ON ANALOGY

1. INTRODUCTORY. The present paper is an attempt to clear up some of the problems involved in the traditional theory of analogy as presented by the Thomistic school. The two main ideas behind the formal developments offered here are: (1) analogy is an important discovery, worthy of a thorough examination and further development, (2) contemporary mathematical logic supplies excellent tools for such work. This paper is, as far as the author knows, the first of its kind; [1] it deals with a difficult subject in a sketchy way; what it contains is, therefore, not meant to be definitive truths, but rather proposals for discussion.

The approach to the problems of analogy used here is the semantic one. This is not the only method, but it would seem to be both the most convenient and the most traditional. As a matter of fact, it is difficult to see how equivocity, which is and must be treated as a relation of the same type as analogy, can be considered except by the semantic method. Also, St. Thomas Aquinas examined analogy in his question concerning divine names and the title of Cajetan's classical work is "*De Nominum Analogia*."

It will be taken for granted that the reader has a good knowledge of classical texts of St. Thomas and Cajetan, and of the content of the *Principia Mathematica*; [2] no reference

[1] The author is, however, indebted to the late Fr. Jan Salamucha and to J. Fr. Drewnowski who were the first to apply recent Formal Logic to Thomistic problems. The present paper may be considered as an attempt to formalize some of the opinions expressed by them. Cf. *Myśl katolicka wobec Logiki współczesnej* (Polish = The Catholic Thought and Contemporary Logic), Poznan 1937 (with French abstracts) and J. Fr. Drewnowski, *Zarys programu filozoficznego* (Polish = A sketch of a Philosophic Programme), Przeglad Filozoficzny, 37, 1943, 3-38, 150-181, 262-292, especially pp. 95-98. (There is a French account of this important work in *Studia Philosophica* (Lwow) I, 1935, 451-454.

[2] A. N. Whitehead and B. Russell, *Principia Mathematica*, 2nd ed., Cambridge 1925-1927.

will be made to these works, except for some laws used in the proofs. Other more recent topics of mathematical logic needed for the theory, as, e. g., plural relations,[3] semantics,[4] etc., will be explained.

The main results of our inquiry are: (1) an exact definition of univocity, equivocity, and analogy of attribution; (2) proof of the principles of contradiction and of excluded middle for univocal and equivocal names; (3) a metalogical examination and exact translation of the formula "analogy itself is analogical"; (4) proof that a syllogism in *Barbara* with analogical middle terms, if analogy is defined according to the alternative theory, is a correct formula; (5) criticism of the alternative theory; (6) definition of analogy of proportionality by isomorphy; (7) proof that a syllogism in *Barbara* with analogical middle terms, if analogy is explained according to the isomorphic theory, is a correct formula; (8) a suggestion that contemporary Logic uses analogy.

Incidentally other results are reached, which may have a more general relevance: (1) the foundations of a semantic system, useful for Thomistic Logic, are sketched; (2) a generalised table of relevant semantic relations between two names is given; (3) the formal validity of a syllogism in *Barbara*, as opposed to its verbal correctness, is defined; (4) a rudimentary analysis of causality, as understood by Thomists, is supplied.

2. Meaning. The fundamental notion of our theory is that of meaning, described by the following formula: "the name a means in the language l the content f of the thing x" (symbolically: "$S(a, l, f, x)$.") The situation symbolized by "$S(a, l, f, x)$" will be called a "semantic complex." In spite of its simpilicity the semanitic complex merits a detailed comment.

(1) By "name" we understand here a written word or other written symbol. It must be emphasized that a written

[3] Cf. R. Carnap, *Abriss der Logistik,* Wien 1929, pp. 43-45.

[4] Cf. A. Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen,* Studia Philosophica (Lwow), I, 1935, 261-405.

symbol is just a black mark (a spot of dry ink) on paper. As such (*materialiter sumptum*) it is a physical object which occupies a given position in space and time. It may happen, therefore, that two names, e. g., a and b have the same graphical form (symbolically $I(a, b)$, where "I" suggests "isomorphy") but we cannot speak correctly of "the same" name which occurs twice, e. g. as middle term in a syllogism. In that case we have always two different names of the same graphical form.

(2) Every relation of meaning implies a reference to a language. This is obvious, for the same name may mean one thing in one language and something quite different in another. Moreover, it may have no meaning at all in another language. If the mention of a language is omitted in classical definitions, it is because the authors writing during the Middle Age and the Renaissance thought of the only one language used at that time, Latin.

(3) What we call "content" is what classical Thomists called "*ratio*." This *ratio* is always conceived as something determining the thing whose content it is; even in case of substantial contents (as "substance" and similars) we conceive them as such and St. Thomas explicitly teaches that in this case we always have to do with a quality in a broader meaning (including "substantial quality").

(4) Finally, the "thing" means the same as the "*res*" of the Thomists, namely the subject to which the content connoted by the name belongs. This is, at least if the logical analysis is pushed sufficiently far, an individual.

The relation S gives rise to several partial relations and partial domains. We are not going to investigate them here, as they are not relevant to our theory. We shall note, however, that the relation S allows some elegant definitions of some important semantic terms. Let $D_n'R$ be the class of all x_n such that there is at least one x_1, one $x_2 \cdots x_{n-1}$, one x_{n+1}, one $x_{n+2} \cdots x_m$ (m being the number of terms of R) such that $R(x_1, x_2, \cdots, x_n, \cdots, x_m)$. We shall call $D_n'R$ "the n-th domain of R." We put now:

2.1. $\text{nom} =_{Df.} D_1{}'S =_{Df.} \hat{a}\{(\exists\, l, f, x)\, S(a, l, f, x)\}$

2.2. $\text{lin} =_{Df.} D_2{}'S =_{Df.} \hat{l}\{(\exists\, a, f, x)\, S(a, l, f, x)\}$

2.3. $\text{rat} =_{Df.} D_3{}'S =_{Df.} \hat{f}\{(\exists\, a, l, x)\, S(a, l, f, x)\}$

2.4. $\text{res} =_{Df.} D_4{}'S =_{Df.} \hat{x}\{(\exists\, a, l, f)\, S(a, l, f, x)\}$.

The above definitions define the classes of names (*2.1*), languages (*2.2*), contents (*2.3*) and things (*2.4*).

3. Analogy a relation involving two names. We contend that analogy, as well as univocity and equivocity, is not an absolute property of *one* name, but a relation involving *two* names at least. If this seems contrary to tradition, it is because of the use the classical authors made of the formula " the same name ": they meant two names of the same form, but spoke, for the reason mentioned above (§ 2), of a single name. If, however, our considerations about the names are admitted, we are compelled to say that no single name is, strictly speaking, univocal, equivocal, or analogical. A single name may have a clear meaning or a confused meaning; but it has always *one* meaning only, and it is not possible to speak about identity or diversity of its meanings, which is required, if we have to define univocity, equivocity, or analogy.

4. The 16 relations between two semantic complexes. Now if our relations involve two meaning names, they must be relations between two semantic complexes; and as the nature of these relations depends on the relations holding between the terms of both complexes, they will be octadic relations, each complex being a tetradic relation. The general form of such relations will be consequently the following:

$$R(a, b, l, m, f, g, x, y),$$

where a and b are names, l and m languages, f and g contents, x and y things, while we have $S(a, l, f, x)$ and $S(b, m, g, y)$.

The question arises now, how many relevant relations are there of the above type. This depends, evidently, on the number of dyadic relations between the terms a-b, l-m, f-g and x-y. Such dyadic relations are very numerous, indeed, infinite in number; but for each couple two relations only are relevant, namely, $I(a, b)$ and $\sim I(a, b)$ for names; $l = m$ and $l \neq m$ for languages; $f = g$ and $f \neq g$ for contents; $x = y$ and $x \neq y$ for things. Thus there are 16 and only 16 relevant relation between two semantic complexes. The following table enumerates them:

No.	a, b	l, m	f, g	x, y	*No.*	a, b	l, m	f, g	x, y
1.	I	$=$	$=$	$=$	*9.*	$\sim I$	$=$	$=$	$=$
2.	I	$=$	$=$	$\neq$	*10.*	$\sim I$	$=$	$=$	$\neq$
3.	I	$=$	$\neq$	$=$	*11.*	$\sim I$	$=$	$\neq$	$=$
4.	I	$=$	$\neq$	$\neq$	*12.*	$\sim I$	$=$	$\neq$	$\neq$
5.	I	$\neq$	$=$	$=$	*13.*	$\sim I$	$\neq$	$=$	$=$
6.	I	$\neq$	$=$	$\neq$	*14.*	$\sim I$	$\neq$	$=$	$\neq$
7.	I	$\neq$	$\neq$	$=$	*15.*	$\sim I$	$\neq$	$\neq$	$=$
8.	I	$\neq$	$\neq$	$\neq$	*16.*	$\sim I$	$\neq$	$\neq$	$\neq$

This table should replace the traditional division of names into univocal, equivocal, and synonyms. As we are, however, not interested in the establishment of a full semantic theory, we shall not define all 16 relations, but only the first four which are directly relevant to the theory of analogy.

5. Definition of univocity and equivocity. These four (octadic) relations, which we shall name "R_1," "R_2," "R_3," and "R_4," are defined as follows:

5.1. $R_1(a, b, l, m, f, g, x, y) \cdot$
$= _{Df.} S(a, l, f, x) \cdot S(b, m, g, y) \cdot I(a, b) \cdot l = m \cdot f = g \cdot x = y$

5.2. $R_2(a, b, l, m, f, g, x, y) \cdot$
$= _{Df.} S(a, l, f, x) \cdot S(b, m, g, y) \cdot I(a, b) \cdot l = m \cdot f = g \cdot x \neq y$

5.3. $R_3(a, b, l, m, f, g, x, y) \cdot$
$= _{Df.} S(a, l, f, x) \cdot S(b, m, g, y) \cdot I(a, b) \cdot l = m \cdot f \neq g \cdot x = y$

5.4. $R_4(a, b, l, m, f, g, x, y) \cdot$
$= _{Df.} S(a, l, f, x) \cdot S(b, m, g, y) \cdot I(a, b) \cdot l = m \cdot f \neq g \cdot x \neq y$

5.1. is the definition of names which are semantically identical in spite of being (physically) two names. We may call them "isosemantic" names. *5.2* is the definition of univocal names: *quorum* (x and y) *nomen est commune* [i. e. $I(a, b)$], *ratio autem significata* (f and g) *est simpliciter eadem* ($f = g$).

5.3 is again the definition of names which have the same denotation, but a different connotation; we may term them "heterologic" from λόγος = *ratio*. Finally *5.4* defines the equivocal names: *quorum* (x and y) *nomen est commune* [i. e. $I(a, b)$], *ratio autem significata simpliciter diversa* ($f \neq g$). In all cases $l = m$, i. e. both languages are identical. This being so, we may drop "$l = m$" and put "l" for "m" in the above definitions. The definitions of univocity and equivocity will now run as follows:

5.5. $$Un(a, b, l, f, g, x, y) \cdot =_{Df.} S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot f = g$$

5.6. $$Ae(a, b, l, f, g, x, y) \cdot =_{Df.} S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot f \neq g$$

We have used "Un" to suggest "*univoca*" and "Ae" to suggest "*aequivoca*"; we also changed, for technical reasons, the order of the two last factors.

The following laws, which are immediate consequences of 5.5, will be needed in the latter parts of this paper:

5.7. $$Un(a, b, l, f, g, x, y) \cdot \supset \cdot S(a, l, f, x)$$

5.8. $$Un(a, b, l, f, g, x, y) \cdot \supset \cdot S(b, l, f, x).$$

6. Partial domains and relata. Each of our relations Un and Ae being heptadic, contains $\binom{7}{6} = 7$ hexadic, $\binom{7}{5} = 21$ pentadic, $\binom{7}{4} = 35$ tetradic, $\binom{7}{3} = 35$ triadic and $\binom{7}{2} = 21$ dyadic partial relations, together 119 (120 with the full relation). We may denote them by "Un" resp. "Ae" followed by two figures: one above, indicating the type of the partial relation (e. g. "Un^5" for a pentadic partial relation of Un), another below, meaning the place which it occupies among partial relations of the given type—the whole between parentheses. E. g. "(Un^5_2)" will mean the second among the pentadic partial relations of Un.

Moreover, each of these partial relations gives rise, exactly as the whole relation does, to many partial domains and relata. The n-th domain of the relation R will be symbolized, as above (par. 2), by "$D_n{}'R$" and the n-th class of relata of R by

" $sg_n{}'R$." There are 120 such domains and 120 such classes of relata. We shall not define them all; the scope of the above remarks was only to show how ambiguous the common language is when we use it to speak about univocity or equivocity and, of course, about analogy.

We shall, however, use our notation in order to define the traditional terms " *univoca* " and " *aequivoca*." We need here first a definition of the following partial dyadic relations:

6.1. $(Un^{2}_{21}) =_{Df.} \hat{x}\hat{y}\{(\exists a, b, l, f, g)\, Un\,(a, b, l, f, g, x, y)\}$

6.2. $(Ae^{2}_{21}) =_{Df.} \hat{x}\hat{y}\{(\exists a, b, l, f, g)\, Ae\,(a, b, l, f, g, x, y)\}$.

We can now define the classes called " *univoca* " and " *aequivoca* " which we shall name " *uni* " or " *aeq* ":

6.3. $\text{uni} =_{Df.} F'(Un^{2}_{21})$

6.4. $\text{aeq} =_{Df.} F'(Ae^{2}_{21})$.

If this would appear too generic, we may use triadic relations, including the language as a term:

6.5. $(Un^{3}_{31}) =_{Df.} \hat{l}\hat{x}\hat{y}\{(\exists a, b, f, g)\, Un\,(a, b, l, f, g, x, y)\}$

6.6. $(Ae^{3}_{31}) =_{Df.} \hat{l}\hat{x}\hat{y}\{(\exists a, b, l, g)\, Ae\,(a, b, l, f, g, x, y)\}$

and consequently:

6.7. $\text{unil} =_{Df.} D'_1(Un^{3}_{31}) \cup D'_2(Un^{3}_{31})$

6.8. $\text{aeql} =_{Df.} D'_1(Ae^{3}_{31}) \cup D'_2(Ae^{3}_{31})$.

7. THE PRINCIPLES OF CONTRADICTION AND EXCLUDED MIDDLE. Other important laws of our theory are two formulae which will be called, respectively, " the law of contradiction " and " the law of excluded middle for univocal and equivocal names." We mean by the first that no two names can be univocal and equivocal in respect to the same language, couples of contents and of things. By the second we mean that if such names are not univocal, they must be equivocal, and conversely. It should be clearly understood that this is true only in respect of some determined contents meant by the names, moreover that these

names must be of the same form and the things they mean must be not-identical. For nothing prevents two names from being univocal in respect of $f - g$ and, at the same time, equi vocal in respect of $h - j$, if $f \neq h$ or $g \neq j$; also, if the names do not mean the contents involved, they are neither univocal nor equivocal in respect of them. The last two conditions follow from our table (in par. 4).

Consequently, we state our principles in the following form:

$$7.1.\quad (a, b, l, f, g, x, y) : S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot \supset \\ \supset \cdot \sim [Un(a, b, l, f, g, x, y) \cdot Ae(a, b, l, f, g, x, y)]$$

$$7.2.\quad (a, b, l, f, g, x, y) : S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot \supset \\ \supset \cdot Un(a, b, l, f, g, x, y) \vee Ae(a, b, l, f, g, x, y).$$

Proofs: [5]

(1) $p \supset \sim (pq \cdot p \sim q)$ (axiom)

(2) $p \supset \cdot pq \vee p \sim q$ (axiom)

(3) $\sim (f = g) \cdot =_{Df.} \cdot f \neq g$ (definition)

(4) $$S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y : \supset \\ \supset : \sim (S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot f = g) : \\ : S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot \sim (f = g) :$$

by (1) putting $\dfrac{S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y}{p}, \dfrac{f = g}{q}$

(5) $= 7.1$

by (4), (3), 5.5 and 5.6 with the rule for adjunction of quantifiers.

(6) $$S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y : \supset \\ \supset : S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot f = g \cdot \vee \\ \vee \cdot S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot \sim (f = g)$$

by (2) with the same substitutions as for (4)

(7) $= 7.2$

[5] The method used is that of the *Principia Mathematica*; therefore what we call a "proof" is rather a sketch of a proof. Rigorous proof could be, however, easily built along the lines given here. (This applies to all proofs contained in the present paper.)

by *(6)*, *(3)*, *5.5* and *5.6* with the rule for adjunction of quantifiers.

The law of excluded middle shows that the classical Thomists were right when they named their *analoga* " *aequivoca a consilio,*" considering them as a subclass of the class of *aequivoca,* and that some modern Thomists are wrong when they put analogy as a third class coordinated to univocity and equivocity. Incidentally it may be remarked that the authors of the *Principia Mathematica* used an exact translation of the " *aequivocatio a consilio* " when they coined the expression " systematic ambiguity." As a matter of fact, they were treating of analogy.

8. ON THE GENERIC NOTION OF ANALOGY. Analogy will be, according to the above analyses, a heptadic relation between two names, a language, two contents and two things (at least). The names will be of the same form; the things must be different. How the contents are related we must still investigate. If we suppose that the answer to that question is expressed by " F," the generic definition of analogy will be the following:

$$8.1. \quad An(a, b, l, f, g, x, y) \cdot = \\ =_{Df.} \cdot S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot F.$$

Moreover, using *7.2* we may say that analogy is either a kind of univocity or a kind of equivocity. According to the Tradition it is certainly not the first. Thus it must be the second. We may put therefore:

$$8.2. \quad An(a, b, l, f, g, x, y) \cdot =_{Df.} \cdot Ae(a, b, l, f, g, x, y) \cdot F.$$

The question arises now, if there is a factor G such that F would be identical with the product of G with another factor, say H_n, G being identical in all kinds of analogy, H_n different for each; the definitions of the successive kinds of analogy would be constructed by putting in *8.2* for " F " first " $G \cdot H_1$," then " $G \cdot H_2$ " and so on. If it be so, we could say that the name " analogy " is univocal; if not, i. e. if there could be no common factor G, it would be equivocal.

As a matter of fact some well known Thomists asserted that the name " analogy " is an analogical name, i. e. (according to *8. 2*) an equivocal one. We are not going to discuss this assertion, but limit ourselves to a correct formulation of it. This requires, however, some preliminary steps.

9. Expansion of the theory to higher levels. We must first note, that we are already dealing with a situation that is far more complex than that which is met in classical Formal Logic. As a matter of fact, all artificial symbols of any system of contemporary Formal Logic belong to the same semantic level, namely to the object language, i. e. each of them means some object, but none of them means a symbol of an object. But in the theory developed above we are using symbols belonging to a higher level, namely our symbols " *a* " and " *b*," which are names of names, i. e. symbols of symbols.

In order to supply the last sentence with a more definite meaning, let us introduce the following recursive definition: (1) the object language is the first level; (2) a language such that at least one term of it is a symbol of a symbol belonging to the n-th level, but none is a symbol of such term, is the $n + 1$ level; (3) a relation holding between objects of which at least one is of the n-th level, and none is of the $n + 1$ level, is of the n-th level.

It will appear that our *a*, *b* and also *S*, *Un*, *Ae* etc. are of the second level; consequently the *names* of these will belong to the third level. Now when we say that " analogy " is an analogical name, the word " analogy " is a *name* of *An*; thus it belongs to the third level. We have to investigate if and how are we allowed to extend our theory to that level, for everything we said until now was clearly situated on the second level.

Let us note first that the laws of the third level would be, as far as structure is concerned, exactly similar to these met on the second. For if we say that " analogy " is analogical, we mean that two names, say *A* and *B* mean in our new language (which is, by the way, the third level), the relations An_1 and An_2 of the objects $(a_1, b_1, l_1, f_1, g_1, x_1, y_1)$ and $(a_2, b_2, l_2, f_2, g_2, x_2, y_2)$. The last two may be considered as classes; but there is nothing to prevent us from considering them as objects, as the relations An_1 and An_2 are true contents of them. Let us put " X " for the first and " Y " for the second. We shall obtain the following exact formulation of the thesis " analogy is analogical ":

$$AN\,(A, B, L, An_1, An_2, X, Y)\,.$$

Here all symbols (except the parentheses and comas) are different from those used in the former paragraphs; and yet the structure is not only similar, but strictly identical with the structure of

$$An\,(a, b, l, f, g, x, y)\,.$$

It is also clear that the whole of our previous analyses might have been repeated on the third level. We would reach a theory, whose terms and meaning would be different from the theory we developed above, but whose structure would be completely identical.

This suggests an important remark. Analyses of such kind involve the use of the idea of structural identity, or isomorphy. Now, according to the theory we shall propose, this means analogy of proportionality. It seems, consequently, that we cannot treat adequately the problem of the generic notion of analogy without a previous examination of analogy of proportionality.

10. Analogy of one-one attribution. Among the several kinds of analogy there are only two that are really relevant: analogy of attribution and analogy of proportionality. Two names which are related by the first will be called " attributively analogous "; similarly, two names related by the latter will be called " proportionally analogous."

We are starting with the first kind. Here again there is one relation called " *analogia unius ad alterum* "—in our terminology " one-one analogy " (symbolically " At ")—and another called " *analogia plurium ad unum,*" here " many-one analogy " (symbolically " *Atm* "). Let us begin with the first, which is the more fundamental.

We have two things, x and y and two contents, f and g; the names a and b are equivocal in regard to them, but there is still another characteristic: x is the cause of y or y the cause of x. Writing " $C(x, y)$ " for " x is the cause of y " we shall have:

$$10.1. \quad At(a, b, l, f, g, x, y) \cdot =_{Df.} \cdot Ae(a, b, l, f, g, x, y) \cdot C(x, y) \vee C(y, x).$$

This is, however, rather unsatisfactory, for the connection of f and g is not shown, the relation of causality being not analysed. We cannot, of course, give a complete analysis of this highly complex notion here. We shall note only that the relation of causality is a pentadic relation which holds between two things, two contents and a peculiar dyadic relation between the things; e. g. the food is the cause of the health of the animal, if and only if there is a content f (health) present in the food (x) such that, if a peculiar relation R (here: of being eaten) is established between x and the animal (y), another content g (the health of the animal) appears in y. Writing " $C(f, x, R, g, y)$ " for this relation we shall have:

$$10.2. \quad At(a, b, l, f, g, x, y) \cdot =_{Df.} \cdot Ae(a, b, l, f, g, x, y) \cdot (\exists R) \cdot C(f, x, R, g, y) \vee \vee C(g, y, R, f, x).$$

The alternative is necessary, according to the traditional doctrine, as there may be an analogy independently from the direction of causality.

11. Analogy of many-one attribution. The second kind of analogy of attribution is clearly derived from the first. The many-one analogy holds, namely, between two names a and b, if and only if there is a third name c, such that both a and b are attributively analogous (according to *10.2*) with c:

$$11.1. \quad Atm(a, b, l, f, g, x, y) \cdot =_{Df.} \cdot (\exists c, h, z) \cdot At(a, c, l, f, h, x, z) \cdot \cdot At(b, c, l, g, h, y, z).$$

Let x be food, y—urine, z—animal, f, g, h—the contents called "health" of, respectively, x, y, z, and a, b, c—the names of these contents. There will be a many-one analogy of a in respect of b.

We may still distinguish four further subclasses of this class of analogical names, for in *11.1* we may have either

(*1*) $C(f, x, R, h, z) \cdot C(g, y, R, h, y)$ —or

(*2*) $C(f, x, R, h, z) \cdot C(h, z, R, g, y)$ —or

(*3*) $C(h, z, R, f, x) \cdot C(g, y, R, h, z)$ —or

(*4*) $C(h, z, R, f, x) \cdot C(h, z, R, g, y)$.

12. Conditions of analogy of proportionality. There are, according to tradition, two conditions for this kind of analogy: the contents must be non-identical, i. e. we must have equivocity; still, the syllogism having as middle terms a couple of proportionally analogous names must be a correct formula. This is secured, according to classical writers, by the fact that these middle terms mean something "proportionally common" in both cases, or that there is an *analogatum commune* containing *in confuso* the contents meant by both names.

It seems at first, that these requirements are contradictory: for, if the meanings of the two names are quite different, one can hardly see how a syllogism with them as middle terms may be a correct formula. As a matter of fact, not only is there a logical theory capable of fulfilling both requirements without contradiction, but it seems even that there are *two* such theories. It seems, namely, that one theory is suggested by the "*proportionaliter commune*," the other by the "*confuse*." We shall call the former "isomorphic," the latter "alternative theory.' As far as is known to the writer, St. Thomas used the isomorphic theory, while the alternative seems to be originated by Cajetan.

13. The alternative theory. The central idea of the alternative theory may be explained as follows: we have to do with three names; one of them means the content f, the other the content g, f and g being the *analogata particularia*; the

third name means the *analogatum commune,* namely, the alternative of f and g, symolically $f \cup g$. We shall give to that expression a sufficiently clear meaning by putting

13.1. $[f \cup g]x \cdot =_{Df.} \cdot fx \vee gx.$

A rather complex situation arises here because of admission of three names: this makes an expansion of our previous formulae to three complexes necessary, and the basic formula for analogy of proportionality becomes a relation of 10 terms. Once a definition of this form is established, the (heptadic) relations analogous to *Un* and *Ae* will appear as partial relations of the general one, and the verbal formulae as elliptic. We shall not, however, define this general relation in that way, as, for several reasons, to be explained later (par. 16), the whole alternative theory appears as inadequate. But we are going to investigate the validity of a syllogism *in Barbara* with proportionally analogous middle terms. For the use in that inquiry we define the analogy of proportionality (*Anp*) according to the alternative theory as a heptadic relation in the following way:

13.2. $Anp(a, b, l, f, g, x, y) \cdot$
$=_{Df.} \cdot Ae(a, b, l, f, g, x, y) \cdot (\exists h) \cdot f = [g \cup h].$

This is a partial relation contained in the full relation of analogy described above.

14. On formal validity of syllogism. If we wish to investigate the validity of a syllogism with analogical middle terms we meet a serious difficulty unknown in current Formal Logic. For in current Formal Logic it is always supposed that a formula which is verbally valid is also formally valid; the reason of this supposition is that all terms used in current Formal Logic are univocal symbols. Here, however, the situation is different, as we have to deal with analogical names. We need, consequently, a distinction between the verbal and formal validity of a formula; moreover we need to know when a verbally valid formula is also formally valid. This is by no

means a universal rule, as the case of the syllogism with equivocal and non-analogical middle terms shows. We are not going to investigate the problem in its full generality, but we will limit ourselves to a single case, the syllogism *in Barbara*.

We shall first construct two languages:

(1) A first-level univocal language. This will be the language of the theory of classes, interpreted as a Logic of contents. In it the mode *Barbara* will run as follows:

$$f \subset g \cdot h \subset f \cdot \supset \cdot h \subset g.$$

(2) A second-level analogical language. This will contain all symbols used until now (small Latin letters being sometimes substituted by small Greek letters and indexes being added to them), with addition of the following: (i) " Π "; a formula composed of " Π " followed by " a," followed by " b " will be interpreted as meaning the formula " $a \subset b$ "; (ii) " $+$ "; a formula such as " $\Pi + a + b$ " will be read: " a formula composed of Π followed by a, followed by b "; (iii) " $\varepsilon\, T$ "; " $F \varepsilon T$ " will be read: " F is a true theorem."

The proofs will be developed in a second-level language, containing as subclasses the above two. We shall proceed as follows. Given the (second-level) premises A and B such that $A \varepsilon T \cdot B \varepsilon T$, we wish to prove that the (verbally correct) conclusion C (of the same level) is a true theorem, i. e. that $C \varepsilon T$. We translate A and B into the first-level language, apply to the result the laws of classical Formal Logic and obtain a conclusion, which we re-translate into the second-level language; if we are able to obtain $C \varepsilon T$ in that way, the formula " if $A \varepsilon T \cdot B \varepsilon T$, then $C \varepsilon T$ " is clearly a valid formula and the formal validity of the mode, whose premises are A and B, and the conclusion is C, is proved.

We put as a law of translation the intuitively evident:

14.1. $S(a, l, f, x) \cdot S(b, l, g, y) : \supset : \Pi + a + b \,\varepsilon\, T \cdot \equiv \cdot f \subset g.$

With the help of *14.1* we can easily prove that a syllogism *in Barbara* with univocal middle terms is a formally valid formula; but we cannot prove it if the middle terms are either purely equivocal or attributively analogical. Alongside of *14.1* we shall need still another law of translation for cases where an existential quantifier is involved:

14.2. $(\exists h) \cdot S(a, l, [f \cup h], x) \cdot S(b, l, g, y) : \supset$
$\supset : (\exists h) \cdot [f \cup h] \subset g.$

This seems to be also intuitively evident.

15. THE VALIDITY OF THE SYLLOGISM IN BARBARA WITH ANALOGICAL MIDDLE TERMS ACCORDING TO THE ALTERNATIVE THEORY. In such a syllogism the middle term of the major premise is analogical with regard to the middle term in the minor premise, the situation being this, that the former means alternatively the content meant by the latter *and* some other content. This syllogism, if *in Barbara*, is a valid formula. The proof is rather cumbersome, because of the existential quantifier; we shall however give here a developed sketch of it.

In the first place we need two theorems analogous to *5.7* and *5.8*. These may be proved as follows:

(1) $Anp(a, b, l, f, g, x, y) \cdot$
$\equiv \cdot Ae(a, b, l, f, g, x, y) \cdot (\exists h) \cdot f = [g \cup h]$
[by *13.2*]

(2) $\equiv \cdot S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot x \neq y \cdot$
$\cdot f \neq g \cdot (\exists h) \cdot f = [g \cup h]$
[by *(1)* and *5.6*]

(3) $\equiv \cdot (\exists h) \cdot S(a, l, f, x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot$
$\cdot x \neq y \cdot f \neq g \cdot f = [g \cup h]$
[by *(2)* and **10.24 Principia Mathematica.*]

(4) $\equiv \cdot (\exists h) \cdot S(a, l, [g \cup h], x) \cdot S(b, l, g, y) \cdot I(a, b) \cdot$
$\cdot x \neq y \cdot f \neq g$
[by *(3)* and **13.12 Principia Mathematica.*]

(5) $\equiv \cdot (\exists h) S(a, l, [g \cup h], x) \cdot (\exists h) S(b, l, g, y) \cdot$
$\cdot (\exists h) \cdot x \neq y \cdot f \neq g$
[by *(4)* and **10.5 Principia Mathematica.*]

15.1. $Anp(a, b, l, f, g, x, y) \cdot \supset \cdot (\exists h) S(a, l, [g \cup h], x)$
[by *(5)* and "$p \equiv qr \cdot \supset \cdot p \supset q$"]

15.2. $Anp(a, b, l, f, g, x, y) \cdot \supset \cdot \dot{S}(b, l, g, y)$ [by *(5)* and "$p \equiv qrs \cdot \supset \cdot p \supset . r$," dropping the quantifier].

We enumerate now the five hypotheses of the syllogism *in Barbara* with analogical middle terms, explained according to the alternative theory:

H1. $\Pi + m_1 + a_1 \varepsilon T$

H2. $\Pi + b_1 + m_2 \varepsilon T$

H3. $Anp(m_1, m_2, l, \mu_1, \mu_2, x, y)$

H4. $Un(a_1, a_2, l, \alpha_1, \alpha_2, z, t)$

H5. $Un(b_1, b_2, l, \beta_1, \beta_2, u, v)$.

The proof of "$\Pi + b_2 + a_2 \varepsilon T$" runs as follows:

(1)	$(\exists h) S(m_1, l, [\mu_2 \cup h], x)$	by *H3* and *15.1*
(2)	$S(a_1, l, \alpha_1, z)$	by *H4* and *5.7*
(3)	$(\exists h) \cdot [\mu_2 \cup h] \subset \alpha_1$	by *(1)*, *(2)*, *H1* and *14.2*
(4)	$S(b_1, l, \beta_1, u)$	by *H5* and *5.7*
(5)	$S(m_2, l, \mu_2, y)$	by *H3* and *15.2*
(6)	$\beta_1 \subset \mu_2$	by *(4)*, *(5)*, *H2* and *14.1*
(7)	$\beta_1 \subset \mu_2 \cdot (\exists h) \cdot [\mu_2 \cup h] \subset \alpha_1$	by *(6)* and *(3)*
(8)	$(\exists h) \cdot \beta_1 \subset \mu_2 \cdot [\mu_2 \cup h] \subset \alpha_1$	by *(7)* and **10.35 PM*
(9)	$(\exists h) \cdot \beta_1 \subset \alpha_1$	by *(8)*, "$f \subset g \cdot [g \cup h] \subset j \cdot \supset \cdot f \subset j$" and **10.28 PM*
(10)	$\beta_1 \subset \alpha_1$	by *(9)*
(11)	$S(b_2, l, \beta_1, u)$	by *H5* and *5.8*
(12)	$S(a_2, l, \alpha_1, z)$	by *H4* and *5.8*
(13)	$\Pi + b_2 + a_2 \varepsilon T \cdot \equiv \cdot \beta_1 \subset \alpha_1$	by *(11)*, *(12)* and *14.1*
(14)	$\Pi + b_2 + a_2 \varepsilon T$	by *(10)* and *(13)*

Q.E.D.

16. Criticism of the alternative theory. It has been shown that a syllogism *in Barbara* with analogical middle terms, defined according to the alternative theory, is a formally

valid formula. This is, however, the only advantage of this theory. Not even all requirements of Theology and Metaphysics in regard to the syllogism can be met by means of it. For a syllogism of these sciences has not only analogical middle terms, but also analogical major terms; e. g. when we write " if every being is good, and God is a being, then God is good," not only " being," but also " good " must be analogical. But this means, according to the alternative theory that *H4* in par. 15 should be replaced by

$$Anp\,(a_1, a_2, l, \alpha_1, \alpha_2, z, t)\,.$$

If so, instead of (3) we would obtain only

$$(\exists\, h) \cdot [\mu_2 \cup h] \subset [\alpha_2 \cup g]$$

which does not allow us to draw the conclusion (*14*). Neither can we try to invert the order of " *f* " and " *g* " in *15.1*; in that case the syllogism would become valid, but the major term in the conclusion would have an alternative meaning, which can hardly be admitted.

Moreover, the theory has other inconveniences. First, the very definition of analogy, as sketched in par. 13, is highly unsatisfactory. By saying that two names are analogical if and only if there is a third name meaning alternatively the contents meant by both, we do not show any intrinsic connection between the contents involved; and every couple of names would be analogical, according to that definition, for we can always introduce into our system a new name, defined precisely as meaning the said alternative. Secondly, there are serious gnoseological difficulties. The situation with which we have to deal, is the following: two names are given, and while we know the meaning of the first by direct experience, we do not know in that way the meaning of the second. In order to be able to use that second name correctly, we must supply it with a meaning correlated in some way with the meaning of the first. Now the alternative theory allows nothing of the sort: it only says how we can deal with middle terms having alternative meanings, when both meanings are already known.

These remarks do not lead to the complete rejection of the alternative theory; but they seem to show that it is at least incomplete and should be completed by another theory. The present author believes that this was the position of Cajetan.

17. THE ISOMORPHIC THEORY. This theory is based on the following considerations: the "*proportionaliter eadem*" suggests that there is an identity, not between the contents meant by both analogical terms, but between some relations holding between the first (f) and its thing (x) on one side, the second (g) and its thing (y) on the other. The texts of St. Thomas Aquinas are clear enough here. The said relations are, however not identical; this is also a traditional thesis, strongly emphasized by all classical Thomists. We may therefore admit, as a first approximation, that, while being non-identical, they are both contained in the same relation. The definition of analogy of proportionality would run, in that case, as follows:

$$17.1.\quad Anp\,(a,b,l,f,g,x,y)\cdot =_{Df.}\cdot Ae\,(a,b,l,f,g,x,y)\cdot (\exists\, P,Q,R)\cdot fPx\cdot gQy\cdot P\neq Q\cdot P\subset R\cdot Q\subset R.$$

This is, however, not satisfactory. For if *17.1* would be the definition of analogy of proportionality, there would be a material univocal element; analogy would allow us to transfer to the other name some material relations found in the meaning of the first. Now St. Thomas Aquinas and Tradition are quite clear as to the negation of such univocity. But *17.1* can be corrected by the affirmation that the common element in both relations is formal, i. e. consists in the isomorphy of these relations. The definitions becomes:

$$17.2.\quad An\,(a,b,l,f,g,x,y)\cdot =_{Df.}\cdot Ae\,(a,b,l,f,g,x,y)\cdot \cdot\,\exists\, P,Q)\cdot fPx\cdot gQy\cdot PsmorQ.$$

This is what we mean by "isomorphic theory." It is strongly supported by the fact that St. Thomas Aquinas uses for illustration of his doctrine mathematical proportionality, the only mathematical function he possessed and a function which makes one immediately think of isomorphy.

One may think, perhaps, that if this be analogy of proportionality, the meaning of our sentences about spirit, God etc., would be extremely poor, indeed limited to some very few formal relations enumerated in the *Principia Mathematica*. But this is not so. It is true that we cannot, as yet, give exact formulations of many formal properties involved in relations used by Metaphysics and Theology; the reason, however, is not the lack of such formal properties, but the very undeveloped state of Biology and of other sciences, from which the Metaphysician and the Theologian must draw his analogical names (and contents). An immense progress in speculative sciences would arise out of a formalization of these disciplines. And yet, even in the actual state of knowledge, where only Mathematics, i. e. the poorest of all sciences, is formalized, we can show, e. g., the difference between the Principle and the Father by purely formal means—as, evidently, the first is transitive, the second intransitive.

18. THE EXISTENTIAL INTERPRETATION OF THE MODE *Barbara*. If the isomorphic theory is admitted, a peculiar interpretation must be given to the mode *Barbara* with analogical middle terms. Let us consider the following substitution: " if all being is good, and God is a being, then God is good." According to the isomorphic theory the only common element meant by the two " being "and the two " good " is a product of some formal relations, say P in the first case and Q in the second. But if it is so, the major must be interpreted as follows: " for all x: if there is an f such that fPx, then there is a g such that gQy "; the minor will be interpreted in the same manner by the formula " for all x: if there is an h such that hRx, then there is an f such that fPx." From this we draw the conclusion " for all x: if there is an h such that hRx, then there is a g such that gQx." This would mean: " if there is an x such that h is the Divinity of x, then there is a g such that g is the Goodness of x." The law used here is:

18.1. $(x) \cdot (\exists f) fPx \supset (\exists g) gQx : (x) \cdot (\exists h) hRx \supset (\exists f) fPx :$
$\supset : (x) \cdot (\exists h) hRx \supset (\exists g) gQx.$

This is a correct formula of the Logic of predicates.

The remarkable result of the existential interpretation is that the Thomistic idea of analogy becomes sharply formulated in a very anti-univocal sense. For, we do not know, as a result of our reasoning according to *18. 1*, anything except that there is something (undetermined as to the content) which has to God the set of quite formal relations Q. And yet, the talk about God's goodness is clearly meaningful; moreover rigorous demonstrations concerning it are possible.

19. THE VALIDITY OF THE SYLLOGISM WITH ANALOGICAL MIDDLE TERMS ACCORDING TO THE ISOMORPHIC THEORY. We are going to show now how, in such theory, a syllogism *in Barbara* is a formally valid formula. We meet here, however, two formal difficulties.

First we note that isomorphy, being a relation between two relations, cannot be, as such, treated as a relation in which these relations are contained; now this seems to be necessary if we wish to construct a correct syllogism with analogical middle terms, interpreted according to the isomorphic theory.

This difficulty may be, however, obviated in the following manner. Isomorphy implies the identity of a series of formal properties of the relations involved. These formal properties are different in each case of couples of isomorphic relations; but for each of them *in concreto* a product of such properties may be determined. E. g., in some cases both relations will be included in diversity and will be transitive; in other cases they will be intransitive and assymetric etc. Now each of these properties may be conceived as a relation in which the given isomorphic relations are contained. This can be done by introducing in the system the name of a new relation, which is treated as a primitive term, but whose meaning is determined by an axiom. E. g. for symmetry we will put a relation S and determine the meaning of " S " by the axiom $(x, y) : xSy \cdot \equiv \cdot xSy \equiv x\breve{S}y$. The product of such relations would constitute the relation in which both isomorphic relations are contained.[6]

[6] The author is conscious that the proposed solution is highly un-orthodox; he

The other difficulty is strictly operational. It will appear that we shall need an expansion of our *17.2* in order that the name of the common relation R, in which the relations P and Q are contained, might be treated as an argument of "*Anp*." If so, a new relation must be defined, namely an octadic relation containing as terms, alongside of the seven stated in *17.2*, also R. We shall define it as follows:

19.1. $Anp(a, b, l, f, g, x, y, R) \cdot =_{Df.} \cdot Ae(a, b, l, f, g, x, y) \cdot$
$\cdot (\exists P, Q, R) \cdot fPx \cdot gQy \cdot P \neq Q \cdot P \neq R \cdot Q \neq R \cdot$
$\cdot P \in R \cdot Q \in R \cdot R \,\varepsilon\, Form.$

By "*Form*" we mean the class of all formal relations, as described in par. 17.

There will be three laws of translation, analogous to *14.1*:

19.2. $Anp(m_1, m_2, l, \mu_1, \mu_2, x, y, P) \cdot Anp(a_1, a_2, l, \alpha_1, \alpha_2, z, t, Q) : \supset$
$\supset : \Pi + m_1 + a_1 \,\varepsilon\, T \cdot \equiv \cdot (x) \cdot (\exists f) fPx \supset (\exists g) gQx.$

19.3. $Anp(b_1, b_2, l, \beta_1, \beta_2, u, v, R) \cdot Anp(m_1, m_2, l, \mu_1, \mu_2, x, y, P) : \supset$
$\supset : \Pi + b_1 + m_2 \,\varepsilon\, T \cdot \equiv \cdot (x) \cdot (\exists h) hRx \supset (\exists f) fPx.$

19.4. $Anp(b_1, b_2, l, \beta_1, \beta_2, u, v, R) \cdot Anp(a_1, a_2, l, \alpha_1, \alpha_2, z, t, Q) : \supset$
$\supset : \Pi + b_2 + a_2 \,\varepsilon\, T \cdot \equiv \cdot (x) \cdot (\exists h) hRx \supset (\exists g) gQx.$

Our hypotheses are

H1. $\Pi + m_1 + a_1 \,\varepsilon\, T$

H2. $\Pi + b_1 + m_2 \,\varepsilon\, T$

H3. $Anp(m_1, m_2, l, \mu_1, \mu_2, x, y, P)$

H4. $Anp(a_1, a_2, l, \alpha_1, \alpha_2, z, t, Q)$

H5. $Anp(b_1, b_2, l, \beta_1, \beta_2, u, v, R)$.

The proof of "$\Pi + b_2 + a_2 \,\varepsilon\, T$" runs as follows:

(*1*) $(x) \cdot (\exists f) fPx \supset (\exists g) gQx$ by *H3, H4, H1* and *19.2*

(*2*) $(x) \cdot (\exists h) hRx \supset (\exists f) fPx$ by *H5, H3, H2* and *19.3*

(*3*) $(x) \cdot (\exists h) hRx \supset (\exists g) gQx$ by (*1*), (*2*) and *18.1*

---

would be glad to find anything better. It must be remembered, however, that the whole difficulty is purely operational; it seems intuitively evident that once there is a common property, the syllogism is valid.

(*4*) $\Pi + b_2 + a_2 \varepsilon T \cdot \equiv \cdot (x)(\exists h) hRx \supset (\exists g) gQx$
by *H5*, *H4* and *19.4*

(*5*) $\Pi + b_2 + a_2 \varepsilon T$ by (*4*) and (*3*)
Q.E.D.

20. ON ANALOGY IN RECENT LOGIC. While the classical Thomists used analogy in Ontology and Theology, but not in Logic, recent writers seem to make a constant use of it in Formal Logic. We noticed already that the authors of the *Principia Mathematica* re-invented the very name used for analogy by the Thomists (par. 7) and that analogy appears in the construction of Semantics (par. 9). The last phenomenon is connected with the theory of types. It is known that, in order to avoid contradictions, we are bound to divide all objects treated by Logic (or all logical expressions) into classes called " types." The formulae used in each type have quite a different meaning, but exactly the same structure as the formulae used in another. This means that the formal properties involved are identical i. e. that we have to do with analogy, at least if the isomorphic theory is accepted.

The question arises as to why analogy has penetrated the domain of Formal Logic. The answer seems to be given by the theory of Prof. H. Scholz, who says that recent Formal Logic is nothing else than a part of classic Ontology.[7] As a matter of fact, recent Formal Logic generally deals, not with rules, but with laws of the being in its whole generality; most of the laws contained in the *Principia Mathematica*, e. g., as opposed to metalogical rules, are such laws. If this is so, it is not to be wondered at that some consideration must have been given to analogy, for " being " is an analogical term and so are the names of all properties, relations, etc., belonging to being as such.

One curious feature of these developments is that the highly trained mathematical logicians who had to speak about analogy, spoke about it in a very loose and inexact way.

[7] H. Scholz, *Metaphysik als strenge Wissenschaft*, Köln 1941.

What, for example, the *Principia Mathematica* contains on the subject is far more rudimentary than the classic Thomistic doctrine. Yet, recent Formal Logic, once applied to the language itself, supplies superior tools for the elaboration of that notion. The present paper is believed to contain only a very small sub-class of the class of theorems on analogy, which may and should be elaborated by means of recent Formal Logic.

I. M. BOCHENSKI, O. P.

University of Fribourg,
Fribourg, Switzerland.