.[9]

In essence, the method, due to LAGRANGE and CHARPIT, consists in determining a function $\varphi(x, y, z, p, q)$ such that
(a) two equations

$$f(x, y, z, p, q) = 0, \qquad \varphi(x, y, z, p, q) = a, \tag{3.2.4}$$

where a is an arbitrary constant, may be solved with respect to p and q:

$$p = f_1(x, y, z, a), \qquad q = f_2(x, y, z, a);$$

(b) the equation

$$dz = f_1(x, y, z, a)\, dx + f_2(x, y, z, a)\, dy = p\, dx + q\, dy \tag{3.2.5}$$

is identically satisfied for all values of a. Integration of it provides a solution which includes not only the arbitrary constant a, but also the arbitrary constant of integration.

Thus, the integration furnishes a complete solution of equation (3.2.3), so the entire problem is reduced to the discovery of an additional equation, the second in system (3.2.4). It is not difficult to prove that in order to determine the function $\varphi(x, y, z, p, q)$, it is sufficient to satisfy identically the relation (3.2.5), leading to an equation

$$P\frac{\partial\varphi}{\partial x} + Q\frac{\partial\varphi}{\partial y} + (Pp + Qq)\frac{\partial\varphi}{\partial z} - (X + pZ)\frac{\partial\varphi}{\partial p} - (Y + qZ)\frac{\partial\varphi}{\partial q} = 0, \tag{3.2.6}$$

$$P = \frac{\partial f}{\partial p}, Q = \frac{\partial f}{\partial q}, X = \frac{\partial f}{\partial x}, Y = \frac{\partial f}{\partial y}, Z = \frac{\partial f}{\partial z}.$$

Indeed, equation (3.2.6) belongs to the type (3.2.1) so that (see above) it is reduced to a system of ordinary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Pp + Qq} = \frac{-dp}{X + pZ} = \frac{-dq}{Y + qZ}. \tag{3.2.7}$$

Any one of its integrals, $\varphi(x, y, z, p, q) = a$, which includes one arbitrary constant, suffices to solve the problem.

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[9] At the beginning of this century CHARPIT's manuscript was found. SALTYKOW [1; 2] described it, but his work passed unnoticed.

Now consider arbitrary equations

$$f(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0 \quad \left(p_i = \frac{\partial z}{\partial x_i}\right) \tag{3.2.8}$$

with more than two ($n > 2$) independent variables. It would have been natural to integrate such equations by discovering one or another generalization of the LAGRANGE-CHARPIT method. However, this approach leads to serious difficulties. Without loss of generality suppose that equation (3.2.8) does not explicitly include z[10]:

$$f(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = 0.$$

Suppose also that there are $n - 1$ more relations between $p_1, p_2, \ldots, p_n$ involving $n - 1$ arbitrary constants $h_1, h_2, \ldots, h_{n-1}$ and such that expression

$$p_1(x_1, x_2, \ldots, x_n, h_1, h_2, \ldots, h_{n-1})\, dx_1 + \ldots \tag{3.2.9}$$
$$+ p_n(x_1, x_2, \ldots, x_n, h_1, h_2, \ldots, h_{n-1})\, dx_n$$

is a total differential. In this case the complete integral sought will be determined by integrating the differential (3.2.9). Now for expression (3.2.9) to be a total differential it is of course necessary and sufficient that

$$\frac{\partial p_i}{\partial x_k} = \frac{\partial p_k}{\partial x_i}$$

for any $i, k = 1, 2, \ldots, n$. Thus, the $n - 1$ functions sought must satisfy $(n - 1)\, n/2$ conditions. Only if $n = 2$ does the number of conditions coincide with that of the unknown functions; otherwise the former number is larger than the latter.

It is likely that many mathematicians of the time (the end of the 18th century) attempted to solve equation (3.2.8).[11] Even so, there was no progress, and the corresponding problem remained one of the most important. Its significance for the beginning of the 19th century is proved by the fact that such eminent scholars as PFAFF, CAUCHY and C. G. JACOBI contributed to its solution.

3.3. Pfaff. PFAFF [1] was the first to integrate equation (3.2.8). In 1815, at a sitting of the Berlin Academy of Sciences, he delivered his report on the subject.

PFAFF shunned the geometric spirit characteristic of the French mathematical school. He was rather attracted by EULER's formal-analytic style and, in actual fact, he kept to the same style, or approach, in his own constructions. Exactly this approach enabled PFAFF to consider the problem as though in a $2n$-dimensional space[12] and, moreover, following EULER, to regard partial derivatives with respect to the unknown function as independent variables.

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[10] It is not difficult to eliminate z from the equation.

[11] According to LACROIX [1, p. 567], CHARPIT himself considered it, but without success.

[12] I say 'as though' since no such geometric interpretation was possible in the beginning of the 19th century.

PFAFF reduced the problem of integrating equation (3.2.8), or, as he himself usually wrote it, equation

$$p_n = \varphi(x_1, x_2, \ldots, x_n, z; p_1, p_2, \ldots, p_{n-1}), \tag{3.3.1}$$

to an equivalent problem of integrating the equation

$$dz - p_1\,dx_1 - p_2\,dx_2 - \cdots - p_{n-1}\,dx_{n-1} - \varphi(x_1, x_2, \ldots, x_n, z; p_1, p_2, \ldots, p_{n-1})\,dx_n + 0\,dp_1 + \cdots + 0\,dp_{n-1} = 0. \tag{3.3.2}$$

However, he actually considered a more general equation in total differentials:

$$A_1(y_1, \ldots, y_k)\,dy_1 + \cdots + A_k(y_1, \ldots, y_k)\,dy_k = 0 \quad (k = 2n). \tag{3.3.3}$$

PFAFF showed that under a particular substitution of variables the latter is replaced by a similar equation in $(2n - 1)$ variables. He noted without proof that the substitution of this kind is possible only for even values of k. As to the case $k = 2n - 1$, equation (3.3.3) might be reduced to a similar equation with $k = 2n - 2$ by a different method, and PFAFF proved that in this instance the equation is finally solved by determining n relations involving $y_1, \ldots, y_k$ and an arbitrary function. For equation (3.3.2) the n relations involve $x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_{n-1}$ and, again, an arbitrary function. Elimination of $p_1, p_2, \ldots, p_{n-1}$ from these relations furnishes the general solution.

Thus PFAFF introduced a method which at least in principle made it possible to integrate equation (3.2.8). Still, his approach led to great difficulties. If $k = 2n$, to diminish by 1 the number of variables in equation (3.3.3) demanded a solution of a system of ordinary differential equations and, accordingly, the problem as a whole involved a solution of n such systems, the first of them being

$$\frac{dx_i}{\dfrac{\partial f}{\partial p_i}} = \frac{dz}{\sum\limits_{k=1}^{n} p_k \dfrac{\partial f}{\partial p_k}} = \frac{dp_j}{-\dfrac{\partial f}{\partial x_j} - \dfrac{\partial f}{\partial z} p_j} \quad (i, j = 1, 2, \ldots, n). \tag{3.3.4}$$

Subsequent research due to CAUCHY and JACOBI (see my §§ 3.4 and 3.5) proved that the integration of system (3.3.4) alone is sufficient for a complete solution of equation (3.2.8). Accordingly, PFAFF'S method was set aside.[13]

3.4. Cauchy. The complicated political situation then prevailing in Europe hindered correspondence. Accordingly, CAUCHY, who investigated equation (3.2.8) for arbitrary values of n, remained unaware of PFAFF'S findings, work he saw only when preparing his own inquiry for publication. CAUCHY published his memoir [1] in 1819. He began it by explaining his point of view: "... since the case concerns one of the most important problems in integral calculus, and since the method due to Mr PFAFF differs from mine," CAUCHY noted, his (CAUCHY'S) own exposition was also important for geometers.

CAUCHY considered only equations with two and three independent variables,

[13] Note that his work proved highly important for the theory of equations in total differentials, equations subsequently called after PFAFF.

but he pointed out that the general case presented no additional difficulties. Here is his method (I restrict attention to an equation with two variables). Given an equation

$$f(x, y, z, p, q) = 0, \tag{3.4.1}$$

it is required to find a solution $z(x, y)$ such that $z(x_0, y) = \varphi(y)$. If x, y and z are functions of two independent variables, then the problem is reduced to discovering five functions x, y, z, p, q of these variables which satisfy equation (3.4.1) and the relation

$$dz - p\,dx - q\,dy = 0. \tag{3.4.2}$$

CAUCHY uses a change of variables introduced by AMPÈRE in 1815 [1],

$$x = x, \quad y = y(u, x), \tag{3.4.3}$$

to arrive at

$$\frac{\partial z}{\partial x} = p + q\frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial u} = q\frac{\partial y}{\partial u}. \tag{3.4.4}$$

He differentiates equation (3.4.1) with respect first to x and then u. Setting

$$X = \frac{\partial f}{\partial x}, Y = \frac{\partial f}{\partial y}, Z = \frac{\partial f}{\partial z}, P = \frac{\partial f}{\partial p}, Q = \frac{\partial f}{\partial q},$$

he gets

$$\begin{cases} X + Y\dfrac{\partial y}{\partial x} + Z\dfrac{\partial z}{\partial x} + P\dfrac{\partial p}{\partial x} + Q\dfrac{\partial q}{\partial x} = 0, \\[2ex] Y\dfrac{\partial y}{\partial u} + Z\dfrac{\partial z}{\partial u} + P\dfrac{\partial p}{\partial u} + Q\dfrac{\partial q}{\partial u} = 0, \end{cases} \tag{3.4.5}$$

so now functions x, y, z, p, q as of variables x and u must satisfy equations (3.4.4) and (3.4.5). CAUCHY chose a function u such that these latter became a system of four equations involving only derivatives with respect to x but not with respect to u:

$$P\frac{\partial y}{\partial x} = Q,\ P\frac{\partial z}{\partial x} = Pp + Qq,\ P\frac{\partial q}{\partial x} = -Y - qZ,\ P\frac{\partial p}{\partial x} = -X - pZ.$$

This system coincides with the system (3.2.7) which appears in the LAGRANGE-CHARPIT method. Its integral under initial conditions $(x_0, u, \varphi(u), p(x_0, u), \varphi'(u))$ furnishes the solution sought.

Thus CAUCHY reduced the integration of equation (3.4.1) to the solution of one system of ordinary differential equations rather than n systems as demanded by the method due to PFAFF (see my § 3.3). CAUCHY also constructed solutions consisting of characteristics[14] passing through curve $x = x_0$, $z = \varphi(y)$. He thus furthered the theory of characteristics originated by MONGE (see § 3.1). However, over a long period of time CAUCHY's memoir, which had been published in an

[14] He did not use the term itself.

inappropriate periodical, remained little known. In particular, C. G. JACOBI (see my § 3.5) knew nothing about it. In 1841 CAUCHY [2] once more described the essence of his method.

3.5. C. G. Jacobi: his "First Method". JACOBI contributed essentially to PFAFF'S method (see my § 3.3), and he published one of his papers [2] on this subject as early as 1827. Ten years later, commencing from this very method, he [3] came to effect the first change of variables into the idea of 'initial values'. This concept was closely connected with the works of W. R. HAMILTON published in 1834–1835 which JACOBI then studied with utmost diligence, using them as a starting point for his own investigations in mechanics.

Considering equation (3.2.8) and following PFAFF, JACOBI wrote the corresponding system of equations as

$$\frac{dx_i}{P_i} = \frac{dz}{\sum_{k=1}^{n} P_k p_k} = \frac{-dp_j}{X_j + Zp_j} \tag{3.5.1}$$

with

$$P_i = \frac{\partial f}{\partial p_i}, \quad X_j = \frac{\partial f}{\partial x_j}, \quad Z = \frac{\partial f}{\partial z}, \quad i, j = 1, 2, \ldots, n.$$

Its integration furnished a system of $2n$ independent integrals

$$u_i(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = a_i, \quad i = 1, 2, \ldots, 2n-1,$$
$$f(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0.$$

JACOBI then assumed $u_1, u_2, \ldots, u_{2n-1}, z$ to be the new variables. From this point on his "first method", as it was subsequently called, differed from the one due to PFAFF. Using the idea of initial values, JACOBI set

$$x_i|_{z=0} = \xi_i, \quad p_i|_{z=0} = \pi_i.$$

The quantities thus introduced were functions of $u_1, u_2, \ldots, u_{2n-1}$ and, therefore, of $z, x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n$. JACOBI showed that

$$-dz + p_1\,dx_1 + \cdots + p_n\,dx_n = \sum_{i=1}^{n} \sum_{j=1}^{2n-1} p_i \frac{\partial x_i}{\partial u_j}\,du_j$$

and that

$$-dz + p_1\,dx_1 + p_2\,dx_2 + \cdots + p_n\,dx_n + 0\,dp_1 + \cdots + 0\,dp_{n-1}$$
$$= \pi_1\,d\xi_1 + \pi_2\,d\xi_2 + \cdots + \pi_n\,d\xi_n.$$

Thus, one single change of variables was enough to reduce an equation with $2n$ variables to another one with only n variables. The complete solution of the initial equation (3.2.8) appeared at once by eliminating $p_1, p_2, \ldots, p_n$ from the system

$$\xi_i(z, x_1, \ldots, {}_n, p_1, \ldots, p_n) = c_i, \quad i = 1, 2, \ldots, n,$$
$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0.$$

JACOBI presented another exposition of the "first method" in his lectures delivered in 1842/43 at the Königsberg University. In 1866, after JACOBI died, CLEBSCH published these lectures (JACOBI [5]).

3.6. Further development of the theory. PFAFF, CAUCHY and JACOBI largely solved the main problem of the theory of first-order partial differential equations, the solution of equation (3.2.8), which challenged mathematicians at the beginning of the century. The principal part of LAGRANGE'S 'theory' attained completeness, a fact which signified the end of the second period. True enough, many particular points within the framework of the 'theory' still remained unstudied while its methods demanded clarification and, moreover, admitted considerable generalization even beyond the theory.

PFAFF regarded partial derivatives of the function sought as additional independent variables, thus effectively introducing a $2n$-dimensional space. The accepted geometric interpretation of the theory could not find a place for his method. CAUCHY outlined the way for a subsequent development of the theory of characteristics for equations of the first order. JACOBI inseparably linked these equations with research in such active directions as analytical mechanics and the calculus of variations. Various transformations (contact transformations, as they were called later on) due to LAGRANGE [2], LEGENDRE [1] and AMPÈRE [2] came into general use. In the 1870s, taking these transformations as a basis, LIE constructed a 'general theory' of equations of the first order (see my § 5).

Thus the studies accomplished in the context of LAGRANGE'S 'theory' possessed an intrinsic potential for further evolution. Even so, their development would hardly have been so impetuous as it occurred in real life were it not for the powerful influence exerted by analytical mechanics. Within the HAMILTON-JACOBI theory, as JACOBI himself proved, it was possible to reduce integrations of equations originating in mechanics to the discovery of complete solutions of one first order partial equation.

Taken in itself, this fact provided no practical benefit since the integration of the latter in its turn came down to a complete integration of a system of ordinary equations equivalent to the original system.

This was a vicious circle, but a method of integrating first-order equations based on a new principle, the so-called "second method" due to JACOBI provided a way out. This method, or rather JACOBI'S work in general, was the essence of the next period under consideration.

4. C. G. Jacobi's 'Theory'

4.1. The "Second Method". JACOBI had to discover a method for integrating equation (3.2.8), a method essentially differing from the one he himself introduced earlier, which had reduced the problem to a complete integration of the system (3.5.1). Such a method, due to LAGRANGE and CHARPIT (see my § 3.2), existed for the case of two independent variables, but its generalization to a larger number of variables ran into grave difficulties (§ 3.2).

"Up to now, this difficulty prevented analysts from extending LAGRANGE'S

[-CHARPIT'S] method to a larger number of variables. On the contrary, it will not scare us; knowing that the problem can still be solved despite the redundant number of conditions, we shall inquire how $n-1$ functions can obey $n(n-1)/2$ equations of condition."

These were the words with which JACOBI introduced the exposition of the new, long sought-for method in his *Vorlesungen* [5, lecture 31]. Even in 1827 he had [1, p. 10] come out in favor of "extending LAGRANGE's method as far as possible."

Not later than in 1838, JACOBI managed to generalize the LAGRANGE-CHARPIT method to an arbitrary number of independent variables. This is the date of his manuscript which CLEBSCH discovered among his posthumous papers and published in 1862 [4].

Essentially, the "second method" is this. Let

$$f_i(x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n) = h_i, \quad i = 1, 2, \dots, n-1,$$

h_i being arbitrary constants, be the $n-1$ sought-for relations. Together with the initial equation

$$f_0(x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n) = 0 \tag{4.1.1}$$

(it is not difficult to represent equation (3.2.8) in this form) these relations determine $p_1, p_2, \dots, p_n$ in terms of $x_1, x_2, \dots, x_n$ for any values of constants $h_1, h_2, \dots, \dots, h_{n-1}$ such that the sum

$$p_1\,dx_1 + p_2\,dx_2 + \cdots + p_n\,dx_n \tag{*}$$

becomes a total differential.

The integration of this differential furnishes the complete integral sought. It depends on the arbitrary constants $h_1, h_2, \dots, h_{n-1}$ and on h_n, the constant of integration. JACOBI proved that for quantities $p_1, p_2, \dots, p_n$ to transform expression (*) into a total differential it is necessary and sufficient that equality

$$(f_i f_k) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}\frac{\partial f_k}{\partial p_l} - \frac{\partial f_i}{\partial p_l}\frac{\partial f_k}{\partial x_l}, \quad i, k = 0, 1, \dots, n-1, \tag{4.1.2}$$

holds identically. Here, $(f_i f_k)$ is POISSON's bracket which he introduced in 1809 [1].

Thus the unknown functions $f_1, f_2, \dots, f_{n-1}$ satisfy conditions

$$(f_0 f_1) = 0, \tag{4.1.2–1}$$

$$\begin{cases} (f_0 f_2) = 0, \\ (f_1 f_2) = 0 \end{cases} \tag{4.1.2–2}$$

$$\dots\dots\dots\dots$$

$$\begin{cases} (f_0 f_{n-1}) = 0, \\ (f_1 f_{n-1}) = 0, \\ \dots\dots\dots\dots \\ (f_{n-2} f_{n-1}) = 0. \end{cases} \tag{4.1.2–(n − 1)}$$

Since f_0 is known, equation (4.1.2–1) is a linear equation of the first order with respect to f_1. After a particular solution involving p_i has been determined, it is possible to solve the system (4.1.2–2) of two linear equations of the first order with respect to f_2 *etc.* All in all, it is thus necessary to solve a system

$$\begin{cases} A'(f) = A_1' \frac{\partial f}{\partial x_1} + A_2^1 \frac{\partial f}{\partial x_2} + \cdots + A_n^1 \frac{\partial f}{\partial x_n} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A^k(f) = A_1^k \frac{\partial f}{\partial x_1} + A_2^k \frac{\partial f}{\partial x_2} + \cdots + A_n^k \frac{\partial f}{\partial x_n} = 0 \end{cases} \quad (4.13).$$

of first-order linear differential equations satisfying, as JACOBI proved, the condition

$$A^k(A^l(f)) - A^l(A^k(f)) = 0.$$

To be more precise, it was sufficient to find one solution of the system which included p_i. JACOBI offered a simple method for obtaining such solutions based essentially on a relation between POISSON brackets:

$$(f(\varphi g)) + (\varphi(gf)) + (g(f\varphi)) = 0.$$

This is JACOBI's identity (as it was subsequently called), which he discovered in passing.

The new method of integrating equation (4.1.1), and, consequently, equation (3.2.8), opened up a new direction of research, *viz*, the study of systems of such equations.

4.2. Jacobi's bequest to the subject. A profound penetration into ideas of his predecessors (LAGRANGE, PFAFF), an indissoluble connection of studies of structures belonging to mathematical analysis and analytical mechanics, and, finally, an exceptional breadth of views—these were the distinctive features of JACOBI's work. They enabled JACOBI to attain important achievements in mechanics and to enrich mathematical analysis advancing the study of equations of the first order. He thus closely approached the ideas which later on constituted the essence of LIE's 'theory'. Judging by his *Vorlesungen* [5] and, also, by his articles [4; 6] posthumously published in 1862 and 1866, respectively, JACOBI came close to the concept of contact transformations, to the understanding of their leading part in the construction of the theory of first-order differential equations. He regarded contact transformations as the most general conversions possible for these equations; moreover, especially in one of his posthumous publications [6, § 42], he offered a general analytical representation of such transformations.

Only one step separated JACOBI from a perfect understanding of contact transformations: he did not reveal their geometric interpretation. Still, his actual use of these transformations proved the need to consider equations (*e.g.*, equation (4.1.1)) in connection with R^{2n+1} rather than R^{n+1}.

Finally, JACOBI came near to the theory of infinitesimal transformations subsequently developed by LIE. Studying systems of linear partial differential equations, he singled out differential expressions of the type

$$A(f) = \sum_{i=1}^{n} A_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i},$$

considering them as differential operators performed on a function f. For two such operators he introduced expressions

$$A(B(f)) - B(A(f))$$

which proved to be operators of the same kind.

JACOBI also noticed the connection of these operators with POISSON brackets:

$$(f\varphi) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial \varphi}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial \varphi}{\partial p_i} \right).$$

In his *Vorlesungen* [5, lecture 34] he treated the latter as a result of the performance of some differential operator on a function f: $A(f) = (f\varphi)$, or, as LIE preferred to say later on, as a result of subjecting f to an infinitesimal transformation associated with φ.

Thus JACOBI's works included all conditions prerequisite to the creation of a 'general theory' of equations of the first order, a theory worked out by LIE. The sole ingredient lacking in the former's construction was a unified geometric view, a fruit yielded only in the 1870s by the entire development of mathematics in the 19th century.

As I mentioned in § 4.1 and above, JACOBI's memoirs on this subject were published posthumuosly, many years after he wrote them. Some of his achievements became known by word of mouth through his former students at Königsberg (K. W. BORCHARDT and others) and from letters written by JACOBI himself.[15] Even so, taken as a whole, JACOBI's findings in the theory of first-order equations and his closely connected achievements in mechanics remained unknown for a long time. Other scholars (OSTROGRADSKY, BERTRAND, LIOUVILLE, E. BOUR, W. P. DONKIN and others) discovered some of them anew.

4.3. The situation just before Lie. Most of JACOBI's works on this subject were published during the 1850s and 1860s, at once attracting general attention. Interest in them revealed the need for a new viewpoint such as to furnish a clear and unified understanding of JACOBI's ideas, to discover interconnections between his constructions and previous theories due to LAGRANGE, PFAFF and CAUCHY.

In 1865 V. G. IMSCHENETSKY [1] published a model description of the achievements (those due to JACOBI in the first place) attained by that time.[16] However,

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[15] LIOUVILLE published one of them in his *Journal de mathématiques pures et appliquées*.

[16] A French edition of his Russian monograph appeared in 1869. One of the motives for its compilation was BERTRAND's influence. At the beginning of the 1860s BERTRAND delivered lectures on partial differential equations and, in one of them, he (IMSCHENETSKY [1, Introduction]) attempted to communicate the "possible degree of simplicity" to JACOBI's theory.

IMSCHENETSKY did not show the way to the construction of a 'general theory', *the theory* which LIE worked out a few years afterwards. Neither did the former describe the notions which were to become the premisses for the development of the central ideas of this theory, *viz*, the notions of

1. A new space element, of a manifold of such elements and of the corresponding generalization of the concepts of an equation and its solution.
2. Contact transformations.
3. Infinitesimal transformations.

J. PLÜCKER was the first to come to the idea of choosing a new space element, the idea which beginning from 1828 [1] runs all through his works including the very last of them [2] published posthumously by CLEBSCH and KLEIN. The notion of a space element is due to LIE, but he himself repeatedly acknowledged (*e.g.*, LIE [9, pp. 1 and 98]) its derivation from PLÜCKER'S ideas. The space element, LIE stated, was a concrete expression of these ideas for the situation under his consideration.

I have noticed (see my § 4.2) that JACOBI came close to the concept of contact transformations. Independently of this fact, two distinct lines of development led to the same idea:

(1) The practice of integrating differential equations. In this connection I mentioned isolated achievements due to EULER (see my § 2.3), LAGRANGE, LEGENDRE and AMPÈRE (§ 3.6) and I now shall additionally refer to A. DE MORGAN and P. DU BOIS REYMOND whose writings on this subject were published in 1849 and 1864 respectively.

(2) Geometric investigations carried out by MONGE (in 1809), M. CHASLES (in 1837) and, especially, PLÜCKER (in 1831).[17] As LIE and ENGEL noted (LIE [7, Bd. **2**, pp. 17–18]), "... very little need be added [to PLÜCKER's achievements] to arrive at the starting point of the geometric theory of contact transformations on the plane."

Thus by the end of the 1860s contact transformations potentially entered mathematics and, in the beginning of the next decade, LIE indeed used them in his works.

The concept of infinitely small transformations has a long history (BOURBAKI [2, pp. 411–412]). At any rate, even DESCARTES discovered the instantaneous centre of rotation assuming that "in the infinitely small" every plane movement might be considered as some rotation. Similar ideas are found in the analytical mechanics of the 18th and 19th centuries.

In 1895 or 1896, in a letter to V. G. ALEKSEEV, LIE (ANDREEV *et al* [1, p. 457] wrote: "IMSCHENETSKY's Arbeit über partielle Differentialgleichungen erster Ordnung war mir soweit bekannt wie die erste systematische Zusammenfassung von LAGRANGE's, CAUCHY's und JACOBI's Untersuchungen auf diesem Gebiete. Jedenfalls lernte ich diese Theorien durch IMSCHENETSKY's Werk kennen, das sich nach meiner Ansicht durch klare Darstellung und exacte Form auszeichnet."

[17] I should also mention DARBOUX. He came to the idea of contact transformations by the end of the 1860s (see his later letters to LIE (LIE [8, p. 18; 9, p. 5]), but he did not then publish his findings.

SYLVESTER (in 1851) and A. CAYLEY (somewhat later) (BOURBAKI [2, pp. 411–412]) used infinitely small transformations in connection with differential operators of the type

$$A(f) = \sum_{i=1}^{n} A_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i}.$$

The commutator

$$A(B(f)) - B(A(f))$$

of such operators also appeared in their works. See my § 3.2 for the use of the same operators and commutator by JACOBI in his study of systems of first-order partial linear equations.

In 1868 JORDAN [1] considered infinitely small transformations from a geometric point of view. He (BOURBAKI [2, p. 412]) came to the idea of one-parameter continuous groups "generated" by such transformations, and he thus anticipated LIE's discovery of the connection of these transformations with finite continuous groups. In its turn, this discovery was conducive to the development, again by LIE, of his concept of infinitely small transformations.

Thus, by the end of the 1860s, all obstacles against the creation of a 'general theory' were surmounted. Only one 'small' point, the development of the most general geometric concepts was left to be achieved. By the beginning of the 1870s two young mathematicians, F. KLEIN and LIE, the latter to become the principal hero of the following period, took this decisive step. But of course it was prepared by revolutionary changes in geometry during the whole 19th century.

5. Lie's 'theory'

5.1. Sophus Lie. LIE combined an unusual creative potential with a keen interest in the works of his predecessors. His penetration into the ideas, and his perfect knowledge of the achievements, of EULER, LAGRANGE, MONGE, PFAFF, HAMILTON, PLÜCKER and JACOBI is witnessed by numerous remarks scattered around in footnotes and in the main body of his works and, also, in his letters. Coming across these comments in LIE's writings, one finds oneself in a state of perpetual astonishment, the more so as his study of the pertinent literature took place at the same time as, and in thorough connection with, the development of his own general geometric views. Thus, forging a new standpoint, LIE added the geometric dimension to previous concepts, combining and relating ideas expressed in various forms or even scarcely outlined during more than a century of developments in the field of partial equations.

LIE's first works devoted to contact transformations appeared in 1871–1872. These included a short note [1] in Norwegian written in 1870, a more detailed version of his [2], again in Norwegian, and an article in German [3]. Then, in 1873, LIE [5] presented an extremely concise and rather obscure outline (written in 1872) of the 'theory' of equations of the first order which he then began to develop. Somewhat later LIE compiled another article [4]. It was published in 1872 and contained a solution of the problem of local equivalence for equations of the first order. Given without proof, the solution constituted one of the principal propositions of the new 'theory' (see below). Finally, a series of publications on the same subject followed in the next years culminating in 1890 in Volume 2 of LIE's remarkable book [7] written in collaboration with ENGEL.

5.2. The Equation and its Solution. Interpreting equation (3.1.1)

$$f(x, y, z, p, q) = 0$$

and its solutions in geometric terms, mathematicians before LIE used the three-dimensional coordinate space R^3. For his part, LIE followed PLÜCKER'S idea of a "generalized space element" supposing it more convenient to consider the space R^5 and regarding the set (x, y, z, p, q) as a point in it. In this case equation (3.1.1) defines a four-dimensional manifold M_4 in space R^5. According to LIE, integration of this equation means the determination of all manifolds M_k $(k \leqq 2)$

$$x = x(t_1, \ldots, t_k), \quad y = y(t_1, \ldots, t_k), \quad z = z(t_1, \ldots, t_k),$$
$$p = p(t_1, \ldots, t_k), \quad q = q(t_1, \ldots, t_k)$$

whose points satisfy both the equation and the condition

$$dz - p\,dx - q\,dy = 0. \tag{5.2.1}$$

However, it can be proved that the integration is reduced to the discovery of manifolds M_2 of only two dimensions.

The notion of a solution of a differential equation also becomes generalized under the interpretation just described. Consider for example the equation

$$\frac{\partial z}{\partial x} = 0.$$

Relations

$$x = t_1, \quad y = 0, \quad z = 0, \quad p = 0, \quad q = t_2$$

define an integral manifold M_2 which in R^3 corresponds to a straight line coinciding with axis Ox and a set of planes passing through it.

Even equations themselves, the object of the theory, assumed a more general meaning. Mathematicians before LIE included in the field of the theory only such equations (3.1.1) as involved at least one derivative, p or q, while the new interpretation made it possible to consider equations $f(x, y, z) = 0$, and in particular, the equation

$$z = 0. \tag{5.2.2}$$

5.3. Contact Transformations. LIE called a transformation a contact transformation if, in R^5,

$$x' = x'(x, y, z, p, q), \quad y' = y'(x, y, z, p, q), \quad z' = z'(x, y, z, p, q),$$
$$p' = p'(x, y, z, p, q), \quad q' = q'(x, y, z, p, q),$$

the relation

$$dz' - p'\,dx' - q'\,dy' = \varrho(dz - p\,dx - q\,dy)$$

being satisfied identically. Here, ϱ, a non-vanishing function of x, y, z, p and q, depended on the transformation. Indeed, the invariant property of such transformations was tangency. Classical transformations introduced by LEGENDRE and AMPÈRE were examples of contact conversions.

One of the first ensuing inquiries was the study of equivalence, *i.e.* of the possibility of reducing a given equation to another one by means of contact transformations. In 1872 LIE himself formulated the corresponding problem in one of his first articles [5] and solved it, at least in principle, in his following work [4]. It turned out that there was a contact transformation such as to reduce a given equation to any other given equation, in particular, to equation (5.2.2). Thus this simplest equation became capable of representing any partial equation of the first order! Integral manifolds of equation (5.2.2) offered an elegant description of LAGRANGE'S 'theory'.

From a modern point of view LIE'S solution of the equivalence problem is unconvincing. First, he did not point out the local nature of the result obtained: contact transformations ensure an isomorphic relation between some neighborhood of a point belonging to the manifold $f(x, y, z, p, q) = 0$ and a certain neighborhood of an arbitrary point on another variety $f(x', y', z', p', q') = 0$. Second, LIE did not notice that contact transformations are impossible in the neighborhood of non-regular points (a necessary condition for non-regularity is $p^2 + q^2 = 0$).

Regarding my first remark, there is no doubt that LIE knew about the restriction just formulated, but he never made any mental reservation. As to the second item, I think that LIE'S failure to state it himself was due rather to a manner typical of mathematicians of the 19^{th} century. Indeed, it was then a prevailing custom to formulate facts correct "in general" and to ignore their being invalid in some isolated cases. It is easy to show that in this particular instance non-regular points belonging to manifold $f(x, y, z, p, q) = 0$ constitute a closed subset of a lesser dimension so LIE'S inference is correct "almost always", *i.e.* "in the general case".

The theory of characteristics and characteristic manifolds made essential progress within the framework of the new 'theory'. Accordingly, LIE developed a general method for solving equations, including as special cases the methods due to CAUCHY and JACOBI (more precisely, JACOBI'S "second method").

The new 'theory' also ensured a still more transparent connection between problems in mechanics and equations of the first order.

5.4. Infinitesimal Transformations. The concept of infinitely small transformations played an important part in LIE'S study of equations of first order. In a work (see for example his and KLEIN'S article (KLEIN & LIE [1])) published in 1871, LIE associated such transformations with systems

$$\frac{dx_i}{dt} = \xi_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n.$$

Indeed, these transformations carried the point $\bar{x}(x_1, x_2, \ldots, x_n) \in R^n$ into $\bar{x}'(x_1', x_2', \ldots, x_n') \in R^n$,

$$x_i' = x_i + \xi_i(x_1, x_2, \ldots, x_n)\,\delta t.$$

Suppose that at the point $\bar{x}(x_1, x_2, \ldots, x_n)$ at least some ξ_i survive. Then a transformation of the described type generates a one-parameter group

$$x_i' = f_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n,$$
$$x_i = f_i(x_1, x_2, \ldots, x_n, 0).$$

Even JORDAN [1, p. 243] (see also BOURBAKI [2, p. 412]) regarded this group as a set of conversions resulting from a "suitably repeated" infinitely small transformation.

Consider an arbitrary function $f(\bar{x})$. If terms which involve powers of δt higher than the first are neglected, its increment under such transformation is

$$\delta f = Xf\,\delta t, \quad Xf = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}.$$

From 1874 onward LIE [6] called operators Xf symbols of infinitely small transformations or simply infinitely small transformations. Studying equations of the first order, LIE discovered the connection between continuous r-parameter groups of transformations and the corresponding totalities of r such operators $X_1 f, X_2 f, \ldots, X_r f$. Denoting by $[X_i X_j]$ the commutator $X_i X_j - X_j X_i$ (first considered by JACOBI; see my § 4.2), LIE established the following relations between the operators of these totalities (of these "groups $X_1, X_2, \ldots, X_r$" as he called them): $[X_i X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$ (c_{ij}^k are constants),

$$[[X_i X_j]\, X_m] + [[X_j X_m]\, X_i] + [[X_m X_i]\, X_j] = 0.$$

The latter equality is JACOBI's identity written out in terms of commutators.

LIE [7, Bd. **3**, pp. 563, 590, 597] explicated the connection between "groups of infinitely small transformations"[18] and finite continuous groups of transformations in three theorems which make the foundation of the theory of LIE algebras. LIE (*ibidem*, p. 665) discovered the connection while studying integration of linear partial homogeneous equations of the first order. Thus these equations came to be the field on which the theory of LIE groups originally rooted itself.

One of the best known facts discovered by LIE (*ibidem*, pp. 708–709; CHEBOTAREV [1, p. 212]) concerning the theory of integration is this: if the equation

$$Af = \sum_{i=1}^{r+1} A_i(x_1, \ldots, x_{r+1}) \frac{\partial f}{\partial x_i} = 0$$

admits an r-term resolvable group G whose infinitely small transformations $X_1 f, X_2 f, \ldots, X_r f$ together with the operator Af constitute an independent system, then the integration of this equation reduces to quadratures.

[18] Or LIE algebras, as they came to be known. Their history is described elsewhere (BOURBAKI [2]; HAWKINS [1]).

5.5. Further Development of the 'Theory'. LIE furthered subjects studied or even scarcely broached by his predecessors during more than a century of intensive work. His 'general theory' is the summit of classical research done in the 19th century in the field of equations of the first order. Moreover, his 'theory', as unfolded in a monograph written by LIE himself in collaboration with ENGEL (LIE [7]), in separate treatises by other authors (notably, by GOURSAT [1]) and in essays (WEBER [1]), became one of the most remarkable mathematical achievements of that century.

Having accomplished his studies of equations of the first order, LIE began his research in the field of second-order equations. Taken as a whole, his work on differential equations opened up an entire direction. E. CARTAN, H. GOLDSCHMIDT, S. STERNBERG and other mathematicians[19] followed in LIE's steps, and their inquiries eventually produced an important part of the theory of smooth manifolds.

Contemporary invariant definitions (independent of the choice of coordinate systems) of partial differential equations and their solutions are connected with the notions of jet (or spray; see BOURBAKI [1]) spaces introduced by CH. EHRESMANN. An equation of the first order is now considered as a closed submanifold E of codimension 1 belonging to manifold $J'(M)$ of 1-jet smooth functions on variety M (VINOGRADOV [1]). A solution of such an equation is a smooth submanifold X which belongs to the equation E and identically (on X) reduces some universal 1-form $U_1 \in \Lambda'(J'_M)$ to zero. VINOGRADOV [1] proposed the problem of classifying nonlinear partial differential equations, and LICHAGIN [1] solved it for equations of the first order. For them the problem is reduced to the classification of germs of hypersurfaces in $J'(M)$ with respect to the group of contact diffeomorphisms.

Acknowledgements. This is a modified version of my article [5] published in 1980 in Russian. With other authors I have described in more detail some of the points outlined in this paper. Thus I have elsewhere [2] studied the role played by D'ALEMBERT and EULER in the origination and initial development of the theory of partial differential equations and also discussed the method of multipliers and its application to specific equations. My article [1] was devoted to D'ALEMBERT's works on partial differential equations, including those of first order. I have treated [4] methods of integrating equation (3.2.8) due to PFAFF and CAUCHY. Finally, I explained [3] the origin and development of LIE's 'theory' with special reference to the prehistory of the notion of contact transformations (in particular, to the first isolated applications of some transformations by EULER, LAGRANGE, LEGENDRE, AMPÈRE, and to JACOBI's gradual approach to the concept).

Dr. S. ENGELSMAN turned my attention to the forgotten publications of SALTYKOW [1; 2], and I have also followed to a considerable extent ENGELSMAN's work [1] in my § 3.1. Likewise, I owe my § 5.4 to the same extent to BOURBAKI [2]. Acknowledgements are also due to Professor A. P. YOUSHKEVITCH for his lasting interest in my work and to Dr. A. N. PARSHIN for his friendly assistance rendered me on many occasions. Dr. O. B. SHEYNIN translated my manuscript from Russian and offered a number of comments.

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[19] I should also mention EGOROV. Along with important mathematical findings, his doctoral dissertation [1] contains an interesting essay on the history of partial differential equations.

Note added in proof: Several points discussed in § 5.4 are somewhat extended and clarified in my latest contribution, *Des paranthèses de Poisson aux algèbres de Lie*, in: *S. D. Poisson et la science de son temps*, edited by M. MÉTIVIER *et al.* Paris, 1981, 133–150.

References

The Russian names in cyrillic characters are subjoined at the end of the list.

AMPÈRE, A. M.

1. *Considérations générales des équations aux différences partielles.*—*J. École Polyt.*, 1815, t. **10**, cahier 17, 549–611.

2. *Mémoire contenant l'application de la théorie exposée dans le* 17e *cahier du J. École Polyt. à l'intégration des équations différentielles de* 1er *et de* 2me *ordre.*—*Ibidem*, 1820, t. **11**, cahier 18, 1–188.

ANDREEV, C. A., P. A. NEKRASSOV & N. E. JOUKOVSKY

1. *Vie et travaux scientifiques de V. G. Imschénetsky.*—*Mat. Sb.*, 1896, t. **18**, 348–467.

BOURBAKI, N.

1. *Variétés différentielles et analytiques. Fascicule de résultats.* §§ 8–15. Paris, 1971.

2. *Lie groups and Lie algebras.* Part 1: Chapters 1–3. Paris, 1975.

CANTOR, M.

1. *Vorlesungen über Geschichte der Mathematik*, Bd. **4**. Leipzig-Berlin, 1924.

CAUCHY, A. L.

1. *Note sur l'intégration des équations aux différences partielles du premier ordre à un nombre quelconque de variables.*—*Bull. Soc. Philomatique Paris*, 1819, 10–21—*Oeuvr. Compl.*, ser. 2, t. **2**. Paris, 1958, 238–252.

2. *Mémoire sur l'intégration des équations aux dérivées partielles du premier ordre.*—*Exerc. d'anal. et de phys. math.* (nouv. exerc.), t. **2**. Paris, 1841—*Oeuvr. Compl.*, ser. 2, t. **12**. Paris, 1916, 272–309.

CHEBOTAREV, N. G.

1. *Theory of Lie groups.* Moscow-Leningrad, 1940 (in Russian).

COUSIN, J. A. J.

1. *Introduction à l'étude de l'astronomie physique.* Paris, 1787.

D'ALEMBERT, J.

1. *Traité de dynamique.* Paris, 1743. 2e éd., 1758.

2. *Réflexions sur la cause générale des vents.* Paris, 1747.

3. *Recherches sur la courbe que forme une corde tendue, mise en vibration.*—*Hist. Acad. Sci. Berlin*, t. **3**, 1747 (1749), 214–219, 220–229.

4. *Addition au mémoire* [3]. *Ibidem*, t. **6**, 1750 (1752), 355–360.

5. *Recherches de calcul intégral.*—*Opusc. math.*, t. **4**. Paris, 1768, 225–253, 254–282.

DEMIDOV, S. S.

1. *Les équations aux dérivées partielles chez D'Alembert.*—*Istoriko-matematicheskie issledovania*, 1974, t. **19**, 94–124. (En russe.)

2. *Les origines de la théorie des équations aux dérivées partielles.*—*Ibidem*, 1975, t. **20**, 204–220. (En russe.)

3. *Sur le développement de la théorie des équations aux dérivées partielles de S. Lie.*—*Ibidem*, 1978, t. **23**, 87–117. (En russe.)

4. *Pour l'histoire de la théorie des équations aux dérivées partielles du premier ordre. Recherches de J. Pfaff et de A. Cauchy.*—*Ibidem*, 1979, t. **24**, 191–217. (En russe.)

5. *Le développement de la théorie des équations aux dérivées partielles du premier ordre au XVIIIe et au XIXe siècles.*—*Ibidem*, 1980, t. **25**, 71–104 (in Russian).

EGOROV, D. F.

1. *Second-order partial differential equations with two independent variables. General theory of integrals.—Uchenye zapiski Imp. Mosc. Univ., Otd. phys.-math.*, 1899, t. **15**, 1–392. (In Russian.)

ENGELSMAN, S. B.

1. *Lagrange's early contributions to the theory of first order partial differential equations.—Hist. Math.*, 1980, vol. **7**, 7–23.

EULER, L.

1. *Additamentum ad dissertationem de infinitis curvis ejusdem generis.—Comment. Acad. Sci. Petrop.* 1734/35, t. **7** (1740), 184–200 (ENESTRÖM 45) = *Opera omnia*, ser. 1, t. **22**. Lipsiae, 1936, 57–75.

2. *Sur la vibration des cordes.—Hist. Acad. Sci. Berlin*, 1748, t. **4** (1750), 69–85 (E 140) = *Opera omnia*, ser. 2, t. **10**. Bernae, 1947, 63–77.

3. *Remarques sur les mémoires précédents de M. Bernoulli.—Hist. Acad. Sci. Berlin*, 1753, t. **9** (1755), 196–222 (E 213) = *Opera omnia*, ser. 2, t. **10**, 233–254.

4. *Investigatio functionum ex data differentialium conditione.—Novi. comment. Acad. Sci. Petrop.*, 1762/63, t. **9** (1764), 170–212 (E 285) = *Opera omnia*, ser. 1, t. **23**. Lipsiae, 1938, 1–41.

5. *Recherches sur l'integration de l'équation* $\frac{ddz}{dt^2} = aa\left(\frac{ddz}{dx^2}\right) + \frac{b}{x}\left(\frac{dz}{dx}\right) + \frac{c}{xx}z$. —*Misc. Taurin.*, 1762–1765, t. **3** (1766), 60–91 (E 319) = *Opera omnia*, ser. 1, t. **23**. Lipsiae, 1938, 42–73.

6. *Institutionum calculi integralis*, t. **3**. Petrop., 1770 (E 385) = *Opera omnia*, ser. 1, t. **13**. Leipzig-Berlin, 1914.

GOURSAT, E.

1. *Leçons sur l'intégration des équations aux dérivées partielles du premièr ordre*. Paris, 1890; 2-e éd., 1921.

HAWKINS, T.

1. *Hypercomplex numbers, Lie groups, and the creation of group representation theory. —Arch. hist. ex. sci.*, 1972, vol. **8**, 243–287.

IMSCHENETSKY, V. G.

1. *Sur l'intégration des équations aux dérivées partielles du premier ordre*. Paris, 1869. (Orig. publ. in 1865, in Russian.)

JACOBI, C. G.

1. *Über die Integration der partiellen Differentialgleichungen erster Ordnung.—J. reine und angew. Math.*, 1827, Bd. **2**, No. 2 = *Ges. Werke*, Bd. **4**. Berlin, 1886, 1–15.

2. *Über die Pfaffsche Methode, eine gewöhnliche lineäre Differentialgleichung zwischen 2n Variabeln durch ein System von n Gleichungen zu integriren.—J. reine und angew. Math.*, 1827, Bd. **2**, No. 2, 347–357 = *Ges. Werke*, Bd. **4**, 17–29.

3. *Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen.—J. reine und angew. Math.*, 1837, Bd. **17**, 97–162 = *Ges. Werke*, Bd. **4**, 57–127.

4. *Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variablium quemcunque propositas integrandi.—J. reine und angew. Math.*, 1862, Bd. **60**, 1–181 = *Ges. Werke*, Bd. **5**. Berlin, 1890, 1–189. German transl.: *Neue Methode zu Integration partieller Differentialgleichungen erster Ordnung zwischen irgend einer Anzahl von Veränderlichen.* Leipzig, 1906 (*Ostwald's Klassiker* No. 156).

5. *Vorlesungen über Dynamik* von C. G. J. Jacobi nebst fünf hinterlassen Abhandlungen desselben. Berlin, 1866.

6. *Über diejenigen Probleme der Mechanik in welchen eine Kräftefunction existirt und über die Theorie der Störungen.*—in [5, 303–468] = *Ges. Werke*, **Bd. 5**, 217–395.

JORDAN, C.

1. *Mémoire sur les groupes de mouvements.*—*Annali math.*, 1868–1869, t. **11**, 167–215, 332–345 = *Oeuvr. Compl.*, t. **14**. Paris, 1964, 231–302.

KLEIN, F., & S. LIE

1. *Über diejenigen ebenen Kurven, welche durch ein geschlossenen System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen.*—*Math. Ann.*, 1871, Bd. **4**, 50–84 = LIE, S., *Ges. Abh.*, Bd. **1**. Leipzig-Oslo, 1934, 229–266.

LACROIX, S. F.

1. *Traité du calcul différentiel et du calcul intégral*, t. **2**. Paris, 1814 (2e éd.).

LAGRANGE, J. L.

1. *Sur l'intégration des équations à différences partielles du premier ordre.*—*Nouv. Mém. Acad. Sci. Berlin*, 1772 (1774), 353–372 = *Oeuvr. Compl.*, t. **3**. Paris, 1869, 549–575.
2. *Sur les intégrales particulières des équations différentielles.*—*Nouv. Mém. Acad. Sci. Berlin*, 1774 (1776), 197–275 = *Oeuvr. Compl.*, t. **4**. Paris, 1869, 5–108.
3. *Sur différentes questions d'analyse relatives à la théorie des intégrales particulières.*—*Nouv. Mém. Acad. Sci. Berlin*, 1779 (1781), 121–160 = *Oeuvr. Compl.*, t. **4**, 585–634.
4. *Méthode générale pour intégrer les équations aux différences partielles du premier ordre, lorsque ces différences ne sont que linéaires.*—*Nouv. Mém. Acad. Sci. Berlin*, 1785 (1787) = *Oeuvr. Compl.* t. **5**. Paris, 1870, 543–562.

LAPLACE, P. S.

1. *Recherches sur le calcul intégral aux différences partielles.*—*Mém. Acad. Sci. Paris*, 1773 (1777) = *Oeuvr. Compl.*, t. **9**. Paris, 1843, 5–68.

LEGENDRE, A. M.

1. *Mémoire sur l'intégration de quelques équations aux différences partielles.*—*Hist. Acad. Sci. Paris*, 1787 (1789), 309–351.

LIE, S.

1. *Om en Classe geometriske Transformationer.*— *Christ. Forh.*, 1870 (1871), 506–509. In Norwegian. Germ. transl.: *Ges. Abh.*, Bd. **1**. Leipzig-Oslo, 1934, 93–96.
2. *Over en Classe geometriske Transformationer.*—*Christ. Forh.*, 1871 (1872), 67–109, 182–245. In Norwegian. German transl.: *Ges. Abh.*, Bd. **1**, 105–214.
3. *Über Komplexe.*—*Math. Ann.*, 1872, Bd. **5**, No. 1, 145–208, No. 2, 209–256 = *Ges. Abh.*, Bd. **2**. Leipzig-Oslo, 1935, 1–121.
4. *Zur Theorie partieller Differentialgleichungen erstern Ordnung.*—*Gött. Nachr.*, 1872, No. 25, 473–489 = [8, pp. 16–26].
5. *Kurzes Résumés mehrerer neuer Theorien.*—*Christ. Fohr.*, 1872 (1873), 24–27 = [8, pp. 1–4].
6. *Verallgemeinerung und neue Verwertung der Jacobischen Multiplikationstheorie.*—*Christ. Fohr.*, 1874 (1875), 255–274 = [8, pp. 188–206].
7. *Theorie der Transformationsgruppen.* (Unter Mitwirkung von F. ENGEL), Bde **1–3**. Leipzig, 1888–1893.
8. *Gesammelte Abhandlungen*, Bd. **3**. Leipzig-Kristinia, 1922.
9. *Gesammelte Abhandlungen*, Bd. **4**. Leipzig-Oslo, 1929.

LYCHAGIN, V. V.

1. *Local classification of first order nonlinear partial differential equations. Dokl. Akad. Nauk. SSSR*, 1973, vol. **210**, No. 3, 525–528 (in Russian).

MONGE, G.

1. *Mémoire sur le calcul intégral des équations aux différences partielles. Hist. Acad. Sci. Paris*, 1784 (1787), 85–117.
2. *Application de l'analyse à la géométrie.* Paris, 1807.

PFAFF, I. F.

1. *Methodus generalis, aequationes differentiarum partialium, nec non aequationes differentialesn vulgares, utrasque primi ordinis, inter quotcumque variabiles, complete integrandi.—Abh. Akad. Wiss.* Berlin, 1814–1815, 76–136. German transl.: *Allgemeine Methode partielle Differentialgleichungen zu integriren.* Leipzig, 1902. (*Ostwald's Klassiker* No. 129).

PLÜCKER, J.

1. *Analytisch-geometrische Entwicklungen*, Bde **1–2**. Essen, 1828–1831.

2. *Neue Geometrie des Raumes, etc.* Leipzig, 1868.

POISSON, S. D.

1. *Mémoire sur la variation des constantes arbitraires dans les questions de la mécanique. J. École Polyt.*, 1809, t. **8**, 266–344,

SALTYKOW, N.

1. *Étude bibliographique sur le mémoire inédit de Charpit.—Bull. sci. math.*, 2e sér., 1930, t. **54**, 255–264.

2. *Étude bibliographique de la seconde partie du mémoire inédit de Charpit.—Ibidem*, 1937, t. **61**, 55–64.

TRUESDELL, C.

1. *The rational mechanics of flexible or elastic bodies*, 1638–1788, this being EULER, L., *Opera omnia*, ser. 2, t. **11**. Turici, 1960, 1–435.

VINOGRADOV, A. M.

1. *Multivalued solutions and the principle of classification of nonlinear differential equations.—Dokl. Akad. Nauk SSSR*, 1973, vol. **210**, No. 1, 11–14. (In Russian.)

WEBER, E.

1. *Partielle Differentialgleichungen*, this being *Enc. math. Wiss.*, Bd. $\mathbf{2}_1$, No. 2–3. Leipzig, 1900.

WILEITNER, H.

1. *Geschichte der Mathematik*, 2. Tl. Leipzig, 1911.

YOUSHKEVITCH, A. P. (ed.)

1. *History of mathematics from the most ancient times to the beginning of the* 19th *century*, vol. **3**. Moscow, 1972. (In Russian.)

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(Received June 5, 1981)