this as a general position. For the devotee of a religious faith, the religious freedom he claims and believes himself to enjoy may be no more than the freedom to practice unmolested a form of worship he has inherited and which he has never felt the faintest temptation to question; in such a case it is a fiction to speak of a process of choice. The same can be said of the man who is content to follow narrowly, uncritically, and unadventurously the established customs and conventions of his society. Even though there may be a sense in which we can intelligibly talk of such men as being slaves to customs, habits, or orthodoxies, it would still be straining the point to maintain that they are not free.

On the other hand, the man who has been so molded and manipulated that he always wants what his ruler or superior wants him to want is scarcely free. This case suggests that freedom will exist only where there exists the *possibility* of choice, and the possibility of choice in turn implies not only the absence of direct coercion and compulsion but also that the availability and the characteristics of alternatives must be capable of being known. Thus, whatever the situation of any particular individual may be, it is most likely that there will be a large measure of individual freedom within a society when there exists what Mill calls a variety of conditions—where a wide variety of beliefs are in fact expressed and where there is a considerable diversity of tastes and pursuits, customs and codes of conduct, ways and styles of living. And, because of the connection between inequality of power and inequality with respect to the enjoyment of freedom, a society in which power is widely distributed is also likely to be the one characterized by the existence of wide possibilities for choice and individual initiative.

***See also*** Authority; Censorship; Democracy; Determinism and Freedom; Liberalism; Liberty; Mill, John Stuart; Power; Rights; Russell, Bertrand Arthur William.

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***P. H. Partridge (1967)***

# FREE WILL

See *Determinism and Freedom*

# FREGE, GOTTLOB

***(1848–1925)***

## LIFE

After studying mathematics, physics, chemistry, and philosophy at the universities of Jena and Göttingen, the German mathematician, logician, and philosopher Gottlob Frege obtained his mathematical doctorate in Göttingen (1873) and his mathematical *habilitation* in Jena (1874). From 1874 to 1879 he taught mathematics at the University of Jena as a lecturer; in 1879 he was promoted to adjunct professor, and in 1896 to associate professor. Frege never obtained a full professorship. He retired from teaching in 1917 because of illness, becoming emeritus in 1918.

While he received little professional recognition during his lifetime, Frege is widely regarded in the early twenty-first century as the greatest logician since Aristotle, one of the most profound philosophers of mathematics of all times, and a principal progenitor of analytic philosophy. His writing exhibits a level of rigor and precision that was not reached by other logicians until well after Frege's death.

## MAIN WORKS

In the monograph *Begriffsschrift* (1879) Frege introduces his most powerful technical invention, nowadays known as predicate logic. In his second book, *Die Grundlagen der Arithmetik* (1884), he discusses the philosophical foundations of the notion of number and provides an informal argument to the effect that arithmetic is a part of logic (a thesis later known under the epithet *logicism*). The pamphlet *Funktion und Begriff* (1891) is an elucidation of Frege's fundamental ontological distinction between functions (with concepts as a special case) and objects; certain difficulties with the views expressed therein are discussed in the essay "Über Begriff und Gegenstand" (1892). Frege's most celebrated achievement in the philosophy of language, the distinction between the sense and the reference of an expression, is expounded in his landmark essay "Über Sinn und Bedeutung" (1892). *Grundgesetze der Arithmetik* (volume 1, 1893; volume 2, 1903), his magnum opus, constitutes his abortive (because of Bertrand Arthur William Russell's antinomy) attempt at rigorously proving the logicist thesis. The essay "Der Gedanke: Eine logische Untersuchung" (1918) is a conceptual investigation of truth and that with respect to which the question of truth arises (called thoughts by Frege).

## FREGE'S LOGIC

By replacing the traditional subject-predicate analysis of judgments with the function-argument paradigm of mathematics and inventing the powerful quantifier-variable mechanism, Frege was able to overcome the limitations of Aristotelian syllogistics and created the first system of (higher-order) predicate logic. He thereby devised a formal logical language adequate for the formalization of mathematical propositions, especially through the possibility of expressing multiply general statements such as "*for every* prime number, *there is* a greater one."

The first presentation of his *begriffsschrift* (concept script—Frege's logical formula language) is contained in the 1879 monograph by the same name. At this time, the linguistic and philosophical underpinnings of begriffsschrift, as well as the description of the language itself, are still somewhat imprecise. There are, for instance, no formation rules given for the formulas of the language; functions seem to be identified with functional expressions; the meanings of the propositional connectives are specified in terms of assertion and denial rather than truth and falsity; and although Frege officially countenances only one inference rule, namely, *modus ponens*, he tacitly uses an instantiation rule for the universal quantifier as well. The first volume of *Grundgesetze*, however, presents a mature and amazingly rigorous version of the system, taking into account the various insights Frege had developed since the publication of *Begriffsschrift*. Unless otherwise noted, the following discussion pertains to this later system; for the time being, one should ignore the course-of-values operator, which is discussed later on in connection with Russell's antinomy.

The primitive symbols of Frege's begriffsschrift are then those for equality, negation, the material conditional, and the first- and higher-order universal quantifiers. In addition, there are gothic letters serving as bound variables (of first and higher orders), as well as Latin letters, whose role one would today characterize as that of free variables (again, of various orders). Disjunction, conjunction, and the existential quantifier are neither primitive, nor are they introduced as abbreviations, as would be customary today; rather, Frege notes that they can be simulated by means of the existing primitives.

Frege carefully distinguishes between basic laws (axioms) on the one hand, and inference rules on the other hand. With respect to a specified set of basic laws and rules of inference, he comes close to a rigorous definition of derivations in the predicate calculus.

The logical connectives, as well as the quantifiers, are taken to be denoting expressions, having as references the requisite truth functions and higher-order functions, respectively. Equality undergoes a radical change in interpretation between the time of *Begriffsschrift* and that of *Grundgesetze*. In the earlier system, assuming that the expression $A$ refers to the object $a$, and the expression $B$ to object $b$, Frege construes identities of the form $A = B$ metalinguistically, taking them to mean that the expressions $A$ and $B$ are coreferential, rather than that $a$ and $b$ are the same object. In *Grundgesetze*, however, identity is conceived of as a binary relation between objects, much as is standard today (this change in interpretation is, incidentally, accompanied by a switch in notation from the triple bar $\equiv$ to the now customary double bar $=$). Arguably, there is an analogous shift in the understanding of the universal quantifier; the formulations in *Begriffsschrift* suggest that it is to be interpreted substitutionally, whereas it is fairly clear in *Grundgesetze* that an objectual interpretation is intended. But the issue is difficult to judge, not only because the language of the earlier work is rather imprecise but also because it is not clear whether Frege was aware of the significance of the distinction between objectual and substitutional quantification.

Frege's perhaps most impressive achievement in pure logic is his celebrated definition (with the proof of its adequacy) of the ancestral (or transitive closure) $R^*$ of a binary relation $R$ with the help of second-order quantification, already contained in *Begriffsschrift* and central to the logicist enterprise. Informally, an object $a$ bears the ancestral $R^*$ of a relation $R$ to an object $b$ if $b$ can be reached from $a$ in a finite (nonzero) number of $R$-steps. That is, whenever there are objects $a_1, a_2, \dots, a_n$ ($n > 1$) such that $a_1Ra_2, a_2Ra_3, \dots, a_{n-1}Ra_n$, then $a_1$ bears $R^*$ to $a_n$. For example, if $R$ is the parenting relation (so that $xRy$ holds if and only if $x$ is a parent of $y$), then $R^*$ is the ancestor relation (i.e., $xR^*y$ holds if and only if $x$ is an ancestor of $y$), because $x$ is an ancestor of $y$ if $y$ is a child of $x$, or a child of a child of $x$, or a child of a child of a child of $x$, and so on. Frege's idea is to define $R^*$ from $R$ as follows: $a$ stands in the relation $R^*$ to $b$ if and only if $b$ has every property $F$ such that (1) all objects to which $a$ bears $R$ have $F$, and (2) $F$ is hereditary with respect to the relation $R$ (meaning that, whenever something $x$ has the property $F$, and $x$ bears $R$ to some $y$, then $y$ also has $F$). Note that this definition employs second-order quantification (over all $R$-hereditary properties $F$).

It is clear that, if $b$ can be reached from $a$ in a finite nonzero number of $R$-steps, then Frege's definition correctly implies that $aR^*b$, for if $F$ is any property and $b$ can be reached from $a$ in one step, then by clause (1) of the definition $b$ must have $F$, and if $b$ can be reached from $a$ by some number of $R$-steps greater than 1, one must have passed through an object to which $a$ bears $R$, and which thus has $F$ by clause (1), and every further object through which one has passed, including the last object $b$, must have $F$ by clause (2). On the contrary, if $b$ cannot be reached from $a$ in a finite nonzero number of $R$-steps, then $b$ lacks just that property of being reachable from $a$ in a finite number of $R$-steps (a property that fulfills conditions [1] and [2]). In modern notation Frege's formal definition is as follows:

$$aR^*b :\leftrightarrow \forall F((\forall x(aRx \rightarrow Fx) \,\&\, \forall x\forall y\,(Fx \,\&\, xRy \rightarrow Fy)) \rightarrow Fb).$$

It should be noted, finally, that Frege did not regard the sentences of his begriffsschrift as mere forms, open to arbitrary interpretation. Rather, he took them to express definite thoughts (i.e., propositions). This is manifest in the presence of a special symbol, the vertical judgment stroke, whose occurrence before a begriffsschrift formula indicates that the formula's content is actually asserted (and not talked about or simply entertained without judgment as to truth and falsity). While Frege did discuss the formal character of logic in terms of preservation of consequence on substituting nonlogical expressions for others (witness his correspondence with David Hilbert and the 1906 essay series "Über die Grundlagen der Geometrie"), he showed little inclination to pursue such investigations himself. Frege also has little to say about the characterization of propositions as logical truths; there is no indication that he had anything like Alfred Tarski's model-theoretic criterion in mind. He occasionally remarks that logical axioms are required to be "obvious," but generally takes it for granted that the specific basic laws he lays down are in fact logical truths.

## FREGE'S ONTOLOGY AND PHILOSOPHY OF LANGUAGE

Frege's mature ontology is characterized by the fundamental dichotomy between saturated entities or objects (*Gegenstände*) on the one hand, and unsaturated entities or functions on the other hand. Functions are unsaturated or incomplete in the sense that they carry argument places that need to be filled; an object is anything that is not a function. Concepts are special functions, namely, functions whose values are always one of the two truth-values: the True and the False (which Frege takes to be objects, as will be explained). The realm of functions is stratified: Unary functions mapping objects to objects are first level, unary functions mapping first-level functions to objects are second level (an instance being the concept denoted by the first-level existential quantifier, which maps every first-level concept under which some object falls to the True, and all other first-level concepts to the False), and so on. The stratification becomes more complicated with functions of more than one argument, since there exist, for instance, functions of two arguments with one argument place for unary first-level functions and one argument place for objects (an instance being the application function, which maps a unary first-level function $f$ and an object $a$ to the result $f(a)$ of applying $f$ to $a$), and so on.

The saturated-unsaturated dichotomy has, for Frege, a parallel in the linguistic realm. Singular terms, such as proper names and definite descriptions, are (linguistically) saturated (or complete) and refer to objects; predicate and functional expressions are incomplete and refer to functions. In determining the ontological status of certain entities Frege often proceeds by analyzing the expressions used to refer to them and takes the saturated or unsaturated nature of the expressions as a reliable guide to their ontological saturation status.

Now since the expression "the concept *horse*" is grammatically a singular term, Frege takes it to refer to an object, which commits him to the paradoxical claim that the concept *horse* is not a concept (compare to "Über Begriff und Gegenstand"). In an attempt to resolve this predicament Frege proposes that with every concept $F$ is associated a certain (proxy) object that serves as the referent of "the concept $F$" (some commentators believe that Frege intended the extension of $F$ to be this proxy object, but the interpretive issue remains contentious). There remains a fundamental problem, however, for on the one hand, objects and concepts belong to distinct ontological categories, so that no predicate can be meaningfully applied to both a concept and an object; but on the other hand, Frege's explanation of this categorial distinction requires him to use the predicates "is an object" and "is a concept" in just this way—as contrasting (nonempty) predicates that can be applied to the same items. This creates some famous difficulties, some of which are discussed in the essay "Über Begriff und Gegenstand," because singular terms such as "the concept *horse*" cannot, according to Frege, refer to concepts, but refer to certain (proxy) objects instead.

Frege's most famous invention is perhaps his distinction between the sense (*Sinn*) and the reference (*Bedeutung*) of a linguistic expression, first introduced in his short 1891 booklet *Funktion und Begriff*, and expounded in detail in the 1892 essay "Über Sinn und Bedeutung." In the case of a singular term its reference is the object denoted by the term, whereas its sense is determined by the way that object is presented through the expression (its mode of presentation). Frege conceives of complete (declarative) sentences, perhaps infelicitously, as peculiar singular terms, so that their references, the special logical objects the True and the False, respectively, are objects. The thought expressed by a sentence is then defined by Frege to be the sentence's sense. The sense of a sentence is thus the mode of presentation of its truth-value; that is, on a natural reading, the sentence's truth-conditions. In the case of incomplete expressions, such as predicates and functional expressions, the references are of course the corresponding unsaturated concepts and functions.

While not explicitly discussed in "Über Sinn und Bedeutung," it becomes clear from the Frege-Husserl correspondence that Frege intended the notion of sense to apply to predicates as well. Scholarly discussion continues whether Frege considered the senses of unsaturated expressions to be functions, or whether he regarded all senses as objects (a stance suggested by the fact that every sense can be referred to by means of a singular nominal phrase of the form "the sense of the expression X"). In the essay "Der Gedanke" Frege expounds a Platonistic view of senses as inhabitants of a "third realm" of nonperceptible, objective entities, as opposed to the (perceptible) objects of the external world and the subjective contents (ideas) of humans' minds.

Frege was motivated to introduce the sense-reference distinction to solve certain puzzles, chief among them (1) the apparent impossibility of informative identity statements and (2) the apparent failure of substitutivity in contexts of propositional attitudes. As for (1), Frege argued that the statements "the morning start is the evening star" and "the morning star is the morning star" obviously differ in cognitive value (*Erkenntniswert*), which would be impossible if the object designated constituted the only meaning of a singular term. The sense-reference distinction allows one to attribute different cognitive values to these identity statements if the senses of the terms flanking the identity sign differ, while still allowing the objects denoted to be one and the same.

Regarding (2), Frege noticed that the sentences "John believes that the morning star is a body illuminated by the sun" and "John believes that the evening star is a body illuminated by the sun" may have different truth-values, although the one is obtained from the other by substitution of a coreferential term. He argued that, in contexts of propositional attitudes, expressions do not have their usual reference, but refer to their ordinary senses (which thus become their indirect references); then since "the morning star" and "the evening star" differ in ordinary sense, they are not, in the context at hand, coreferential, having distinct indirect references. Debate continues as to Frege's intentions concerning indirect senses of expressions, in particular whether iterated propositional attitude contexts give rise to an infinite hierarchy of indirect senses.

In the introduction to *Grundlagen* Frege enunciates "three fundamental principles" for his investigations. The first of these is an admonition to separate the logical from the psychological (a motif that runs through all of Frege's works); the third demands observance of the concept-object distinction. But it is the second of these principles that has drawn most attention and interpretation: "never to ask for the meaning of a word in isolation, but only in the context of a proposition." Other (not obviously equivalent) formulations of the principle occur in sections 60, 62, and 106 of *Grundlagen*; some authors take Frege to express a precursor of this principle in section 9 of *Begriffsschrift*, and some see an echo of it in *Grundgesetze*, volume 1, section 29.

The proper interpretation of the context principle continues to be contentious. While some philosophers regard it as being of the utmost importance to an understanding of Frege's philosophy, others view it as a rather ill-conceived and incoherent doctrine that he appears to have given up in later works. Those who take the context principle seriously mostly take it to claim some sort of epistemological priority of sentences (or perhaps the thoughts expressed by such) over subsentential linguistic items (or perhaps their senses). It is easy to see why one might have misgivings about such an interpretation; after all, it at least appears to conflict with another Fregean principle, namely, that of compositionality (according to which the sense/reference of a compound expression is determined by the senses/references of its constituent expressions), which he held in high regard throughout his life.

## FREGE'S PHILOSOPHY OF MATHEMATICS

Frege was, first and foremost, a philosopher of mathematics. While he followed Immanuel Kant in taking the truths of (Euclidean) geometry to be synthetic and knowable *a priori* (forcefully defending this view against Hilbert's axiomatic method in geometry), he vigorously argued, against Kant, for the logicist thesis, that is, the claim that the arithmetic truths, presumably including real and complex analysis, are analytic. In comparing Frege's views with Kant's it is however important to keep in mind that Frege was operating with his own technical definitions of analyticity and syntheticity, which are not obviously equivalent to Kant's: According to Frege (*Grundlagen* §3), a mathematical truth is analytic if it is derivable by means of logical inference rules from the general logical laws (and definitions) alone, whereas it is synthetic if it cannot be proved without recourse to truths belonging to a particular area of knowledge. Thus, analyticity and syntheticity are, for Frege, logico-epistemic notions, while Kant took them to be part semantic (analytic judgments are those whose predicate is contained in the subject, they are true by virtue of the meanings of their terms) and part epistemic (synthetic judgments extend one's knowledge, analytic ones do not).

In the preface to *Begriffsschrift* Frege makes it clear that it was the question of the epistemic status of arithmetic truths that prompted him to develop his new logic. At this time, Frege still avoids outright endorsement of the logicist thesis, stating only that he intends to investigate how far one may get in arithmetic with logical inferences alone. But there can be little doubt that he already envisages a definite path along which the ultimate proof of logicism is to proceed. Thus, he notes in part 3 of this work that mathematical induction rests on the *Begriffsschrift* theorem that, if an object $x$ bears the transitive closure $R^*$ of a binary relation $R$ to an object $y$, and if $x$ has a property $F$ that is inherited along $R$, then $y$ has $F$ as well. It therefore seems clear that Frege already understood the possibility of logically proving the mathematical induction principle once the number 0 and the successor relation among natural numbers had been suitably defined, for the natural numbers could then be given as just those objects following 0 in the transitive closure of the successor relation.

By the time of *Grundlagen* the doctrine of logicism is firmly in place. Having vigorously criticized a selection of philosophical views about the notion of number (notably John Stuart Mill's empiricist and Kant's transcendentalist views), Frege, in the second part of that work, provides an informal, yet rigorous outline of how the reduction of arithmetic to logic may actually be carried out. He begins this endeavor by insisting that (1) ascriptions of number involve assertions about concepts and (2) the numbers themselves must be construed as objects. Frege argues for (1) by noting first that certain statements, like universal categoricals such as "all whales are mammals" and existential statements such as "there are books on the shelf," predicate something of concepts (rather than individuals). The first example statement is clearly not about any individual whale, but says of the concept *whale* that it is subsumed under the concept *mammal*; the second example predicates nonemptiness of the concept *book on the shelf*. The point is even clearer with respect to negated existential statements; "there are no Venus moons" is obviously not about any moon of Venus (if the statement is true, there are none), but denies that something falls under the concept *Venus moon*. Indeed, Frege notes, saying that there are no Venus moons amounts to the same thing as ascribing the number zero to the concept *Venus moon*. And just as in these examples, the numerical statement "there are four books on the shelf" clearly does not predicate anything of any particular book; instead, it, too, is a statement about the concept *book on the shelf*.

The thesis that ascriptions of number are best understood, in analogy with these examples, as assertions about concepts, is further bolstered by the observation that everyday numerical statements invariably involve common nouns or predicates, which, according to Frege, refer to concepts. Moreover, faced with the fact that one may with equal justice say "there is one deck of cards on the table," "there are fifty-two cards on the table," and "there are four suits of cards on the table," one is led to the recognition that there are different standards of unit involved in these assertions, and it seems perfectly natural to identify the respective concepts as these standards of unit. Thesis (2) is a consequence of Frege's view that the ontological category of an entity may be read off reliably from the linguistic category of expression that denotes the entity: According to Frege number terms typically appear as singular terms in natural languages, for example, as "*the* number of cards on the table" or "*the* number four." Furthermore, pure arithmetic number terms typically flank the equality symbol, positions that, in Frege's view, are reserved for singular terms. Hence, Frege concludes, numbers must be objects.

Thus on the one hand, numbers, qua properties of concepts, would seem to be (higher-order) concepts; yet on the other hand, they must be construed as objects. Frege solves this apparent difficulty by suggesting that attributive uses of number words, as in "Jupiter has four moons," can always be paraphrased away, as in "the number of moons of Jupiter is four" (or, even more explicitly, "the number belonging to the concept *moon of Jupiter* is four"). In the latter statement, Frege claims, the *is* must denote identity and cannot function merely as a copula, since *four* is a singular term, and singular terms cannot follow the *is* of predication. The paradigmatic ascription of number then has the form "the number belonging to $F = x$," where $F$ represents a predicate and $x$ a singular term. Thus, the number term only forms part of the (higher-order) property ascribed to the concept, so that the objectual nature of number and the attributive character of ascriptions of number are compatible after all.

Frege next identifies a constraint that his reconstruction of arithmetic will have to abide by. Of fundamental importance for arithmetic are judgments of recognition, that is, identities, and so the definitions of the number-theoretic notions required for a proof of the logicist thesis must ensure that, in particular, identities of the form "the number belonging to $F$ = the number belonging to $G$" receive the proper truth conditions. For this special type of identity statement, the truth conditions can readily be formulated in (dyadic second-order) logical terms, namely, the number belonging to $F$ is the same as the number belonging to $G$ if and only if there exists a binary relation $R$ that correlates the objects that are $F$ one-one and onto with the objects that are $G$. Since Frege quotes a somewhat obscure passage from David Hume at this point in *Grundlagen*, the principle has, perhaps infelicitously, come to be known as Hume's principle (HP).

Frege rejects HP as a definition of "the number belonging to $F$" on the grounds that it fails to specify truth conditions for contexts of the form "the number belonging to $F = x$," where $x$ is a term that does not have the form "the number belonging to $G$," for example, when $x$ is an individual variable. (This objection is now usually referred to as the Caesar problem—somewhat inaccurately, as Frege uses Julius Caesar as an example in arguing against a slightly different proposal for a definition). Some commentators maintain that Frege's only point in bringing up this objection is to show how HP is inadequate as a definition of *number* as described earlier. Other commentators see Frege as struggling here to arrive at adequacy conditions for the introduction of new sortal concepts into a language. On such a reading, however, it is difficult to see why Frege was not troubled by the obvious analogous problem arising for extensions of concepts in the *Grundgesetze*.

In any case Frege proposes an explicit definition of "the number belonging to $F$" that in effect amounts to taking this number to be the equivalence class of $F$ under the equivalence relation of equinumerosity (which is explained in terms of the existence of a one-one and onto correlation): the number belonging to $F$, Frege stipulates, is the extension of the concept "concept equinumerous with $F$." Frege relies on a naive understanding of the notion of extension (later, in *Grundgesetze*, extensions themselves would be governed by an axiom that was to prove fatal for Frege's project). Frege then defines an object $a$ to be a (cardinal) number if there exists a concept $F$ such that $a$ is the number belonging to $F$.

From the explicit definition of the number belonging to a concept, Frege proceeds to show that HP becomes derivable by means of pure logic and defines 0 as the number belonging to the concept "is an object not identical with itself" and 1 as the number belonging to the concept "is an object identical with 0." The successor relation among cardinal numbers is defined as follows: $n$ succeeds $m$ if $n$ is the number belonging to some concept $F$ under which some object $a$ falls, and $m$ is the number belonging to the concept "is an object falling under $F$, but not identical to $a$." Without proof Frege mentions the theorems that every number has at most one successor and one predecessor, and that every number except 0 succeeds some number. Making use of his definition of the ancestral (transitive closure) of a binary relation (as developed in *Begriffsschrift*), he defines the finite or natural numbers as those objects standing to 0 in the transitive reflexive closure of the successor relation, that is, informally, as those numbers than can be reached from 0 by taking successors finitely many times. Frege observes that this definition allows for a rather straightforward proof of the mathematical induction principle for natural numbers.

At this point, he has effectively recovered all the axioms of (second-order) Peano arithmetic from his definitions, except the one requiring every natural number to have a successor. Frege sketches a proof for this remaining axiom, which ultimately consists in showing by means of induction that, for any natural number $n$, the number belonging to the concept "object to which $n$ bears the transitive reflexive closure of the successor relation" (i.e., informally, "natural number being less than or equal to *n*") succeeds *n* (a fully detailed proof is carried out in *Grundgesetze*, although it is not entirely clear whether this is the same proof Frege intended in *Grundlagen*).

While the exposition of *Grundlagen* is entirely informal, *Grundgesetze*, which Frege hoped to be the final word on the logical nature of arithmetic, carries out the earlier sketch with full rigor, containing pages and pages of formal deductions in begriffsschrift notation. The crucial element added in *Grundgesetze* is the rigorous treatment of extensions of concepts (more precisely, of courses-of-values of functions, of which concept extensions are a special case). These are governed by Frege's basic law V, whose special case for concepts says that the extensions of concepts F and G coincide if and only if the same objects fall under F as fall under G. The use of extensions allows for the technique of type-lowering: First-level concepts can be simulated by their extensions, second-level concepts H can be simulated by the first-level concepts under which fall precisely the extensions of concepts falling under H, and so on. Frege makes extensive use of this technique; in particular, instead of defining the number belonging to F as the extension of the second-level concept "concept equinumerous with F," he is now able to take numbers to be extensions of first-level concepts. Otherwise, he follows the sketch of *Grundlagen* closely.

As Russell pointed out in a letter to Frege in 1902, the theory expounded in *Grundgesetze* is inconsistent, since it allows for the derivation of Russell's antinomy: Letting R be the first-level concept "*x* is the extension of some concept under which *x* does not fall," and *r* its extension, it follows easily from Frege's rules of inference, together with basic law V, that *r* both does and does not fall under R. Frege immediately realized that the antinomy threatened to undermine his life's work. While the second volume of *Grundgesetze* was in press, he hastily devised a quick fix that has come to be known as Frege's way out and added an appendix to the book, expressing both confidence that the revised system would prove capable of reconstructing arithmetic and worries about the philosophical underpinning of his revised basic law V. Frege's way out proved not to be a way out, since it was inconsistent with the existence of more than one object. The genesis of the antinomy in Frege's system is by now well understood; it arises through interplay of two principles that are individually consistent, namely, basic law V as mentioned earlier and *impredicative* second-order comprehension (roughly, statements to the effect that there exists a concept with a certain property, where that property is itself specified with the help of quantification over concepts); Frege's system with basic law V but only predicative instances of comprehension is now known to be consistent, but too weak to allow for a reconstruction of substantial mathematics.

Frege's work on the logical foundation of real analysis remained fragmentary; the second volume of *Grundgesetze* contains only preliminary definitions and theorems. Presumably he had planned a third volume, which, however, never appeared. Toward the end of his life, Frege seems to have abandoned logicism altogether, suggesting that arithmetic was instead based entirely on geometry, and hence synthetic, as Kant had held. His ideas on how such a claim might be proved were, however, never worked out.

## NEO-FREGEANISM

Frege himself, and generations of philosophers and logicians after him, considered the mathematical content of *Grundlagen* and *Grundgesetze* largely obsolete because of the inconsistency of Frege's theory of extensions of concepts. In the 1980s, however, it began to be recognized that Frege had indeed hit on an exciting fact: If one takes the framework of Frege's theory to be essentially second-order predicate logic and adopts HP (with a primitive operator "the number belonging to," attaching to concept expressions) as an axiom, all of second-order Peano arithmetic becomes derivable, using the exact definitions and proofs employed by Frege (who used the explicit definition of "the number of F" only to prove HP from it, obtaining all further results directly from HP). This fact has become known as Frege's theorem. Importantly, it was soon observed that Frege arithmetic (i.e., full axiomatic second-order logic plus HP) is consistent, in contradistinction to the system of *Grundgesetze* (indeed, consistent relative to second-order Peano arithmetic).

It is still being debated whether, and to what extent, these discoveries have any bearing on the validity of the logicist thesis (restricted to arithmetic proper). While no one has seriously suggested that HP could be regarded as a principle of logic, some argue that it nevertheless enjoys some privileged epistemological status akin to analyticity, the principle being, in some sense, "analytic *of*" number. There are, however, serious difficulties in defending Frege arithmetic as being analytic. To start with, there is the familiar problem about the status of second-order logic itself, quite independently of HP. But even granting that second-order logic may count as logic in the requisite sense, further objections apply to HP. First, the principle is not ontologically innocent, since it requires the first-order domain to be infinite, which is usually taken to be incompatible with analyticity. Second, any attempt to ground a privileged logical status of HP on its logical form (of an abstraction principle) runs afoul of the "bad company objection": There are abstraction principles of the same general logical form as HP that are inconsistent (such as Frege's basic law V). What is more, there are abstraction principles (like Boolos's parity principle) that hold only in finite domains, which makes them incompatible with HP, and hence it cannot be the logical form of an abstraction principle alone that could make HP analytic. Research on abstraction principles has increased significantly as a consequence of this discussion, as has work on the general logical and mathematical features of Frege's systems.

## FREGE'S INFLUENCE

Through his publications, as well as through personal correspondence, Frege exerted a profound influence on Russell, who appears to have been the first major thinker to appreciate Frege's achievements in logic. Russell took over the logicist torch from Frege, and although Alfred North Whitehead and Russell's *Principia Mathematica* differs in many ways from Frege's work (it is much wider in scope, considerably less rigorous, and, in view of Russell's antinomy, takes a different approach to classes), it is clearly also heavily influenced by Frege (e.g., in imposing a structure of levels, or types, on the underlying ontology, and in the definition of number, nowadays often referred to as the Frege-Russell definition of cardinal number). It is known that Russell had read "Über Sinn und Bedeutung" and at least parts of *Grundgesetze* when he developed his celebrated theory of descriptions; and while there is no direct evidence for such a claim, it seems plausible to assume that Frege's discussion of definite descriptions in these works (especially the fully worked out formal theory of *Grundgesetze*) provided a helpful foil for Russell's own theory.

The degree to which Frege influenced Edmund Husserl is a more contentious matter. It is known that Husserl read all of Frege's major works and that the two corresponded extensively (except in the aftermath of Frege's rather hostile review [1894] of Husserl's *Philosophie der Arithmetik* [1891]). It seems fair to say that Frege (in particular, through the aforementioned review, as well as the preface to the first volume of *Grundgesetze*) is at least partly responsible for Husserl's antipsychologistic turn.

While Frege met neither Russell nor Husserl in person, he did have personal interactions with both Rudolf Carnap and Ludwig Josef Johann Wittgenstein. As a student, Carnap enrolled in various classes on begriffsschrift taught by Frege in Jena between 1910 and 1914; surely it was Frege who instilled in Carnap the idea that mathematics was reducible to logic, a view that was to become central to the Vienna Circle's philosophy. More generally, Frege shaped Carnap's whole attitude toward philosophy. After his immigration to the United States, Carnap, with Alonzo Church, was instrumental in keeping Fregean ideas in logic alive in the United States (where they came to flourish, for instance, in the work in semantics of David Kaplan and Richard Montague). Wittgenstein first visited Frege in Jena in 1911, and then at least two more times, in 1912 and 1913, while he was Russell's student in Cambridge. In addition, the two corresponded rather extensively from 1911 to 1920; it is clear from this correspondence that Frege and Wittgenstein thought highly of each other (the end of the correspondence is marked by an exchange of rather critical remarks by Frege on the *Tractatus* and by Wittgenstein on "Der Gedanke"). Fregean themes pervade the work of both the early and the late Wittgenstein, and it appears that Wittgenstein's intellectual respect for Frege never subsided.

In spite of this illustrious group of correspondents, Frege was for many years regarded as a somewhat obscure and ultimately failed predecessor of Russell's, possibly because few philosophers fully acknowledged Frege's influence on them (of course, the extent of this influence may not have been clear to them at the time). In the 1930s Heinrich Scholz and his school in Münster, Germany, rediscovered Frege and began work on an edition of his works, but that never materialized. The situation changed somewhat in the wake of John Langshaw Austin's English translation of the *Grundlagen*, which appeared in 1950; Frege was read, at that time, mainly as a philosopher of language, and as such influenced, among others, the British philosopher Peter Geach. The originality and independence of Frege's work (especially from Russell's), as well as his important role as a progenitor of analytic philosophy, was brought to prominence through the writings of Michael Dummett in the 1970s, who was himself heavily influenced by Frege's methodology and interests. In the United States, besides those mentioned earlier, Donald Davidson's work also revived discussion of Fregean themes. Crispin Wright's neologicism, especially as subsequently articulated and criticized by George S. Boolos and others, caused a veritable renaissance of interest in Frege's logical and mathematical work, beginning in the 1980s and continuing to this day.

***See also*** Analytic and Synthetic Statements; Analyticity; Aristotle; Austin, John Langshaw; Carnap, Rudolf; Categories; Church, Alonzo; Davidson, Donald; Dummett, Michael Anthony Eardley; Geometry; Hilbert, David; Hume, David; Husserl, Edmund; Identity; Kant, Immanuel; Kaplan, David; Logic, History of; Logical Positivism; Mathematics, Foundations of; Mill, John Stuart; Montague, Richard; Peano, Giuseppe; Propositions; Russell, Bertrand Arthur William; Scholz, Heinrich; Whitehead, Alfred North; Wittgenstein, Ludwig Josef Johann.

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***Kai F. Wehmeier (2005)***

## FRENCH ENCYCLOPEDIA, THE

See *Encyclopédie*

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See *Clandestine Philosophical Literature in France; Encyclopédie*

## FREUD, SIGMUND

***(1856–1939)***

Sigmund Freud was the father of psychoanalysis, but—contrary to much apocryphal lore that dies hard—certainly not the originator of the hypothesis that unconscious ideation is essential to explain much of human overt behavior.

The generic doctrine of an unconscious domain of the mind has a venerable, long pre-Freudian history. Indeed, many of the most important doctrines commonly credited to Freud as his creations were tenets of his intellectual patrimony. Thus, as we recall from Plato's dialogue *The Meno*, Plato was concerned to understand how an ignorant slave boy could have arrived at geometric truths under mere questioning by an interlocutor with reference to a diagram. Plato argued that the slave boy had not acquired such geometric knowledge during his life. Instead, he explained, the boy was tapping prenatal but unconsciously stored knowledge, and restoring it to his conscious memory.

At the turn of the eighteenth century, Gottfried W. Leibniz gave psychological arguments for the occurrence of subthreshold sensory perceptions and for the existence of unconscious mental contents or motives that manifest themselves in our behavior (Ellenberger 1970). Moreover, in his *New Essays on Human Understanding* (1981), Leibniz pointed out that when the contents of some forgotten experiences subsequently emerge in our consciousness, we may misidentify them as new experiences, rather than recognize them as having been unconsciously stored in our memory.

Historically, it is more significant that Freud also had other precursors who anticipated some of his key ideas with impressive specificity. As he himself acknowledged ([the abbreviation "S.E." will be used to refer to the Standard Edition of Freud's complete psychological works in English] S.E., 1914, 14:15–16), Arthur Schopenhauer and Friedrich Nietzsche had speculatively propounded major psychoanalytic doctrines that he himself reportedly developed independently from his clinical observations only thereafter. Indeed, in a 1995 German book, *Die Flucht ins Vergessen: Die Anfänge der Psychoanalyse Freuds bei Schopenhaeur*, the Swiss psychologist Marcel Zentner traces the foundations of psychoanalysis to the philosophy of Schopenhauer.

But, as Freud then pointed out illuminatingly, it is one of the greatest threats to human self-esteem to face that "*the [human conscious] ego is not master in its own house*" (S.E., l917, 17:143; emphasis in original). On the other hand, it is evasive to dismiss substantive criticisms of Freudian theory as being due to fears induced by psychoanalytic accounts of presumed unconscious motivations. Such a dismissal does not address the merits of the strictures directed against psychoanalysis.

Freud was born in Freiberg, Moravia, then part of the Austro-Hungarian Empire, in 1856. But when he was three years old, his family moved to Vienna, where he entered the University of Vienna in 1873 to study medicine. He lived there until he was expelled by the Nazis, when he moved to London, where he died in l939.

It is important to distinguish between the validity of Freud's work qua psychoanalytic theoretician, and the merits of his earlier work. The zealous Freudian partisan