MATERIALISM, DIALECTICAL

See *Dialectical Materialism*

MATERIALISM, HISTORICAL

See *Historical Materialism*

MATHEMATICS, FOUNDATIONS OF

The study of the foundations of mathematics comprises investigations, though probably not all possible investigations, that consist of general reflection on mathematics. The subject naturally proceeds by singling out certain concepts and principles as "fundamental" and concentrating attention on them, but of course the identification of fundamental concepts and principles is itself based on foundational research or may be revised in the light of it.

In this entry considerable emphasis will be placed on philosophical questions about mathematics, which undoubtedly belong to foundations. However, many, perhaps most, foundational investigations are mainly mathematical. In the last hundred years an important role has been played by mathematical logic. We shall not give a detailed exposition of mathematical logic, but we hope that our discussion will give an idea of the relation between the logical problems and results and the philosophical problems and an idea of some of the results of recent work in logic.

Two of the main qualities for which mathematics has always attracted the attention of philosophers are the great degree of systematization and the rigorous development of mathematical theories. The problem of systematization seems to be the initial problem in the foundations of mathematics, both because it has been a powerful force in the history of mathematics itself and because it sets the form of further investigations by picking out the fundamental concepts and principles. Also, the systematic integration of mathematics is an important basis of another philosophically prominent feature, its high degree of clarity and certainty. In mathematics systematization has taken a characteristic and highly developed form—the axiomatic method—which has from time to time been taken as a model for systematization in general. We shall therefore begin our main exposition with a discussion of the axiomatic method.

Foundational research has always been concerned with the problem of justifying mathematical statements and principles, with understanding why certain evident propositions are evident, with providing the justification of accepted principles that seem not quite evident, and with finding and casting off principles which are unjustified. A natural next step in our exposition, then, will be to consider mathematics from an epistemological point of view, which leads us to examine mathematics as a primary instance of what philosophers have called a priori knowledge. In this connection we shall give some logical analysis of two very basic mathematical ideas, class and natural number, and discuss the attempts of Gottlob Frege and Bertrand Russell to exploit the intimate relation between these two ideas in order to prove that mathematics is in some way a part of logic. We shall also discuss Immanuel Kant's views on the evidence of mathematics and other conceptions of a priori knowledge. (The word *evidence* will often be used in this entry in a way that is unusual outside philosophical writings influenced by the German tradition, to mean "the property of being evident"—German, *Evidenz*.)

The growth of modern mathematics, with its abstract character and its dependence on set theory, has caused the problem of evidence to be focused on the more particular problem of platonism. It is in this development and the accompanying growth of mathematical logic that modern foundational research has centered.

Throughout the nineteenth century, mathematicians worked to make arithmetic and analysis more rigorous, which required axiomatization and an attempt to use the concepts of the theory of natural numbers as a basis for defining the further concepts of arithmetic and analysis. The manner in which this axiomatization and definition was undertaken was platonist, in the sense that both numbers and sets or sequences of numbers were treated as existing in themselves. The development of set theory by Georg Cantor provided a general framework for this work and also involved even greater abstraction and even stronger platonist assumptions.

The growth of mathematical logic introduced as further elements the axiomatization of logic (the basic step in which was completed by Frege in 1879), the effort to incorporate the axiomatization of logic into that of mathematics, and the accompanying tendency, on the part of Frege and Giuseppe Peano, to interpret rigorous axiomatization as formalization. Frege carried the development much further by undertaking to develop the whole of arithmetic and analysis in a formal system that is essentially a system of set theory.

At the turn of the twentieth century the entire development reached a crisis with the discovery of the paradoxes of set theory, which showed that the concept of class or set as it was then being used had not been sufficiently clarified. Much of the foundational research of the early twentieth century—and not only in the axiomatization of set theory—was directed at problems posed or believed to have been posed by the paradoxes.

In that period emerged three general viewpoints, each of which had its own program based on a distinctive attitude toward the question of platonism. The most radical was intuitionism, based on L. E. J. Brouwer's critique of the whole idea of platonism. In contrast to Brouwer, David Hilbert had a firm commitment to the patronizing tendency in mathematics, but he held epistemological views that were fundamentally in accord with Brouwer's critique of platonism. Making use of the fact that no matter how platonist the mathematics formalized, questions of provability in a formal system are meaningful from a narrow constructivist point of view, Hilbert's school sought to secure the foundations of platonist mathematics by metamathematical investigation of formalized mathematics—in particular, by a proof of consistency. This viewpoint was called formalism, although the designation is misleading, since Hilbert never maintained that even platonist mathematics could be simply defined as a "meaningless" formal system.

Proponents of the third viewpoint, logicism, whose leading figure was Russell, continued to believe in Frege's program of reducing mathematics to logic. Accepting this program involved taking some platonist assumptions as intuitively evident.

A great deal of work in mathematical logic was directed toward clarifying and justifying one or another of these points of view. We might mention Brouwer's (informal) results on the impossibility of constructively proving certain theorems in analysis, Arend Heyting's formalization of intuitionist logic, the development of finitist proof theory by Hilbert and his coworkers, and Russell and A. N. Whitehead's *Principia Mathematica* as a much further development of mathematics within a system of set theory.

Nonetheless, the trichotomy of logicism, formalism, and intuitionism has probably never been the best classification of points of view in foundations. It does not take account of one of the philosophically most important problems, that of predicativity, or of some mathematical developments—such as the development of the semantics of logic by Leopold Löwenheim, Thoralf Skolem, Kurt Gödel, and Alfred Tarski—which were crucially important for later work. At any rate the schools no longer really exist. All of them had programs that encountered serious difficulties; further experience with set theory and the axiomatizations of Ernst Zermelo and Russell deprived the paradoxes of their apparently apocalyptic character; and specialized work in mathematical logic led more and more to the consideration of problems whose significance cut across the division of the schools and to looking at the results of the schools in ways which would be independent of the basic controversies. A decisive step in this development came in the early 1930s, with the discovery of Gödel's incompleteness theorem and the coming of age of formal semantics.

Some areas of the foundations of mathematics will be passed over here—in particular, we shall not go far into the significance of the fact that mathematics has applications to the concrete world, although historically the relation between mathematics and its applications has been very close, and the present sharp distinction between pure and applied mathematics is a rather recent development. For instance, we shall omit a special consideration of geometry. If the pre-twentieth-century view that geometry is a purely mathematical theory that nonetheless deals with actual space is correct, then the omission is unjustified. However, even the question whether this view still has something to be said for it is more intimately related to the philosophy of physics than to the problems on which we shall concentrate. Geometry as understood today by the pure mathematician, as the general study of structures analogous to Euclidean space, raises no philosophical problems different from those raised by analysis and set theory.

§1. THE AXIOMATIC METHOD

As we said, we shall begin our discussion with the axiomatic method. Consideration of the notion of an informal axiomatic system leads to the notions of formalization and formal system. Through this process, especially through the last step, mathematical theories become themselves objects of mathematical study. The exploitation of this possibility is perhaps the specifically modern move in the study of the foundations of mathematics and has led to an enormous enrichment of the subject in the last hundred years.

1.1. AXIOMATIZATION. Ever since Euclid, axiomatizing a theory has meant presenting it by singling out certain propositions and deducing further ones from them; if the presentation is complete, it should be the case that all statements which could be asserted in the theory are thus deducible. Axiomatization has also come to mean a similar reduction of vocabulary, in that certain notions should be taken as primitive and all further notions which are introduced in the development of the theory should be defined in terms of the primitive ones. In essence this is the conception of an axiomatized theory that prevails today, although it has been developed in different directions.

There are important ambiguities concerning the means of deduction and definition to be admitted in the development of the theory. Here informal axiomatics always makes use of some general background that can be used in developing the theory but is not itself included in the axiomatization. In modern mathematics this background typically includes logic and arithmetic and usually also analysis and some set theory. For example, in an axiomatic theory concerning objects of a certain kind, one permits oneself very quickly to make statements about sequences and sets of those objects, to introduce concepts defined in terms of the primitives of the theory by means of these general mathematical devices, and to make inferences that turn on laws of arithmetic, analysis, or set theory. Such notions often enter into the statement of the axioms themselves. We shall presently say more about the significance of this procedure.

It might seem natural to require provisionally that the means of deduction and definition be restricted to those of pure logic, for logic is supposed to contain those rules of correct inference which have the highest degree of generality and which must be applied in all sciences. We would then regard an axiomatization as only partial if deductions from it required the use of methods of the special sciences—in particular, branches of mathematics (likewise if, in addition to the primitives, notions other than purely logical ones entered into the definitions). An axiomatic theory would then consist of just those statements that are deducible by purely logical means from a certain limited set of statements and of the statements that can be obtained from these by definitions expressible purely logically in terms of the primitives.

It seems possible that such an axiomatic system was the objective toward which Euclid was striving. He evidently did not intend to allow himself general mathematical notions, such as arithmetical ones, for he included propositions involving such notions among his axioms and undertook to develop some of number theory from the axioms in Books VII–IX. Even some of Euclid's well-known failures to achieve this degree of rigor—for example, his assuming in his very first proof that two circles with the center of each lying on the circumference of the other will have two points of intersection—might have arisen because he saw them as immediate deductions from the meaning of the concepts involved. Of course, a rigorous theory of definition would require definitions to be given or axioms to be explicitly stated in such a way that such deductions do proceed by mere logic.

A perfectly satisfactory axiomatization in this form certainly was not possible in Euclid's time; it probably had to wait for two developments that did not take place until the late nineteenth century, Frege's discovery and axiomatization of quantification theory and the Dedekind-Peano axiomatization of arithmetic. (Nonetheless, considerable progress was made prior to these developments.)

This remark points to a limitation of the conception we are considering, for it does not give a meaning to the idea of an axiomatization of logic itself, although such axiomatization has played a vital role in modern foundational studies. Appreciation of this point leads to the concept of a formal system, but before we consider this concept let us observe a consequence of the axiomatization of a theory.

1.2. THE ABSTRACT VIEWPOINT. Suppose a theory is so completely axiomatized that all concepts of the special theory which are used in statements and deductions are explicitly given as primitives and all special assumptions underlying the proofs are disengaged and either stated among or deduced from the axioms. This means that the validity of the deductions does not at all depend on the actual meaning of the primitive terms of the special theory. It follows that the formal structure determined by the primitive concepts and the axioms can have a more general application than they have in the given special theory, in the sense that we could by any choice of interpretation of the primitive terms obtain a deductive system of hypotheses concerning some subject matter, even though the hypotheses will in many cases be false.

This fact is of crucial importance in the study of axiom systems. We can then think of a *model* of an axiomatic theory as a system of objects and relations that provides references for the primitive terms so that the axioms come out true. We can think of axiomatization as having proceeded with a particular model in mind, but this need not have been the case; at any rate, interest attaches to the study of other possible models. (Although we may, in this discussion, allow means of deduction that go beyond pure logic, it ought to be the case that if a proposition is deducible from the axioms of the theory, then it must be true in all models of the theory. It might be reasonable to take this as a sufficient condition of deducibility, but if so it seems that the notion of model will have to have a relativity comparable to that of the notion of deducibility.)

For example, suppose we consider absolute geometry—that is, Euclidean geometry without the parallel postulate. Then any model either of Euclidean geometry or of the standard non-Euclidean geometries will be a model of absolute geometry. If the parallel postulate is deducible from the other axioms of Euclidean geometry—that is, from the axioms of absolute geometry—then it must be true in every model of absolute geometry. The construction of models for non-Euclidean geometries showed that this is not the case. We call an axiom of a system independent if it is not deducible from the others. Thus, if the theory obtained by dropping an axiom $\mathscr{A}$ has a model in which $\mathscr{A}$ is false, then $\mathscr{A}$ is independent.

Another possibility, which has been much exploited in modern mathematics, is to replace a system of primitive terms and axioms by what amounts to an explicit definition of a model of the axioms. Thus, suppose Euclidean geometry is formulated with two primitive predicates (following Alfred Tarski in "What Is Elementary Geometry?," 1959):

$$\text{“}\beta(x,y,z)\text{”},$$

meaning "x, y, and z are collinear, and y lies between x and z or $y = x$ or $y = z$," and

$$\text{“}\delta(x,y,z,w)\text{”},$$

meaning "x is the same distance from y as z is from w." (The variables here range over points, which in the informal theory must be thought of as a primitive notion.) Then we can define a *Euclidean space* as a triple $\langle S,B,D\rangle$, where S is a set of entities called "points," B a ternary relation on S, and D a quaternary relation on S, such that the axioms of Euclidean geometry hold. Then to any theorem proved from these axioms corresponds a statement of the form "Every Euclidean space is such that … ." A number of attempts to characterize mathematical structures axiomatically have led in a similar way to explicit definitions of abstract types of structure. This is regarded, for more than historical reasons, as a fruit of the axiomatic method. The search for an axiomatic basis for a mathematical theory is also the search for a formulation of the arguments in a fashion which will make them more generally applicable, giving them a generality which can be expressed in the definition of a general type of structure.

1.3. FORMALIZATION. Whereas one development of the axiomatic method tends to the replacement of axioms by definitions, another leads to the conception of a formal system. One result of the axiomatization of a theory was that the meaning of the primitive terms became irrelevant to the deductions. If we carry this abstraction from meaning to its limit, we can cover the case of axiomatizations of logic and resolve once and for all the question of what means of deduction are to be allowed. That is, we put into the construction of an axiom system a complete specification of all the means of inference to be allowed (for example, logic and basic mathematics) in the form both of further axioms and of rules of inference that allow us to infer from statements of certain given forms a statement of another given form. If this is done with utmost rigor, so that use can be made of only as much of the meaning of the terms as is specified in axioms and explicit definitions, then the system is specified simply in terms of the designs of the "linguistic" forms in which it is expressed. "Linguistic" is put in quotation marks because, invariably, much of the language has been replaced by an artificial syntax. We are left with a specification of certain strings of symbols as "axioms" and certain rules, each of which allows us to "infer" a new string from certain prior ones. The strings which we can obtain from axioms by successive application of the rules can be called *theorems*.

A proper explanation of the concept of a formal system requires somewhat more apparatus. The exactness of this procedure requires that the strings of symbols used be constructed out of preassigned material, which we can assume to be a finite list of symbols. Among the strings of these symbols we single out a subclass that we call formulae (or well-formed formulae, wffs), which are those strings to which, in an interpretation, we would give a meaning. (The non-wffs correspond to ungrammatical sentences.) Then a certain class of formulae is singled out as the axioms. The class of theorems can be defined as the closure of the axioms under certain operations; that is, rules of the following form are specified:

(R_i). If $\mathscr{A}_1, \cdots, \mathscr{A}_{r_i}$ are theorems and $\mathscr{R}_i(\mathscr{A}_i, \cdots \mathscr{A}_{r_i})$, then $\mathscr{B}$ is a theorem, where $\mathscr{R}_i$ is some relation on strings of the symbols of the system.

So the definition of *theorem* is an inductive definition with the clauses (R_i) and

every axiom is a theorem.

In this setting we can resolve another ambiguity of our original rough conception of axiomatization. The question arises concerning what conditions a class of statements must satisfy to be appropriate as the axioms of an axiomatic theory. Various epistemological desiderata, such as self-evident truth for the intended model, are put aside once we take the abstract point of view. Another requirement that has been found natural in the past is that both individual axioms and the class of axioms as a whole should have a certain simplicity. What there is in the way of general theory about the simplicity of individual axioms has not played much of a role in investigations of the foundations of mathematics, although much effort has been expended in replacing individual axioms with simpler ones or in finding systems of axioms which have particular advantages of "naturalness" for intended applications.

In order to characterize the important axiom systems which have been used in the past we shall have to place some limitation on the class of axioms. In the traditional cases the class has been finite. However, the formalization of such an axiomatic system can give rise to an infinite system—for example, if we take as axioms all instances of a certain schema.

The limitation which is used instead of a finite class of axioms is based on the fact that the notions of formula, axiom, and theorem are to be syntactically specified. Then the requirement is that there be a mechanical, or effective, procedure for deciding whether a given formula is an axiom and whether a given inference (of a formula from finitely many premises) is correct according to the rules of inference. This requirement is natural in the light of the idea that a proof of a statement in an axiomatic theory should contain all the mathematically significant information needed to show that the statement is indeed assertible in the theory. That would not be the case, it is argued, if something beyond mechanical checking were needed to determine the correctness of the proof. (It should be pointed out, however, that generalizations of the concept of formal system in which this condition is not satisfied are frequently used in mathematical logic.)

The notion of a formal system gives the highest degree of generality, in that there is no element of the symbolism whose interpretation is restricted. Indeed, it permits much of what we might want to say about an axiomatic theory to be formulated without reference to interpretation, since the formulae, axioms, and rules of inference are specified without reference to interpretation, and what is a theorem is then defined, again without such reference. An entire division of the theory of formal systems—what is usually called syntax—can thus be built up with no more than a heuristic use of interpretation. In particular, the intensional notions—concept, proposition, etc.—relied on so far in the informal exposition can be eliminated.

The concept of a formal system also brings to the formulation of the theory the highest degree of precision, at the cost of a still further idealization in relation to the concrete activities of mathematicians. Furthermore, the concept not only gives a refined formulation to axiomatizations and allows a mathematical study of axiom systems of a more general scope than was possible without it but also makes possible a precise formulation of differences about mathematical methods. Carrying the axiomatic method to this limit makes possible a new approach to a wide variety of questions about the foundations of mathematics.

Inasmuch as axiomatization is a rendering of a theory in a more precise formulation (if not a singling out of some particular aspect of the theory), the axiomatized theory cannot be identified in every respect with what has gone before. It can replace, however, what has gone before and actually has done so in many cases. The passage from axiomatization to formalization is in an important respect more radical than the various stages of informal axiomatization, and we can therefore regard a formalization of a theory as not so much a more precise formulation of the theory as an idealized representation of it. The process of replacing expressions of natural language by artificial symbols, which goes on in all mathematical development, is here carried to an extreme. For example, we lay down by a definition what are "formulae" and "proofs" in the system, whereas informally we rely for the notion of sentences on our more or less unanalyzed linguistic sense, and for proofs we rely on this sense, on mathematical tradition, and on intuitive logic. In particular, formulae and formal proofs are of unbounded length and complexity, without regard to the limits of what we can perceive and understand.

With this goes the fact that the basic general notions with which we operate in formulating and reflecting on theories—sentence, proposition, deduction, axiom, inference, proof, definition—are replaced in the formalized version by specifically defined, more or less simplified and idealized substitutes. In particular, although we "interpret" formalized theories, the relation between a sign or a formal system and its reference in some model is a "dead" correspondence, an aspect of a purely mathematical relation between two systems of objects. This enables one to avoid the intractable problems of how linguistic expressions come to have "meaning" and, with it, reference and is therefore an extremely valuable piece of abstraction. But it is an abstraction; moreover, it does not mean that the informal linguistic and intellectual apparatus disappears altogether, since it will still be used in the setting up and investigation of the formalized theory. In fact, one of the results of formalization is a sharper separation between what is within the theory and what belongs to discourse about it—that is, to the metatheory. If the metatheory is in turn axiomatized and then formalized, the same situation arises at the next-higher level.

The importance of this observation is difficult to assess, but it is relevant to a number of problems we shall discuss later—in particular, attempts to argue from results of mathematical logic to philosophical conclusions.

§2. EPISTEMOLOGICAL DISCUSSION

2.1. A PRIORI KNOWLEDGE. We shall now put the matter of axiomatization and formalization aside and consider mathematics from the point of view of general epistemology. The guiding thread of our discussion will be the fact that a powerful tradition in philosophy has regarded mathematics, or at least a part of it, as a central case of a priori knowledge. This means that reflection on mathematics has been at the center of philosophical discussion of the concept of a priori knowledge.

The characteristics of mathematics which have led to the conclusion that mathematics is a priori are its abstract character and accompanying enormous generality and its great exactitude and certainty, which, indeed, have traditionally been considered absolute. Thus, even before setting forth a developed logical analysis of the concept of number, we find that the effort to interpret "2 + 2 = 4" as a hypothesis that can be checked by observation runs into obvious obstacles. It is perhaps not so vital that the statement refers to abstract entities, numbers, which are not the sort of thing we observe. The concept of number certainly does apply to empirically given objects, in the sense that they can be counted and that the numbers thus attributed to them will obey such laws as "2 + 2 = 4." Therefore, the proposition could so far be taken as a law concerning such entities. Even then its range of application is so enormous, extending over the entire physical universe, that it seems evident that if it were taken as a hypothesis, it would be stated and used in a more qualified way, at least by critically minded scientists. In other words, the certainty that we attribute to elementary arithmetical propositions would be quite unwarranted if they were laws based on observation. Even in the case of mathematical principles to which we do not attribute this degree of certainty, such as the axiom of choice and the continuum hypothesis, the possible "contrary evidence" would arise from the deductive development of the theory involved (in the examples, set theory), not from observation.

Moreover, it seems that we ought to be able to conceive of a possible observation which would be a counterinstance. Although it is perhaps not evident that this is impossible, the ideas that come to mind lead either to descriptions of doubtful intelligibility or to the description of situations where it seems obviously more reasonable to assume some other anomaly (such as miscounting or the perhaps mysterious appearance or disappearance of an object) than to admit an exception to "2 + 2 = 4."

Another difficulty is that the concept of number must apply beyond the range of the concrete entities which are accessible to observation; such abstract entities as mathematical objects must be subject to counting, and this seems also to be the case for transcendent entities.

The foregoing considerations could be developed into decisive arguments only with the help of both a more developed formal analysis of number and a more detailed discussion of the relation between arithmetical laws and actual counting and perhaps also of the role of mathematics in empirical science. In any case, they do not tell against another form of the denial that arithmetic is a priori, the view that arithmetical laws are theoretical principles of a very fundamental sort, which we are therefore far more "reluctant to give up" in a particular situation than more everyday beliefs or impressions or even than fundamental theoretical principles in science. Such a view would nonetheless take it to be conceivable that in response to some difficulty in, say, particle physics a new theory might be formulated which modified some part of elementary arithmetic.

2.2. MATHEMATICS AND LOGIC. The above considerations show why it is necessary to add technical analysis to the epistemological discussion. We shall take as our guiding thread the attempt to show that mathematics—in particular, arithmetic—is a part of logic. This attempt has led to some of the most important results in the logical analysis of mathematical notions. The view that mathematics can be reduced to logic is one of the principal general views on the foundations of mathematics which we mentioned earlier; it goes generally by the name of logicism, and its classic expression is in the writings of Frege and Russell.

Even if successful, the reduction of mathematics to logic could not by itself give an account of how there can be a priori knowledge in mathematics, for it would only reduce the problem of giving such an account to the corresponding problem with regard to logic. Nonetheless, the a priori character of mathematics has traditionally been found perhaps slightly less certain than that of logic. The obvious fact that one of the primary tasks of mathematics is the deductive development of theories has been found to be one of the most powerful supports of the claim that mathematics is a priori. We can expect that a successful reduction of mathematics to logic will simplify the problem of a priori knowledge, and not only by replacing two problems by one. Logic is more unavoidable: We cannot get anywhere in thinking without using logical words and inferring according to logical rules. This would suggest that logic is in fact more basic than mathematics and more certainly a priori. (It would also suggest that philosophical treatments of logic are more liable to circularity.) Moreover, in the course of history philosophers have invoked sources of evidence for mathematics which are at least apparently special, such as Kant's pure intuition. Thus, a reduction of mathematics to logic might make superfluous certain difficult epistemological theories.

The claims of logicism are based in large part on mathematical work in axiomatics. A number of nineteenth-century investigations showed that the basic notions of analysis—for example, rational, real, and complex number—could be defined, and the basic theorems proved, in terms of the theory of natural numbers and such more general notions as class and function. At the same time, axiomatic work was done in the arithmetic of natural numbers, culminating in the axiomatization of Richard Dedekind (1888) and Peano (1889). The movement toward formalization began somewhat later, with the work of Frege and of the school of Peano.

Thus, the effort to reduce mathematics to logic arose in the context of an increasing systematization and rigor of all pure mathematics, from which emerged the goal of setting up a comprehensive formal system which would represent all of known mathematics with the exception of geometry, insofar as it is a theory of physical space. (But of the writers of that generation only Frege had a strict conception of a formal system.) The goal of logicism would then be a comprehensive formal system with a natural interpretation such that the primitives would be logical concepts and the axioms logical truths.

We shall be guided by Frege's presentation, although he did not go very far in developing mathematics within his system and of course the system turned out to be inconsistent. Nonetheless, it is already clear from Frege's work how to define the primitives and prove the axioms of a standard axiomatization of arithmetic. We shall begin with some discussions of the notions of number and class, which are crucial for the reduction and for the foundations of mathematics generally.

2.3. COUNTING AND NUMBER. In order to be clearer about the concept of number, we might start with the operation of counting. In a simple case of carefully counting a collection of objects, we perhaps look at and point to each one successively, and with each of these directions of the attention we think of or pronounce one of a standard series of symbols (numerals) in its place in a standard ordering of these symbols. We are careful to reach each of these objects once and only once in the process. We thus set up a one-to-one correspondence between the objects and a certain segment of the series of numerals. We say that the number of objects in the collection is __________, where the blank is filled by the last numeral of the series.

Before pursuing this matter further, let us examine the series of numerals itself. We have certain initial symbols and rules for constructing further symbols whose application can be iterated indefinitely. We could simplify the situation in actual language and suppose that there is one initial symbol, say "|," and a generating operation, concatenation of another "|," so that the numerals will be |, ||, |||, ||||, · · ·, It is not clear, however, that it is merely a matter of "practical convenience" that ordinary numerals are, in the long run, considerably more condensed: If a string of several million "|'s" were offered as a result of counting, one would have to count them to learn what the number was.

However, it is worth asking whether the pure notion of natural number requires more than the possibility of generating such a string of symbols. By "symbols" do we mean here blobs of ink? Only with certain reservations. The particular blobs which we have produced are not at all essential; if we write others—|, ||, |||, ||||, · · · —they will do just as well. In fact, we could have chosen symbols of quite different forms and still have produced something equivalent for our purposes, such as +, ++, +++, · · ·, or something not consisting of marks on paper at all, such as sounds, which are, of course, actually used. As long as it is capable of representing to us the process of successive generation by which these sequences of symbols are produced, anything will do—any collection of perceptible objects that can be placed in one-to-one order-preserving correspondence with our first sequence of symbols.

Thus, the blobs of ink serve as the representatives of a quite abstract structure. This abstraction allows us (even on a subordinate level) to disregard some limitations of the blobs besides their particularity and accompanying boundedness to a particular place and time. They are constructed according to a procedure for generating successive ones, and what matters is the structure embodied in the procedure, not any particular limitations that might be encountered in carrying it out. On a sufficiently abstract level we say that we can continue to generate symbols indefinitely, although life is too short, paper and ink run out, the earth perhaps disintegrates, etc.

Here we have already taken the step of introducing abstract entities. In a weak form this could be represented as taking certain abstract equivalence relations between entities (e.g., marks on paper) as criteria of identity for new kinds of entities (e.g., symbols as types or, further, numbers). But we have already reached a point where more is involved, since the abstract entities which are represented by all the marks of a given equivalence class belong to a series which can be continued far beyond any practical possibility of constructing representatives. We can create a "pseudo-concrete" model by appealing to space, time, and theoretical physics, but then we are already depending on abstract mathematical objects. Given that we do think of numerals as referring to numbers, it is natural to introduce the apparatus not only of identity but also of quantification. Certain uses of such quantification, however, will involve still stronger presuppositions than we have uncovered up to now, and we shall discuss these when we consider platonism and constructivism.

2.4. **AXIOMS OF ARITHMETIC**. We have so far taken for granted that the natural numbers are obtained by starting with some initial element 0 and iterating an operation of "successor" or "adding 1." This is the basis for an especially simple axiomatization of the theory of natural numbers, that of Dedekind and Peano, in which the primitives are "0," "number" ("*NNx*"), and "successor" (which we shall give as a relation: "*Sxy*" means "*y* is successor of *x*"). Then the axioms are

(1) $NN0.$

(2) $NNx \supset (\exists!;y)(Nny \ \& \ Sxy).$

(3) $\neg S0x.$

(4) $Sxz \ \& \ Syz. \supset x = y.$

(5) $(F)[F0 \ \& \ (x)(y)(Fx \ \& \ Sxy \,.\supset Fy) \,.\supset (x)(NNx \supset Fx)].$

In (5), "(*F*)" may be read "for all properties *F*," but for the present we shall not discuss just what this means. We do not need to suppose that precisely what properties there are is determined in advance, but we have to acknowledge that if it is not determined what properties there are, then it may not be determined precisely what natural numbers there are.

We could think of the natural numbers as given by a kind of inductive definition:

(***a***) $NN0.$

(***b***) If *NNx*, then *NN*(*Sx*).

(***c***) Nothing is a natural number except by virtue of (*a*) and (*b*).

However, in this case we have to suppose that the successor relation is given in such a way that axioms (2), (3), and (4) are evident. We might think of "0" as represented by "|" and the successor function as represented by the addition of another "|" to a string. Then there is apparently an appeal to spatial intuition in regarding these axioms as evident. In that event the induction principle (5) will be in some way a consequence of (*c*). It could be regarded simply as an interpretation of (*c*), or one might argue, as Ludwig Wittgenstein apparently did at one time (see Friedrich Waismann, *Introduction to Mathematical Thinking*, Ch. 8), that the meaning of all natural numbers is not given to us by such specifications and our independent concept of "all" and that the induction principle functions as a criterion for a proposition's being true of all natural numbers.

2.5. **THE CONCEPT OF CLASS (SET)**. Before we discuss further the notion of number it is necessary to give some explanation of the notion of class or set. We shall consider two explanations, one suggested by Cantor and one suggested by Frege.

2.5.1. *Frege's explanation.* Instead of the term *class* or *set*, Frege used the phrase "extension of a concept." Frege's usage is based on the tendency to regard the predicates of a language as standing in quantifiable places—

John is a Harvard man.

Henry is a Harvard man.

∴ John and Henry have something in common—

and the tendency to derive from general terms abstract singular terms, which are usually explained as referring to properties or attributes.

These two tendencies can be separated. Frege regarded predicates in context as in fact referring, but to concepts, not to objects. Concepts, like the predicates themselves, have argument places; Frege called both predicates and concepts "unsaturated" because only with the argument place filled by an object (in the case of a predicate, a proper name) could they "stand by themselves." A notation which expresses his conception is that of the second-order predicate calculus, in which the above conclusion might be symbolized (misleadingly) as $(\exists F)[F(\text{John}) \ \&\ F(\text{Henry})]$. An expression which is syntactically appropriate for denoting an object cannot denote a concept, and vice versa.

The extension of a concept, then, is simply an object associated with the concept in such a way that if two concepts apply to the same objects, they have the same extension—that is,

(6) $$\hat{x}Fx = \hat{x}Gx. \equiv (x)(Fx \equiv Gx),$$

where $\hat{x}Fx$ is the extension of the concept F. This is essentially Frege's famous axiom V (*Grundgesetze der Arithmetik*, Vol. I, p. 36; Frege's notion of concept can interpret the quantifiers in our axiom 5).

2.5.2. *Cantor's explanation.* Cantor characterized a set as "jedes Viele, welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann" ("every many, which can be thought of as one, that is, every totality of definite elements which can be combined into a whole by a law"; *Gesammelte Abhandlungen*, p. 204). "Unter einer 'Menge' verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denkens (welche die 'Elemente' von *M* genannt werden) zu einem Ganzen" ("By a 'set' we understand any collection *M* of definite well-distinguished objects of our intuition or thought, which are called the 'elements' of *M*, into a whole"; p. 282).

It is virtually impossible to explain Cantor's idea of set without using words of the same general type, only vaguer ("collection," "multitude," *Inbegriff*). We can perhaps approach it by mentioning a few ways in which multitudes are thought of as unities: by being thought of by means of a predicate—that is, by being brought under a concept in Frege's sense—so that Frege's extensions could perhaps be regarded as sets, or by being in some way brought to the attention at once, even without the intervention of language; in particular, a finite number of objects of perception can constitute a set. That the objects must be "determinate and well-distinguished" means that it must be determinate what the elements are, that identity and difference be well-defined for the elements, and that a set must be determined by its elements.

One is inclined in this connection to think of a set as "composed" of its elements, but this is not essential and might lead to confusion of a set with a spatiotemporal sum, but a portion of space or time (for example, a geometric figure) can be partitioned in a number of ways, so the sets of the parts will be different but the sum will always be the same.

The picture of finite sets can be extended in such a way that one might imagine an "arbitrary" infinite set independent of any predicate. Suppose it is to be a set *S* of natural numbers. We go through the natural numbers one by one deciding for each *n* whether *n* is a member of *S* ($n \in S$) or not. Although the determination takes infinitely long, it is determined for each *n* whether $n \in S$. (Or we might imagine its being done all at once by God.)

2.5.3. *Difficulties in these conceptions.* Both Cantor's and Frege's conceptions of sets have difficulties which did not come clearly to the consciousness of logicians and set-theorists until the discovery of the set-theoretical paradoxes, discussed below. We shall merely mention here a source of difficulty. In both theories a set or extension is supposed to be an object, capable of being itself a member of sets. Cannot this give rise to circularities—that is, that a set is formed from or constituted by certain objects, among them itself?. (Or, in Frege's terms, among the objects in the range of the quantifiers on the right side of formula 6 are $\hat{x}Fx$ and $\hat{x}Gx$ themselves, so that the identity condition for these objects, which from Frege's point of view was part of their essence, seems to depend on particular facts about them.)

We shall not say anything at the moment about the particular form the difficulties take or about how to resolve them. We shall continue to use second-order quantification somewhat vaguely; one can interpret the variables as ranging over Frege's concepts, in most cases over classes or even over intensional entities, as might have been suggested by our original word "property."

2.6. FREGE'S ANALYSIS OF NUMBER. We can now proceed to the main steps of Frege's argument for the thesis that arithmetic is a part of logic. Frege observed that a necessary and sufficient condition for, say, the number of *F*'s (which we shall write as "N_xFx") to be the same as the number of *G*'s is that there should be a one-to-one correspondence of the F's and the G's. (In that case we say they are numerically equivalent.) This criterion, which is quite general—that is, not restricted to the case where there are only finitely many F's or G's—had already been exploited by Cantor to generalize the notion of cardinal number to infinite classes. It can be justified by our discussion of counting and number, above.

On the basis of a one-to-one correspondence between the F's and $\{1, \cdots, n\}$ we are prepared to say that the number of F's is n. But no such correspondence can then exist with $\{1,\cdots,m\}$ for any $m \neq n$, and if by the same criterion there are n G's, then by composition we can set up a one-to-one correspondence between the F's and the G's. If there are m G's for $m \neq n$, we cannot. So we say that there are n F's if and only if a one-to-one correspondence exists between the F's and $\{1, \cdots, n\}$, and in that case there are n G's if and only if there is a one-to-one correspondence between the F's and the G's. Writing "there are n F's" as "$(\exists x)_n Fx$," we have that if $(\exists n)[(\exists x)_n Fx]$,

(7) $N_x Fx = N_x Gx. \equiv$ the F's and the G's are numerically equivalent.

Since we have no independent criterion for the case where there are infinitely many F's, we take (7) to be true by definition in that case. We then have Frege's criterion.

Frege then defined a relation H as a one-to-one correspondence of the F's and the G's if and only if for every F there is exactly one G to which it bears the relation H and vice versa—in symbols,

(8) $(x)[Fx \supset (\exists! y)(Gy \;\&\; Hxy)] \;\&\; (y)[Gy \supset (\exists! x)(Fx \;\&\; Hxy)]$,

where "$(\exists! x)(\cdots x \cdots)$" can be defined in first-order logic:

(9) "$(\exists! x)(\cdots x \cdots)$" for "$(\exists x)[\cdots x \cdots \;\&\; (y)(\cdots y \cdots \supset y = x)]$".

Thus, numerical equivalence can be defined by a formula "$(\exists H)\mathcal{S}(H,F,G)$," where "$\mathcal{S}(H,F,G)$" is an abbreviation for a first-order formula, namely, the expansion of (8) in terms of (9).

The relation of numerical equivalence is an equivalence relation; Frege's idea was, in effect, to define cardinal numbers as the equivalence classes of this relation. This definition, however, requires a powerful use of the notion of extension which is allowed by his axiom (6). In other words, $N_x Fx$ is to be the extension of the concept concept numerically equivalent to the concept F—that is, we define

(10) "$N_x Fx$" for "$\hat{G}(\exists H)\mathcal{S}(H,G,F)$".

(In fact, in the *Grundgesetze*, Frege avoided applying the extension operator to a second-order variable by appeal to formula 6: G can be replaced by its extension. We define "$\hat{G}\mathcal{F}(G)$" as $\hat{y}(\exists G)[y = \hat{x}Gx \,.\, \mathcal{F}(G)]$".)

Formula (10) gives a definition of Cantor's general concept of cardinal number, so we can prove (7); no further use of axiom V is needed for the definition of the natural numbers and the proof of the axioms (1)–(5). We now define Peano's primitives—"0," "Sxy" ("y is the successor of x"), and "NNx" ("x is a natural number"):

(11) "0" for "$N_x(x \neq x)$,"

for then (7) yields $N_x Fx = 0 \equiv \neg(\exists x)Fx$.

Intuitively, $n + 1 = N_x(x = 0 \vee \cdots \vee x = n)$; this result will be reached if we define "Sxy" as follows:

(12) "Sxy" for "$(\exists F)\{y = N_w Fw \;\&\; (\exists z)[Fz \;\&\; N_w(Fw \;\&\; w \neq z) = x]\}$".

Intuitively, the number of F's is one more than the number of G's if there is an F such that the number of the *rest* of the F's is precisely $N_x Gx$. Definition (12) implies that in this case $S(N_x Gx, N_x Fx)$.

The remaining primitive is defined by an ingenious device (already present in Frege's *Begriffsschrift*), which yields mathematical induction: we want to define "NNx" so that something true of 0 and of the successor of anything of which it is true is true of every natural number—that is,

(13) $F0 \;\&\; (x)(y)(Fx \;\&\; Sxy. \supset Fy) \,.\supset (x)NNx \supset Fx)$.

But this will be immediate if we define "x is a natural number" as "x falls under every concept F which 0 falls under and which is such that any successor of whatever falls under it also falls under it"—that is,

(14) "NNx" for "$(F)\{F0 \;\&\; (x)(y)(Fx \;\&\; Sxy. \supset Fy) \,.\supset Fx\}$".

To prove the other axioms: (1) is immediate from (14); that S is one-to-one and that 0 is not the successor of anything follow from (12) together with (7).

2.7. DIFFICULTIES IN LOGICISM. The first difficulty with Frege's construction is certainly the use Frege made of the notion of extension. We have alluded to difficulties with the ideas of set theory; they affected Frege's system through Russell's deduction in 1901 of a contradiction from (6). (For Russell's initial exchange of letters with Frege, see van Heijenoort, 1967). We shall discuss Russell's paradox and other paradoxes and the difficulties of the concept of class below.

Nonetheless, it turns out that a reasonably secure system of set theory can be developed in any one of a number of ways that are more than sufficient for the definition of Peano's primitives and proof of his axioms. In fact, no part of the axiomatic apparatus of a system of set theory which gives rise to any doubts as to consistency is really necessary for this reduction; we can say that if the development in set theory of a branch of mathematics necessarily involves the stronger and more problematic parts of set theory, this is due to the nature of the branch of mathematics itself, not the reduction to set theory.

This success is not without loss for the development of arithmetic: it seems that in the more natural set-theoretical systems (the theory of types, Zermelo's set theory) no definition of "N_xFx" can be given with the same appearance of naturalness as in (10). The consequences of Russell's theory of types are more serious: The numbers must be duplicated at each type. What one usually ends up doing is identifying the numbers in a somewhat arbitrary way with a sequence of sets of the required order type.

Given that all this has been done, in what sense is the enterprise a reduction of arithmetic to set theory, and in what sense is it a reduction to logic? To take up the last question first, obviously the construction does not reduce arithmetic to logic unless the principles of the set theory involved can count as logical principles. The notion of class is not very far removed from concepts which played a role in traditional logic; from that point of view it is not at all evident why the first-order predicate calculus, which is already a considerable extension of the traditional formal apparatus, should count as logic and the theory of classes should not.

One difference is that whereas a valid formula of first-order logic will yield a truth if the quantifiers are interpreted to range over any domain of objects whatsoever, and without regard to its cardinal number in particular, set theory involves existence assumptions, so the domain over which the quantifiers range must be large enough to contain representatives for the sets whose existence is implied by the formula in question. In Frege's procedure these assumptions were embodied in the admission as a term of an abstract "$\hat{x}Fx$" for any predicate "F," and simple nonparadoxical instances of (6) already require that Frege's universe contain infinitely many objects.

Frege, of course, regarded (6) as a logical principle, a view which was fairly well refuted by its inconsistency. It would be much more reasonable to regard set theory as logic if its existence assumptions all followed from a single general principle, such as (6). But the analysis of the foundations of set theory stimulated by the paradoxes points to the opposite conclusion: Any very definite system of existential postulates will prove incomplete in the sense that it is always possible to construct further existential postulates that are stronger (in the sense of first-order, or even second-order, logic). Moreover, these postulates assume a character not unlike principles of construction, so it is at least as natural to consider them hypothetical and analogical extensions of "constructions in pure intuition" as it is to consider them principles of logic. At any rate, if logic consists of the necessary principles of all coherent reasoning, then it seems evident that the stronger principles of set theory do not have this character; it is far from certain even that the weaker ones have it (perhaps even that all of first-order logic does). This being so, a reduction of arithmetic to set theory does little to increase the security and clarity of the foundations of arithmetic.

2.8. KANT'S VIEW. One of the purposes that Frege, Russell, and many later proponents had in mind in seeking to reduce arithmetic to logic was to show that no appeal to sensible intuition was necessary in arithmetic, as had been claimed by such empiricists as John Stuart Mill and by Kant in his theory of a priori intuition. Let us consider whether this purpose has been accomplished. Since Kant's view constitutes an independent effort to explain the a priori character of arithmetic, and since it is part of an extremely influential general philosophy, it deserves special mention.

Kant began by insisting that mathematical judgments (at least the most characteristic ones) were synthetic, rather than analytic. We shall not enter into the question of just what he meant by that. Provided that one remembers that the scope of logic was much narrower for Kant than it is for us, it is plausible to suppose that his claim that mathematical judgments are synthetic implies that the propositions of a mathematical theory cannot be deduced from logical laws and definitions. The case of Kant's principal example, the geometry of space, seems clear, given, for instance, the fact that there are consistent geometrical theories which differ with respect to certain fundamental principles, such as the parallel postulate. (Even here, however, one might claim that the difference in principles corresponds to a difference in the meanings of the primitive terms. In application to real space this comes down to the question of "conventionalism" in geometry. W. V. Quine is probably right in holding that one cannot, in general, decide the question whether such a difference is merely a difference of meaning.)

The case of arithmetic presents a certain similarity if we deny that set theory is logic. The proofs in the set-theoretic development even of such elementary arithmetical laws as "2 + 2 = 4" depend on existential axioms of these theories. However, this does not mean that we can come as close to clearly conceiving the falsity of these principles as we can for the principles of geometry. Although we can easily enough set up a domain in which the existence postulates will fail, it is not clear that this counts as conceiving that the numbers 0, 1, 2, · · · should not exist.

Kant went on to maintain that the evidence of both the principles of geometry and those of arithmetic rested on the "form of our sensible intuition." In particular, he said that mathematical demonstrations proceeded by "construction of concepts in pure intuition," and thus they appealed to the form of sensible intuition. Mathematical proof, according to Kant, required the presentation of instances of certain concepts. These instances would not function exactly as particulars, for one would not be entitled to assert anything concerning them which did not follow from the general concept. Nonetheless, conclusions could be drawn which were synthetic, because the construction of the instance would involve not merely the pure concept as of an abstract structure but also its "schematism" in terms of the general structure of our manner of representing objects to ourselves.

Thus, geometric figures would obey the axioms of geometry even though these axioms were not provable by analysis of the concepts. At the same time, the constructions would serve to verify any existence assumptions involved. (Indeed, instead of existential axioms Kant spoke of postulates asserting the possibility of certain constructions.)

In the case of arithmetic Kant argued that in order to verify "7 + 5 = 12" one must again consider an instance, this time in the form of a set of five objects, and add each one in succession to a given set of seven. It seems that although the five objects may be quite arbitrary, even abstract, they will, if not themselves present to perception, be represented by symbols which are present and which exhibit the same structure. In fact, we find this structure even in the symbolic operations involved in the formal proofs of "7 + 5 = 12" either within a set theory or directly from axioms for elementary number theory—or even in the proof of the formula of *first*-order logic

$$\textbf{(15)}\ (\exists x)_7 Fx\ \&\ (\exists x)_5 Gx\ \&\ (x)\neg(Fx\ .\ Gx)\ .\supset (\exists x)_{12}(Fx \vee Gx),$$

which is the key to the proof of "7 + 5 = 12" in Frege's construction. We think of "$(\exists x)_n(Fx)$" expanded as follows:

"$(\exists x)_0 Fx$" for "$\neg(\exists x)Fx$".

"$(\exists x)_{n+1} Fx$" for "$(\exists x)[Fx\ \&\ (\exists y)_n(Fy\ \&\ y \neq x)]$".

The arguments for the claim that intuition plays an essential role in mathematics are inevitably subjectivist to a degree, in that they pass from a direct semantical consideration of the statements and of what is required for their truth to a more pragmatic consideration of the operations involved in understanding and verifying them (and perhaps even "using" them, in a broad sense) and to a metalinguistic reflection on formulae and proofs as configurations of symbols. Gottfried Wilhelm Leibniz had already emphasized the essential role of calculation with symbols in mathematics, and to Kant this role became an argument for the dependence of mathematics on sensible intuition.

We can see why the arguments must have this subjectivist character if we notice the complete abstractness of both set theory and arithmetic, which talk of objects in general in terms of logical operations (propositional combination, quantification) which are equally general. Even the specifically mathematical objects (sets and numbers) are subjected by the theory only to certain structural, relational conditions, so that they are not, as it were, individually identified by the theory. The content thus does not suggest any direct sensory verification; indeed, it seems that any proposition which is susceptible of such verification must contain some particular reference to space or time or to objects or properties which by nature occur only in space and time. Although it is Frege's construction and the development of set-theoretic mathematics which make this fact clear, Kant apparently was aware of it in the case of arithmetic, which he related closely to the pure categories and therefore to logic.

Nevertheless, it does not seem, at least in the light of philosophical and mathematical experience, that we can directly verify these propositions, or even understand them, independently of the senses. Determining the precise nature of the dependence of the operations of the mind in general on the senses is one of the central difficulties of all philosophies. But it is hard to maintain that we understand mathematical structures, or even the general notion of object which underlies them, without at least starting with a sensible representation, so that concrete explanations make use both of embodiments of the structures by perceptible objects and of reflection on symbolism. For instance, explanations of the notion of class can either make use of an appeal to language, as Frege's explanation does, or begin with the notion of a group of perceptible objects. (Indeed, it seems that even in the second case an appeal to language is sooner or later indispensable.)

Perhaps more decisive than these rather vague considerations is the fact that we cannot carry on any even fairly elaborate reasoning in mathematics without, as it were, placing ourselves at the mercy of a symbolic representation. Prior to the construction of a proof or calculation we do not know the answer to any substantial mathematical question. That the proof can be constructed, that the calculation turns out as it does, is, as it were, brute fact without which one cannot see any reason for the mathematical state of affairs being what it is. In *Über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral*, Kant gave this as his principal reason for asserting that mathematics proceeds by representing concepts in intuition, and in the *Critique of Pure Reason* the idea is again suggested in the discussion of "7 + 5 = 12" and the remarks about "symbolic construction" in algebra.

One might argue that the existence of a natural number *n* is verified by actually constructing a sequence of numerals up to that point. Such a construction provides a representation for the numbers up to *n*. It is noteworthy that either it or a mental equivalent is necessary for a full and explicit understanding of the concept of the number *n*. This gives some plausibility to the view that the possibility of such a representation rests on the "form of our sensible intuition," since everything belonging to the content of the particular realization is nonessential. It is perhaps permissible to speak, as Kant did, of "pure intuition," because we are able to take the symbols as representing or embodying an abstract order. This conception could be extended to the intuitive verification of elementary propositions of the arithmetic of small numbers. If these propositions really are evident in their full generality, and hence are necessary, then this conception gives some insight into the nature of this evidence.

However, the above description already ceases to apply when we pass to the construction, by a general rule, of the sequence of natural numbers and therefore when we consider large numbers, which we must describe in terms of general rules. Besides the "factor of abstraction" signalized in our being able to use sensory representations in thinking about the abstract structures they embody, there is also a factor of higher generality and the accompanying possibility of iteration, so that the sequence of natural numbers extends far beyond those represented by numerals it is possible actually to construct. Here the sense of the notion of "form of intuition" is less clear. Kant's idea, however, must surely be that the larger numbers are conceived only as an extension of the structures of our actual experience. The fact that the forms in question are, according to Kant, those of space and time means that the abstract extension of the mathematical forms embodied in our experience parallels an extension of the objective world beyond what we actually perceive.

Kant connected arithmetic with time as the form of our inner intuition, although he did not intend by this to deny that there is no direct reference to time in arithmetic. The claim apparently was that to a fully explicit awareness of number goes the successive apprehension of the stages in its construction, so that the structure involved is also represented by a sequence of moments of time. Time thus provides a realization for any number that can be realized in experience at all. Although this view is plausible enough, it does not seem strictly necessary to preserve the connection with time in the necessary extrapolation beyond actual experience. However, thinking of mathematical construction as a process in time is a useful picture for interpreting problems of constructivity (discussed below).

Kant's view enables us to obtain a more accurate picture of the role of intuition in mathematics, but, at least as developed above, it is not really satisfying, because it takes more or less as a fact our ability to place our perceptions in a mathematically defined structure and to see truths about this structure by using perceptible objects to symbolize it. The great attraction of Kantianism comes from the fact that other views seem unable to do any better: Frege, for example, carried the epistemological analysis less far than Kant in spite of his enormously more refined logical technique.

2.9. **CONVENTIONALISM**. Attempts to avoid dogmatism completely while still affirming the existence of a priori knowledge in mathematics have been made on the basis of conventionalism, the characteristic logical positivist view of a priori knowledge. This view in effect rejects the question of evidence in mathematics: Mathematical statements do not need evidence because they are true by fiat, by virtue of the conventions according to which we specify the meanings of the words occurring in mathematics. Mathematics is therefore "without factual content" or even "empty."

Before we proceed to discuss this view we should distinguish it from two others which are associated with logical positivism, the view that mathematical statements are true by virtue of the meanings of the words in them and the view that they are analytic. The doctrine that mathematical statements are true by virtue of the meaning of the words they contain is somewhat vague and is likely to reduce to the doctrine that they are analytic, to conventionalism, or to something compatible with Kantianism or even with some form of direct realism. If there are objective relations of meaning which hold not merely by fiat, then there is as much need in this view for an account of the evidence of our knowledge of them as there is for the evidence of mathematics itself.

The view that mathematics is analytic has generally been associated on one side with logicism and on the other with conventionalism. The definitions of "analytic" that have been given have been such that logical truths were automatically analytic. If the thesis that mathematics is analytic was to say more than the thesis of logicism, the definitions had to be taken as explicating a concept which had a more direct epistemological significance, usually truth by virtue of meanings or truth by convention. (Once this has been done, the connection with logicism seems less important, in spite of the importance that the logical positivists attributed to it. Thus, one may explain the claim that the axioms of set theory are analytic by saying that they are "meaning postulates" in Carnap's sense, but one could argue equally well that the axioms of number theory are meaning postulates. Logicism was important to the logical positivists for other reasons: the reduction served as a methodological paradigm; it served the "unity of science.")

That the propositions of mathematics should be true by convention in a strong sense, that one should actually have set up conventions which determine that they should be true, seems possible only for "rational reconstructions" of mathematics by explicit construction of an axiom system and identification of the system with mathematics. If such a procedure could be carried out, there would still be room for discussion of the sense in which it showed that the mathematics practiced by those who are not interested in foundations is true by convention.

The usual conventionalist position appeals to rules specifying that certain propositions are to be true by convention or, more often, to rules of another sort (such as semantical rules of an interpreted formal system), from which it can be deduced that certain statements are true, the nature of the premises being such that they can be called conventions governing the use of expressions. (For example, the truth of any statement that is a substitution instance of a theorem of the classical propositional calculus can be deduced from the information contained in the truth tables for the propositional connectives. Then if the truth tables are regarded as semantical rules specifying the meanings of the connectives, then the theorems of classical propositional logic thus become true by virtue of these rules.)

In the simplest case—that of simply laying down, by rules or in individual instances, that certain sentences are to be taken as expressing true statements—something more seems to be required to justify this procedure as attributing "truth" to "statements." No serious philosopher, however, has been content to leave the matter at that.

Nonetheless, the procedure of specifying by rules runs into a difficulty essentially independent of the form of the rules and the manner in which they are interpreted. This difficulty, which was pointed out forcefully by Quine early in his career (in "Truth by Convention") and is perhaps implicit in remarks by Frege, is that the passage from the general statements which are the actual explicit conventions to the truth by convention of specific statements involves inference. So something essentially logical is not, on the face of it, reduced to convention by the analysis. The inferences will assume properties of generality (for example, the properties of the universal quantifiers) and of the conditional, since the rules will in all probability be of the form of conditionals—for instance, they may say that if a statement satisfies certain conditions, then it is true by convention. In the example that we gave, one needs in addition the laws of contradiction and of excluded middle: Application of the truth tables already supposes that each statement has one, and only one, of the two truth-values.

Quine showed that the attempt to regard the rules by which this inference proceeds as themselves valid by convention leads to an infinite regress. For example, suppose a rule is *modus ponens:* from "p" and "$p \supset q$" infer "q". This could be stated as the convention:

(16) If A and C are true and C is the result of substituting A for "p" and B for "q" in "$p \supset q$", then B is to be true.

Now, suppose that for some A' and B' we have proved that A' and C' are true by convention, where

(17) C' is the result of substituting A' for "p" and B' for "q" in "$p \supset q$".

Then we have also

(18) A' is true;

(19) $A' \supset B'$ is true.

Therefore, by (16) and *modus ponens, B*' is true. However, in order to represent this inference as proceeding according to the convention, it is necessary to make another application of *modus ponens*, and so on.

The above argument would not prevent this form of conventionalism from being applied to further parts of mathematics, particularly to existential axioms. In view of the equivalences between derivability statements in logic and elementary propositions in number theory, as well as the above-mentioned element of brute fact in the existence of a derivation, it is not likely that such an approach will work for elementary number theory. But with the stronger axiom systems for set theory the view is on somewhat firmer ground, in that such axioms are often not justified by appeal to direct evidence and "pragmatic" criteria have played a role in the selection of axioms.

Nonetheless, the procedure also has much in common with the setting up of a hypothetical theory in science, and, indeed, as Alfred North Whitehead and Russell already emphasized, the axioms are subject to a sort of checking by their consequences, since some propositions deducible from them are decidable by more elementary and evident mathematical means. It is not evident that if a system of axioms is replaced by another because its consequences come into conflict with intuitive mathematics, the meaning of "set" has changed and the original axioms can be interpreted according to a previous meaning so as to remain true. Moreover, set theory proceeds on the assumption that the truth-value of statements is determinate in many cases where it is not determined by the axioms—that is, by the conventions.

Quine, in fact, now argues, apparently even in the case of elementary logic, that there is no firm ground for distinguishing between making such principles true by convention and adopting them as hypotheses ("Carnap and Logical Truth"). This is as much an extension of conventionalism to the whole of science as a rejection of it in application to mathematics.

2.9.1. ***Wittgenstein's view.*** At this point we must consider the possibility that a priori truths, even the elementary ones, are thought of as true by convention, not in the sense that they may be made so by an explicit convention actually set up but in the sense that the conventions are, as it were, implicit in our practice with the logical and mathematical vocabulary. It might still be argued that the principles of mathematics are not in that way sufficiently distinguished from the principles of natural science or from other rather deep or fundamental principles that we firmly accept. But this objection could be met by a more detailed descriptive analysis of how logical and mathematical words are used.

However, this type of conventionalism must be careful not to slip into the situation of the more explicit conventionalism of requiring a necessary connection between general intentions and their application in particular statements which is not itself accounted for by the conventions. It appears that the only philosopher who has really faced these challenges has been Ludwig Wittgenstein, in his later period. In connection with Wittgenstein it would probably be better to speak of "agreement" than convention, since the reference to explicit conventions or to "decisions" seems metaphorical, as a picture which is contrasted with that against which he is arguing rather than as a fundamental theoretical concept. It is agreement in our actions—e.g., what we say follows from what—that is essential. We should also be cautious in attributing to Wittgenstein any explanatory theory of logical and mathematical knowledge, in view of his disclaimers of presenting a theory.

Even with these qualifications Wittgenstein's view seems highly paradoxical, for in order to avoid the above-mentioned pitfall the analysis in terms of agreement must extend even to the connection between general rules and their instances. This seems to be the point of the famous discussion of following a rule in Wittgenstein's *Philosophical Investigations*. What ultimately determines what is intended in the statement of a rule are facts of the type of what is actually accepted in the course of time as falling under it.

Wittgenstein (I, 185) gave the example of instructing someone in writing down the terms of the sequence of natural numbers 0, 2, 4, · · ·, $2n$, · · ·. At the start the instructor does not actively think that when the time comes the pupil is to write 1,000, 1,002, 1,004, · · ·, rather than 1,000, 1,004, 1,008, · · ·. Wittgenstein regarded it as conceivable that the pupil might do the second on the basis of a misunderstanding which we just could not clear up. Moreover, it is, as it were, just a fact of natural history that normally, in such a case, we accept the first and reject the second—indeed, continue in that way ourselves. It appears, further, that the same issue can arise for steps in the sequence which have been written before, since the recognition of symbols as tokens of an already understood type is itself an application of a rule (see I, 214).

Wittgenstein's criticism seems directed particularly against certain psychological ideas associated with platonism and Kantianism. The manner in which the steps of writing numerals are determined by the rule cannot be explained by appealing to one's understanding of the relations of abstract entities expressed in the rule or even to the intentions of the instructor. According to Wittgenstein the criterion of how the pupil does understand the rule lies in the steps which he in fact takes. And what makes them right or wrong is their agreement or disagreement with what we do.

The steps are indeed determined by the rule, in the sense that at each stage there is only one number we accept as correct, and the force of social custom directs us to expand the series in the way we do. But this does not mean that Wittgenstein considered his appeals to custom and training as constituting a fully satisfactory explanation of either the agreement that exists or the fact that we feel "compelled" by the rule, for it is because we are made as we are that we react to custom and training as we do.

The paradoxical nature of Wittgenstein's position can perhaps be brought out by considering the case of a complex mathematical proof which contains steps which no one has thought of before. The proof may lead to a quite unexpected conclusion. Yet each step is recognized by every trained person as necessary, and their combination to form the proof is entirely convincing. (This is, of course, not inevitably the case: proofs as published can be obscure or doubtful and can rest on principles about which there are difficulties.) In spite of the fact that it is in principle possible for an irresolvable disagreement to arise at each point, this does not happen: Irresolvable disputes among mathematicians are only about fundamental principles and about taste. Nonetheless, Wittgenstein, in *Remarks on the Foundations of Mathematics*, used the metaphor of decision in speaking of our acceptance of the proof and spoke of the proof as providing a new criterion for certain concepts; his terminology suggests change of meaning.

The vast extent of the agreement on which mathematics rests seems to have astonished Wittgenstein; indeed, it is hard to understand, on his view, how such agreement is possible and why contradictions arise so seldom. We may be faced here with natural facts, but they are facts which show an extremely regular pattern.

Wittgenstein devoted a good deal of attention in the *Remarks* to discussions of calculation and proof, their relation to mathematical truth, and the ways in which they resemble and differ from experiment. In a number of examples he revealed an outlook which resembles Kant's in seeing a construction either of figures or of arrangements of formulae or propositions as essential to a proof. To the problem concerning how such a singular construction can serve to establish a universal and necessary proposition Wittgenstein suggested a quite different answer: In accepting the proof we accept the construction as a paradigm for the application of a new concept, so that, in particular, we have new criteria for certain types of judgments. (For example, if we have determined by calculation that $25 \times 25 = 625$, then a verification that there are 25×25 objects of a certain kind is also accepted as verifying that there are 625.) The same question arises in connection with the possibility of conflict in these criteria as arose in connection with agreement.

We shall close at this point our discussion of the a priori character of mathematics and the attempts to justify and explain it. In the sense that the concepts of mathematics are too general and abstract to refer to anything particular in experience, their a priori character is evident, at any rate after a certain amount of logical analysis of mathematical concepts. The a priori evidence of mathematics, on the other hand, is perhaps not raised, by our discussion, above the level of a somewhat vague conviction. In the case of the more powerful forms of set theory one is probably forced to admit that the evidence is less than certainty and therefore to admit that there is an analogy between the principles involved and the hypotheses of a scientific theory. In the case of arithmetic and elementary logic, however, this conviction can withstand the objections that might be posed, but in view of the difficulties we have discussed in relation to various accounts, it seems still not to have been analyzed adequately.

§3. PLATONISM AND CONSTRUCTIVISM

The discussion in the preceding section suggests that the problem of evidence in mathematics will appear to differ according to the part of mathematics being emphasized. The form which discussion of these differences has tended to take is a distinction between two broad methodological attitudes in mathematics, which we shall call platonism and constructivism. This section will be devoted to a discussion of these attitudes.

3.1. PLATONISM. We begin with platonism because it is the dominant attitude in the practice of modern mathematicians, although upon reflection they often disguise this attitude by taking a formalist position. Platonism is the methodological position that goes with philosophical realism regarding the objects mathematics deals with.

Mathematical objects are treated not only as if their existence is independent of cognitive operations, which is perhaps evident, but also as if the facts concerning them did not involve a relation to the mind or depend in any way on the possibilities of verification, concrete or "in principle."

This is taken to mean that certain totalities of mathematical objects are well defined, in the sense that propositions defined by quantification over them have definite truth-values. Thus, there is a direct connection between platonism and the law of excluded middle, which gives rise to some of platonism's differences with constructivism.

It is clear that there is a connection between platonism and set theory. Various degrees of platonism can be described according to what totalities they admit and whether they treat these totalities as themselves mathematical objects. These degrees can be expressed by the acceptance of set-theoretic existence axioms of differing degrees of strength.

The most elementary kind of platonism is that which accepts the totality of natural numbers—i.e., that which applies the law of excluded middle to propositions involving quantification over all natural numbers. Quite elementary propositions in analysis already depend on this law, such as that every sequence of rational numbers either tends to the limit 0 or does not, which is the basis for the assertion that any real number is either equal to 0 or not. We shall see that not even this assertion is immune to constructivist criticism.

What is nowadays called classical analysis advances a step further and accepts the totality of the points of the continuum or, equivalently, the totality of subsets of the natural numbers. The equivalence between these totalities and their importance in mathematics were brought out by the rigorous development and "arithmetization" of analysis in the nineteenth century. We recall that the theories of (positive and negative) integers and rational numbers can be developed from the theory of natural numbers by means of the notion of ordered pair alone and that this notion can in turn be represented in number theory. A general theory of real numbers requires general conceptions of a set or sequence of natural numbers to which those of a set or sequence of rational numbers can be reduced.

Following Paul Bernays ("Sur le platonisme dans les mathématiques") we can regard the totality of sets of natural numbers on the analogy of the totality of subsets of a finite set. Given, say, the numbers $1, \cdots, n$, each set is fixed by n independent determinations of whether a given number belongs to it or not, and there are 2^n possible ways of determining this. An "arbitrary" subset of the natural numbers is fixed by an infinity of independent determinations fixing for each natural number whether it belongs to the subset or not. Needless to say, this procedure cannot be carried out by a finite intelligence. It envisages the possibility of sets which are not the extensions of any predicates expressed in a language.

3.1.1. ***Impredicative definitions.*** The strength of the assumption of the totality of arbitrary subsets of the natural numbers becomes clear if we observe that it justifies impredicative definitions, definitions of sets or functions in terms of totalities to which they themselves belong. A predicate of natural numbers involving quantification over all sets of natural numbers will have a well-defined extension, which will be one of the sets in the range of the quantifier.

Such definitions have been criticized as circular (for example, by Henri Poincaré), but they do not seem so if we understand the sets as existing independently of any procedure or linguistic configuration which defines them, for then the definition picks out an object from a preexisting totality. The resistance that impredicative definitions met with arose partly because their acceptance clashes with the expectation that every set should be the extension of a predicate, or at least of a concept of the human mind.

Given any definite (formalized) notation, we can by Cantor's diagonal method define a set of natural numbers which is not the extension of a predicate in the notation. Thus, no procedure of generating such predicates by continually expanding one's notation can possibly exhaust the totality. And the idea that every set is the extension of a predicate has little sense if it is assumed that in advance of the specification of notations there is a totality of possible predicates which can be arrived at by some generating procedure.

If the statements of classical analysis are interpreted naively, then quite elementary theorems, such as that every bounded set of real numbers has a least upper bound, require impredicative definitions. Nonetheless, in *Das Kontinuum*, Hermann Weyl proposed to construct analysis on the basis of mere platonism with respect to the natural numbers. He proposed an interpretation under which the least upper bound theorem is true. Later interpretations have preserved more of the statements of classical analysis than Weyl's, and it is an involved technical question how much of it can be given a natural predicative interpretation (see below).

3.1.2. ***Set theory and the paradoxes.*** Set theory as developed by Cantor and as embodied in the present standard systems involves a higher degree, or variety of degrees, of platonism. The axiom system of Zermelo and its enlargement by Fraenkel (which is called the Zermelo-Fraenkel system), for example, allows the iteration of the process of forming the set of all subsets of a given set and the collection into a set of what has been obtained by iterated application of this or some other generating procedure. This latter allows the iteration into the transfinite. If we assume we have transfinite ordinal numbers, then we can generate a transfinite succession of "universes" U as follows: Let $\mathcal{P}(A)$ be the set of all subsets of the set A.

U_0 = a certain class, perhaps empty, of "individuals."

$$U_\alpha + 1 = \mathcal{P}(U_\alpha) \cup U_\alpha.$$

U_α = the union of all U_β, for $\beta < \alpha$, if α is a limit ordinal.

Then for certain ordinals α the U_α will form models for the different systems of set theory ($U_\omega + \omega$ for Zermelo's set theory, without Fraenkel's axiom of replacement).

The paradoxes of set theory imply that we must accept some limitations on forming totalities and on regarding them in turn as mathematical objects—that is, as sets. If, for example, the totality of sets is a well-defined set, then it seems that it will be reasonable to ask of each set x whether it is a member of itself ($x \in x$) or not and to form $\hat{x}(x \notin x)$, the set of all sets which are not members of themselves. This will satisfy

$$(y)[y \in \hat{x}(x \notin x) \,.\!\equiv y \notin y],$$

which implies

$$\hat{x}(x \notin x) \in \hat{x}(x \notin x) \,.\equiv.\, \hat{x}(x \notin x) \notin \hat{x}(x \notin x).$$

a contradiction. This is Russell's paradox, the most shocking, because the most elementary, of the paradoxes of set theory.

On the same basis one can ask for the cardinal number of the set of all sets, which we shall call S. Then $\mathcal{P}(S)$, the set of all subsets of S, will have a cardinal number no greater than that of S, because $\mathcal{P}(S) \subseteq S$. But by Cantor's theorem the cardinal number of $\mathcal{P}(S)$ is properly greater than that of S (Cantor's paradox, 1895).

If the totality O of ordinals is a set, then, since it is well-ordered, there will be an ordinal number γ that represents its order type. But then O will be isomorphic to the set of ordinals less than γ—that is, to a proper initial segment of itself. This is impossible: γ must be the greatest ordinal, but there is no obstacle to forming $\gamma + 1$ (Burali-Forti's paradox, 1897).

These paradoxes do not imply that we have to stop or otherwise limit the process, described above, of generating larger and larger universes. On the contrary, we must never regard the process as having given us "all" sets. The totality of sets, and hence the totality of ordinal numbers, cannot be the terminus of a well-defined generating process, for if it were we could take all of what we had generated so far as a set and continue to generate still larger universes.

Thus, suppose we consider the arguments for the paradoxes applied to a particular U_α, as if it were the universe of all sets. The construction precludes $x \in x$, so $\hat{x}(x \notin x)$ is just U_α itself. But $U_\alpha \notin U_\alpha$ and hence is disqualified as a set. The same consideration applies to Cantor's paradox. Burali-Forti's paradox is avoided because the passage from U_α to $U_{\alpha+1}$ always introduces well-orderings of higher order types. Thus, for no α can U_α contain "all" ordinals, no matter how the ordinals are construed as sets. (A very natural way of construing them would be such that α occurs in $U_{\alpha+1}$ but not in U_β for any $\beta \leq \alpha$. But then only for certain ordinals will U_α contain an ordinal for each well-ordered set in U_α.)

For some time after they were first discovered, the paradoxes were viewed with great alarm by many who were concerned with the foundations of mathematics. In retrospect this seems to have been because set theory was still quite unfamiliar; in particular, the distinction between the customary reasonings of set theory and those that led to the paradoxes was not very clear. The opposition that set theory had aroused had not yet died down. However, the marginal character of the paradoxes has seemed more and more evident with time; the systems which were soon devised to cope with the paradoxes (Russell's theory of types and Zermelo's set theory, both published in 1908) have proved satisfactory in that they are based on a reasonably clear intuitive idea, and no one today regards it as a serious possibility that they (or the stronger Zermelo-Fraenkel system) will turn out to be inconsistent. This does not mean that the security and clarity of set theory are absolute; in the sequel some of the difficulties will become apparent.

The above-described sequence of universes uses general conceptions of set and ordinal but applies the characteristic move of platonism only one step at a time. It renounces what Bernays calls "absolute platonism," the assumption of a totality of all mathematical objects which can be treated as itself a customary mathematical object—for example, a set. Such a conception seems definitely destroyed by the paradoxes. The totality of sets can be compared with Kant's "Ideas of Reason": it is an "unconditioned" or absolute totality which just for that reason cannot be adequately conceived by the human mind, since the object of a normal conception can always be incorporated in a more inclusive totality. From this point of view there is an analogy between the set-theoretic paradoxes and Kant's mathematical antinomies.

If we assume that every set will appear in one of the U_α, we have a conception which is adequate for all of modern mathematics except, perhaps, the recent theory of categories. The conception is by nature imprecise: there are limitations on our ability to circumscribe both what goes into the power set of a given set and what ordinals there are. It is perhaps unreasonable to apply classical logic to propositions involving quantification over all sets, since such an application seems to presuppose that it is objectively determined what sets (and a fortiori, on this conception, what ordinals) there are. Nonetheless, this additional idealization does not seem to have caused any actual difficulties.

This way of conceiving sets combines two of Russell's early ideas for resolving the paradoxes—the theory of types and the theory of "limitation of size." What are rejected as sets are the most inclusive totalities, such as the entire universe. (Our talking of "totalities" while rejecting them as sets is not incompatible with our conception; as John von Neumann observed, all that is necessary is to prohibit them from belonging to further classes. Von Neumann's observation was the basis for some new set theories, the principal one being that of Bernays and Gödel.) Moreover, the sets are arranged in a transfinite hierarchy: One can assign to each set an ordinal, its type or, as it is now called, rank, which will be the least ordinal greater than the ranks of its members. We have thus a transfinite extension of the cumulative theory of types. But we have dropped the more radical idea from which Russell proceeded: that each variable of a system of set theory should range over objects of a specified type, and that "$x \in y$" is meaningless unless the range of "y" is of a type one higher than that of "x," so that, in particular, "$x \in x$" is meaningless.

3.1.3. *Predicativism.* In the first twenty-five years or so after the discovery of the paradoxes a number of more radical proposals for their elimination were presented. These generally amounted to some further attenuation of platonism. We shall first consider the program of eliminating impredicative definitions, which amounts to a restriction of platonism to the natural numbers. This was the outcome of the general views of Poincaré and Russell. Russell's original theory, the ramified theory of types, which formed the basis of *Principia Mathematica*, was directed to the elimination of impredicative definitions, which he held to involve a "vicious circle" and to be responsible for the paradoxes. The effect was, however, nullified by his axiom of reducibility.

A greatly simplified version of the ramified theory is as follows: One has variables, each of which is assigned a natural number as its level, and the predicates of identity and membership. The logic is the usual quantification theory, except that in the rules for quantifiers allowance must be made for levels. Since the levels can be cumulative, we could have for the universal quantifiers the following:

(20) $(x^i)Fx^i \supset Fy^j$ if $j \leq i$;

(21) From "$p \supset Fy^i$" infer "$p \supset (x^i)Fx^i$," where for "p" only something not containing free "y^i" can be substituted.

The axioms are those of identity, extensionality, and the following schema of class existence:

(22) If "F" represents a predicate which does not contain free x^{i+1}, any free variables of level $> i + 1$, or any bound variables of level $> i$,

$$(\exists x^{i+1})(y^i)(y^i \in x^{i+1} \equiv Fy^i).$$

One effect of this axiom is that a predicate involving quantification over objects of level n need not have an extension of level n. Therefore, the axiom does not assert the existence of any impredicative classes; in fact, it is compatible with the idea that classes are constructed by the construction of predicates of which they are the extensions.

Russell's actual theory combined that of a hierarchy of levels, applied in this case to "propositional functions," the objects over which the variables of a higher-order logic were to range, with the "no class" theory, the introduction of locutions involving classes by contextual definition in terms of propositional functions. In order to derive classical mathematics, however, he wanted to avoid dividing the classes into levels. This he did by postulating the axiom of reducibility, which asserts that for every propositional function there is a function of the lowest possible level (compatible with the nature of its arguments) extensionally equivalent to it. Russell admitted that this axiom was equivalent to the existence of classes, and he has never been satisfied with it. In effect, it yields even impredicatively defined classes and destroys the effect of the hierarchy of levels.

A formalization of mathematics on the basis of the ramified theory is the most natural formalization if a platonist theory of classes is repudiated but classical logic admitted. The construction of the natural numbers leads to the difficulty that the class quantifier needed to reduce induction to an explicit definition is no longer available. One must either assume the natural numbers or have a hierarchy of different concepts of natural number.

A ramified theory with the natural numbers as individuals and the Peano axioms would be a natural formalization of the mathematics allowed by platonism with respect to the natural numbers. But there is in principle no reason not to extend the hierarchy of levels into the transfinite. The question of the limits of predicative mathematics has become identical with the question of the transfinite ordinals that can be predicatively introduced.

We have said that quite elementary proofs in analysis already require impredicative definitions when naively interpreted. Nonetheless, from recent work it appears that a good deal of classical analysis is susceptible of a natural predicative interpretation, which, however, fails for some theorems. One can, on this basis, give a good approximation to classical analysis, but not to the whole of it. That part of mathematics which depends essentially on still more powerful set theory is completely lost. It seems that it would not be reasonable to insist on this limitation unless there were some quite powerful reason for rejecting platonism. We shall discuss some possible reasons later.

3.2. **CONSTRUCTIVISM**. We shall now consider the complete rejection of platonism, which we shall call constructivism. It is not a product of the situation created by the paradoxes but rather a spirit which has been present in practically the whole history of mathematics. The philosophical ideas on which it is based go back at least to Aristotle's analysis of the notion of infinity (*Physics*, Bk. III). Kant's philosophy of mathematics can be interpreted in a constructivist manner, and constructivist ideas were presented in the nineteenth century—notably by Leopold Kronecker, who was an important forerunner of intuitionism—in opposition to the tendency in mathematics toward set-theoretic ideas, long before the paradoxes of set theory were discovered.

Our presentation of constructivism relies heavily on the "intuitionism" of Brouwer, presented in many publications from 1907 on, but the ideas can also be found to some extent in other critics of platonism, including the French school of Émile Borel, Poincaré, and Henri Lebesgue, although in their work predicativity played a greater role than constructivity. These writers did not arrive at a very consistent position, but they contributed mathematically important ideas. L. E. J. Brouwer reached and developed a conclusion from which they shrank: that a thoroughgoing constructivism would require the modification of classical analysis and even of classical logic.

3.2.1. *Intuitionism.* Constructivist mathematics would proceed as if the last arbiter of mathematical existence and mathematical truth were the possibilities of construction. "Possibilities of construction" must refer to the idealized possibility of construction mentioned in the last section. Brouwer insisted that mathematical constructions are mental. The possibilities in question derive from our perception of external objects, which is both mental and physical. However, the passage from actuality to possibility and the view of possibility as of much wider scope perhaps have their basis in intentions of the mind—first, in the abstraction from concrete qualities and existence; second, in the abstraction from the limitations on generating sequences. In any case, in constructive mathematics the rules by which infinite sequences are generated are not merely a tool in our knowledge but part of the reality that mathematics is about.

Why this is so can be seen from the problem of assertions about the infinite. We have suggested that the generation of a sequence of symbols is something of which the construction of the natural numbers is an idealization. But "construction" loses its sense if we abstract further from the fact that this is a process in time which is never completed. The infinite in constructivism must be "potential" rather than "actual." Each individual natural number can be constructed, but there is no construction which contains within itself the whole series of natural numbers. To view the series *sub specie aeternitatis* as nonetheless determined as a whole is just what we are not permitted to do.

Perhaps the idea that arithmetic rests on time as a form of intuition lies behind Brouwer's insistence on constructivity interpreted in this way. One aspect of sensibility from which we do not abstract in passing from concrete perception to its form is its finite character. Thus, whatever one may think of the notion of form of intuition, Brouwer's position is based on a limitation, in principle, on our knowledge: Constructivism is implied by the postulate that no mathematical proposition is true unless we can in a nonmiraculous way know it to be true.

Because of its derivation from his own philosophical account of mathematical intuition Brouwer called his position, and the mathematics which he constructed on the basis of it, intuitionism. We shall use this name for a species of constructivism which answers closely to Brouwer's ideas.

In spite of the "potential" character of the infinite in mathematics, we shall not renounce assertions about all natural numbers or even, with some reservations, talk of infinite classes. A proposition about all natural numbers can be true only if it is determined to be true by the law according to which the sequence of natural numbers is generated. This Brouwer took to be equivalent to its possessing a proof. Thus, the intensional notions of "law" and "proof" become part of the subject matter of mathematics.

A consideration of existential propositions connects the broad philosophical notion of constructivity with the general mathematical notion. Roughly, a proof in mathematics is said to be constructive if wherever it involves the mention of the existence of something, it provides a method of "finding" or "constructing" that object. It is evident that the constructivist standpoint implies that a mathematical object exists only if it can be constructed; to say that there exists a natural number x such that Fx is to say that sooner or later in the generation of the sequence an x will turn up such that Fx. If x depends on a parameter y, this x must be determinable from y on the basis of the laws of the construction of the numbers and of the constructions involved in F. Proving $(\exists x)Fx$ means showing how to construct x, so one can say that the proof is not complete until x has been exhibited. (But then "proof" is used in an idealized sense.) To prove $(y)(\exists x)Fxy$ must involve giving a general method for finding x on the basis of y.

This point of view leads immediately to a criticism of the basic notions of logic, particularly negation and the law of excluded middle. That "$(x)Fx$" is true if and only if it can be proved does not mean that "$(x)Fx$" is a statement about certain entities called proofs in the way in which, on the usual interpretation, it is a statement about the totality of natural numbers. According to Brouwer we can assert "p" only if we have a proof; the hypothesis that $(x)Fx$ is the hypothesis that we have a proof, and it is a reasonable extrapolation to deny that we can say more about what "$(x)Fx$" asserts than is said in specifying what is a proof of it. The explanation of "$\neg(x)Fx$" as "$(x)Fx$ cannot be proved" does not satisfy this condition. Brouwer said instead that a proof of "$\neg p$" is a construction which obtains an absurdity from the supposition of a proof of "p."

An immediate consequence of this interpretation is that the law of excluded middle becomes doubtful. Given a proposition "p," there is no particular reason to suppose that we shall ever be in possession either of a proof of "p" or of a deduction of an absurdity from "p." Indeed, if the general statement of the law of excluded middle is taken as a mathematical assertion, a proof of it will have to yield a general method for the solution of all mathematical questions. Brouwer rejected this possibility out of hand.

It is evident that such a point of view will lead to changes in quite basic parts of mathematics. Many instances of the law of excluded middle, where the propositions involved can be shown constructively to be systematically decidable, will be retained. But Brouwer rejected even very elementary instances in classical analysis. Let the sequence r_n of rational numbers be defined as follows: if there is no $m \leq n$ such that the mth, $(m + 1)$st, $(m + 2)$d terms of the decimal expansion of π are each 7, then $r_n = 1/2^n$; if there is such an m, then $r_n = 1/2^k$, where k is the least such m. Then r_n constructively defines a real number r. But a proof of either $r = 0$ or $r \neq 0$ would tell us whether or not there are three 7's in the decimal expansion of π. Thus, we cannot assert either $r = 0$ or $r \neq 0$.

For a satisfactory constructivist theory of analysis, an analysis is needed of the notion of an arbitrary set or sequence of natural numbers. Brouwer's analysis gives additional distinctiveness to intuitionism. Such a sequence is thought of as generated by a succession of independent determinations or "free choices," which may be restricted by some law. Obviously the succession of choices must be thought of as never being complete. In the absence of a law a statement about a sequence can be true only if it is determined to be true by some finite initial segment of the sequence. The consequence of this is that a function defined for all sequences of natural numbers whose values are integers must be continuous. It also leads to sharper counterexamples to the law of excluded middle: It is absurd that for all sequences α, either $(x)(\alpha(x) = 0)$ or $\neg(x)(\alpha(x) = 0)$. We can also sharpen the result of the preceding paragraph and state generally that not every real number is equal to or different from 0.

The intuitionist point of view thus leads to a distinctive logic and to a distinctive theory of the foundations of analysis. The latter contains another distinctive principle, the bar theorem, obtained by analyzing the requirement that if a function is defined for all sequences, there must be a constructive proof of this fact. It is roughly equivalent to the proposition that if an ordering is well-founded, transfinite induction holds with respect to it. Nonetheless, intuitionism is far from having shown itself capable of the same rich development as classical mathematics, and it is often very cumbersome. Important as it is in itself, it does not provide a sufficient motive for renouncing platonism.

3.2.2. ***Finitism.*** So far our account of constructivism has been based entirely on Brouwer's intuitionism. However, intuitionism is not the only possible constructivist development of mathematics. Indeed, it makes some quite powerful assumptions of its own. As we have said, the intuitionists make the notions of construction and proof a part of the subject matter of mathematics, and the iteration of logical connectives, especially, renders it possible to make quite elaborate and abstract statements involving construction and proof. Thus, intuitionist mathematics seems to rest not merely upon intuition but upon rather elaborate reflection on the notion of intuitive construction. (It also does not obviously exclude impredicativity, since what counts as a proof of a given proposition can be explained in terms of the general notion of proof.) A constructivist might feel that intuitionism leads from the Scylla of platonist realism to the Charybdis of speculative idealism.

A weaker and more evident constructive mathematics can be constructed on the basis of a distinction between effective operation with forms of spatiotemporal objects and operation with general intensional notions, such as that of proof. Methods based on operation with forms of spatiotemporal objects would approximate to what the mathematician might call elementary combinatorial methods or to the "finitary method" which Hilbert envisaged for proofs of consistency. Formal systems of recursive number theory, in which generality is expressed by free variables and existence by the actual presentation of an instance or (if the object depends on parameters) a function, will accord with this conception if the functions admitted are sufficiently elementary—for example, primitive recursive functions. In such formalisms any formula will express a general statement each instance of which can be checked by computation. For this reason classical logic can be used. Moreover, the concept of free choice sequence can be admitted so that some analysis can be constructed.

The precise limits of this conception are perhaps not clear, although it is evident that some constructive arguments are excluded. The conception does not allow full use of quantifiers but probably does allow a limited use of them.

3.2.3. ***The Hilbert program.*** If one accepts the idea that from a philosophical point of view constructivist conceptions are more satisfactory than platonist conceptions—more evident or more intelligible—one is not necessarily constrained to abandon classical mathematics. The way is still open to investigating classical mathematics from a constructive point of view, and it may then prove to have an indirect constructive sense and justification.

Such an investigation was the objective of the famous program of Hilbert, which was the third main animating force—with logicism and intuitionism—in foundational research in the period before World War II. The possibility arises first from the fact that classical mathematics can be formalized (though not completely; we shall consider this fact and its implications later). Once it has been formalized, one can in principle drop consideration of the intended meaning of the classical statements and simply consider the combinations of the symbols and formulae themselves. Thus, if the proof of a certain theorem has been formalized in a system S (say Zermelo-Fraenkel set theory), it is represented as a configuration of symbols constructed according to certain rules. Whether a configuration is a proof can be checked in a very elementary way.

The concepts by which a formal system is described belong, in effect, to finitist mathematics. For example, the consistency of the system is the proposition that no configuration which is a proof will have a last line of a certain form—for example, $\mathcal{A}$ & $\neg\mathcal{A}$. Nonetheless, although in the mathematical study we abstract from the intended interpretation, this interpretation certainly guides the choice of the questions in which we are interested.

Hilbert sought to establish classical platonist mathematics on a firm foundation by formalizing it and proving the consistency of the resulting formalism by finitist means. The interest of the question of consistency depends on the fact that the formulae of the system represent a system of statements; that is, even if the meanings of the platonist conceptions are highly indeterminate, statements in terms of them are introduced according to an analogy with "real" (i.e., finitist) statements which is intended to preserve at least the notions of truth and falsity and the laws of logic.

In fact, Hilbert had a further motive for his interest in consistency: the fact that platonist mathematics is an extension of an extrapolation from finitist mathematics. Certain elementary combinatorial notions are also embodied in the formalism; formulae involving them express "real statements." Hilbert thought of the other formulae as expressing "ideal statements"—analogous to the ideal elements of projective geometry—introduced to give greater simplicity and integration to the theory. Within the system they have deductive relations to the real statements. It would be highly undesirable that a formula of the system should be seen by elementary computation to be false and yet be provable. One might hope to prove by metamathematical means that this would not happen. In the central cases a proof of consistency is sufficient to show that it would not. Thus, suppose we extend a quantifier-free recursive number theory by adding quantifiers and perhaps also second-order quantifiers. A proof of the consistency of the resulting system will show that no false numerical formula (stating a recursive relation of particular integers) will be provable. In fact, it will yield a constructive proof of any formula of the original system provable in the extension, in this sense showing the use of "ideal" elements to be eliminable. Since Hilbert it has been pointed out (chiefly by Georg Kreisel) that many further results relevant to the understanding of nonconstructive mathematics from a constructivist point of view can be obtained from consistency proofs.

Hilbert hoped to settle the question of foundations once and for all, which for him meant establishing the platonist methods of set theory on a firm basis. His hope was founded on two expectations: that all of mathematics (at least all of analysis) could be codified in a single formal system and that the consistency of this system could be proved by methods so elementary that no one could question them. He was disappointed of both these expectations as a result of Gödel's incompleteness theorems (1931). Work on the program has nonetheless continued, with the limitations that one has to work with formalisms which embody only part of the mathematics in question and that the proofs must rely on more abstract, but still constructive, notions; and the work in finitist proof theory has achieved valuable results, some of which will be discussed later.

§4. MATHEMATICAL LOGIC

Our remaining considerations on the subjects of the two preceding sections fit best into an independent discussion of mathematical logic as a factor in the study of the foundations of mathematics. Before World War II an important part of the work in logic was directed toward establishing, in the service of some general position such as logicism or intuitionism, a more or less final solution to the problems of foundations. Certain particular results, and probably also a more diffuse evolution of the climate of ideas, have discouraged this aim. Today nearly all work in mathematical logic, even when motivated by philosophical ideas, is nonideological, and everyone acknowledges that the results of this work are independent of the most general philosophical positions.

Starting from the axiomatic method in a more general sense, mathematical logic has become the general study of the logical structure of axiomatic theories. The topics selected from the great variety of technical developments for discussion here are Gödel's incompleteness theorems, recursive function theory, developments related to Hilbert's program, foundations of pure logic, and axiomatic set theory.

4.1. GÖDEL'S INCOMPLETENESS THEOREMS. Research in mathematical logic took quite new directions as a result of the discovery by Kurt Gödel, in 1930, of his incompleteness theorems. According to the first theorem (as strengthened by J. B. Rosser in 1936) any formalism *S* that is sufficiently powerful to express certain basic parts of elementary number theory is incomplete in the following sense: A formula $\mathscr{A}$ of *S* can be found such that if *S* is consistent, then neither $\mathscr{A}$ nor $\neg\mathscr{A}$ is provable in *S*. The conditions are satisfied by very weak systems, such as the first-order theory *Q* whose axioms are the Peano axioms for the successor function and the recursion equations for addition and multiplication. (This system is formalized in first-order logic with equality, having successor, addition, and multiplication as primitive function symbols. The axioms are versions of our axioms (1)–(4), recursion equations for addition and multiplication, and an axiom which says that every number not equal to 0 is the successor of something.) They are satisfied by extensions of systems that satisfy them and therefore by the full elementary number theory *Z* (the first-order version of the Dedekind-Peano axiomatization, obtained from *Q* by adding induction: in place of the second-order axiom (5) one adds all results of substituting a predicate of the formalism for "*F*" in (7), by analysis, and by axiomatic set theories in which number theory can be constructed. They are also satisfied by formalizations of intuitionist theories. Evidently adding further axioms offers no escape from this incompleteness, since the new theories will also satisfy the conditions of the theorem.

One of the conditions necessary for some general statements of the theorem is that which we mentioned earlier, that proofs can be checked mechanically. This must be interpreted more precisely in terms of one of the concepts of recursive function, discussed below.

The technique of Gödel's proof is of great interest and has since found wide application. It consists of a mapping of the syntax of the theory into the theory itself, through assigning numbers to the symbols and formulae of the system. Any syntactical relation will then be equivalent to some relation of natural numbers. For the crucial relation "$\mathscr{X}$ is a proof in S of the formula $\mathscr{A}$" the corresponding relation $P(x,a)$ can be expressed in the theory, and certain things about it can be proved in S. Then the undecidable formula $\mathscr{A}$ is a formula which has a number k such that what $\mathscr{A}$ says (about numbers) is equivalent to the unprovability of the formula number k, i.e., $\mathscr{A}$. (1) Then if only true formulae are provable, $\mathscr{A}$ is unprovable. But then $\mathscr{A}$ is true. Therefore, (2) by the same assumption $\neg\mathscr{A}$ is also unprovable. This appeal to the notion of truth was replaced in Gödel's detailed argument by the condition that S be consistent for (1) and ω-consistent for (2). By changing the formula Rosser showed that the assumption of ω-consistency could also be replaced by that of consistency.

The proof that if S is consistent, then $\mathscr{A}$ is unprovable is finitist. If S and the mapping of its syntax into S satisfy some further conditions, the argument can be formalized in S. This yields the second theorem of Gödel. If S is consistent, then the formula which, under the above mapping, corresponds to the consistency of S is unprovable in S.

The first theorem implies not only that mathematics as a whole cannot be codified in a single formal system but also that the part of mathematics that can be expressed in a specific formal notation cannot be so codified. This fact undermines most attempts at a final solution to the problem of foundations by means of mathematical logic. The second theorem was a blow to the Hilbert program in particular. The methods that the Hilbert school envisaged as finitary could apparently be codified in first-order number theory *Z;* indeed, that they can be so codified seems fairly certain, even though the notion of finitary methods is not completely precise. Therefore, not even the consistency of *Z* is provable by finitary means. Moreover, the consistency of stronger and stronger systems requires stronger and stronger methods of proof.

There has been much discussion of the broader philosophical implications of Gödel's theorem. We shall not enter into the discussion of such questions as whether the theorem shows the falsity of any mechanistic theory of mind. It should be remarked that there are a number of connections between the surpassing of any given formal system by possible means of proof and the inexhaustibility phenomena in the realm of mathematical existence. Gödel's argument can be viewed as a diagonal argument parallel to that by which Cantor proved that no countable set of sets of natural numbers can exhaust all such sets. Peano's axioms are categorical if the range of the quantifiers in the induction axiom (5) includes all classes of natural numbers, but in the context of a formal system one can use only the fact that induction holds for classes definable in the system, of which there are only countably many. In set theory the addition of axioms asserting the existence of very large classes can make decidable previously undecidable arithmetical formulae.

4.2. RECURSIVE FUNCTION THEORY. A number of problems in mathematical logic require a mathematically exact formulation of the notion of mechanical or effective procedure. For most purposes this need is met by a concept of which there are various equivalent formulations, arrived at by several writers. The concept of (general) recursive definition, introduced in 1931 by Jacques Herbrand and Gödel, was the first. A function of natural numbers which is computable according to this conception (the "computation" consists of the deduction of an evaluation from defining equations by simple rules) is called a general recursive, or simply a recursive, function. Other formulations are that of λ-definability (Alonzo Church), computability by Turing machine (A. M. Turing), algorithms (A. A. Markov), and different notions of combinatorial system (Emil Post and others).

The concept of recursive definition has proved essential in decision problems. Given a class of mathematical problems defined by some parameter, is there an effective algorithm for solving each problem in the class? As an example consider the tenth problem of Hilbert: Given a polynomial with integral coefficients, is there a general method that tells us whether it has a zero among the integers? If such a question can be resolved in the affirmative, the resolution can generally be reached on the basis of the intuitive conception of an algorithm: If one can invent the procedure, then it is generally clear that the procedure is effective. But to give a negative answer to such a question one needs some idea of the possible effective procedures. The development of recursive function theory has made possible a large number of results asserting the nonexistence of decision procedures for certain classes of problems. This way of interpreting the results depends on a principle known as Church's thesis, which says that the mathematical conception of an effectively computable function in fact corresponds to the intuitive idea—i.e., that a number-theoretic function is (intuitively) effectively computable if and only if it is recursive.

An important type of decision problem is that concerning provability in formal systems. Given a formal system S, is there an algorithm for deciding whether a given formula $\mathscr{A}$ is a theorem of *S*? If there is, then *S* is said to be decidable. Although quite interesting examples of decidable systems exist, the systems to which Gödel's first incompleteness theorem applies are undecidable. In fact, Gödel's type of argument can also be used to prove that first-order logic is undecidable (as by Church in 1936).

Another important aspect of recursive function theory is the classification of sets and functions according to different principles related to recursiveness. One such principle, stated in terms of the complexity of possible definitions by recursive predicates and quantifiers (the Kleene-Mostowski hierarchy), not only is of wide application in logic but is closely related to older topological classifications. One can single out the arithmetical sets (those sets definable from recursive predicates by quantification over natural numbers alone), the hyperarithmetical sets (a certain transfinite extension of the arithmetical hierarchy—in effect, those sets definable in ramified analysis with levels running through the recursive ordinals), and the analytic sets (those sets definable from recursive predicates by quantification over numbers and functions, or sets, of natural numbers). The recursive ordinals, singled out by Church and Kleene, can most readily be characterized as the order types of recursive well-orderings of the natural numbers.

The theory of recursive functions is evidently valuable for explicating different notions of constructivity and for comparing classical and constructive mathematics. A constructive proof of a statement of the form "$(x)(\exists y)Fxy$" should yield an effective method of obtaining y from x. For example, Kleene and his collaborators have shown that any statement provable in formalized intuitionist number theory and analysis has a property called "realizability," which amounts roughly to interpreting "$(x)(\exists y)Fxy$" as asserting the existence of a recursive function giving y in terms of x. Although it is also intuitionistically meaningful, the construction gives a classical interpretation of the intuitionist formalisms. It also allows a sharpening and extension of Brouwer's counterexample technique. Certain classically provable formulas can be shown not to be realizable and therefore not to be provable in the intuitionist formalisms Kleene considers.

A problem arises with regard to the relation between the concept of recursive function and the fundamental concepts concerning constructivity—for instance, the concept of intuitionism. One cannot interpret Church's thesis as explicitly defining "effectively computable function" and therefore as giving the meaning of the intuitionist quantifiers. For by definition a function is general recursive if there is a set of equations from which for each possible argument one can compute the value of the function for that argument, a statement of the form "$(x)(\exists y)Fxy$." If this is interpreted constructively, the proposed definition is circular. The relation between "function constructively proved to be everywhere defined" and "general recursive function" is still not clear. One can ask whether every intuitionistically everywhere-defined number-theoretic function is general recursive or whether every (classically) general recursive function can be proved constructively to be such. Neither question has yet been resolved.

4.3. DEVELOPMENT OF THE HILBERT PROGRAM. For the study of constructivity it is also important to study more restricted types of recursive definition that can be seen by definite forms of argument to define functions. This is particularly important for the extended Hilbert program.

Gödel's second incompleteness theorem meant that the consistency even of elementary number theory Z could not be proved by the methods envisaged by Hilbert. A number of consistency results of the sort envisaged by Hilbert have since been obtained by stronger constructive methods. Gödel and Gentzen proved independently (and finitistically) that if intuitionistic first-order arithmetic is consistent, then so is classical first-order arithmetic. The proofs were based on a quite simple method of translating classical theories into intuitionist theories which is of wide application—for example, to pure logic. One renders an atomic formula P by $\neg\neg P$ (in elementary number theory, equivalent to P itself). If $\mathscr{A}$, $\mathscr{B}$ are translated into $\mathscr{A}°$, $\mathscr{B}°$, respectively, then $\mathscr{A} \vee \mathscr{B}$ is translated by $\neg\neg(\mathscr{A}° \vee \mathscr{B}°)$, $(\exists x)\mathscr{A}$ by $\neg\neg(\exists x)\mathscr{A}°$, $\mathscr{A} \supset \mathscr{B}$ by $\neg(\mathscr{A}° \ \& \neg\mathscr{B}°)$, $\mathscr{A} \ \& \ \mathscr{B}$ by $\mathscr{A}° \ \& \ \mathscr{B}°$, $\neg\mathscr{A}$ by $\neg \mathscr{A}°$, and $(x)\mathscr{A}$ by $(x)\mathscr{A}°$. Evidently the translation not only proves relative consistency but also gives each provable formula an intuitionist meaning according to which it is intuitionistically true. If $\mathscr{A}$ is a quantifier-free formula of number theory, or if it is composed with conjunction, negation, and universal quantification only, then if it is provable in Z, it is intuitionistically provable. This translation can easily be extended to ramified analysis. Since intuitionistically the consistency of the intuitionist systems follows from their soundness under the intended interpretation, the consistency of the classical systems has been intuitionistically proved.

A sharper result was obtained in 1936 by Gerhard Gentzen. New proofs, with various advantages and refinements, have since been found by several workers.

Gentzen proved the consistency of Z by adding to finitist arithmetic the assumption that a certain recursive ordering of natural numbers, of order type ϵ_0 (the least ordinal greater than $\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots$), is a well-ordering. This assumption could be proved in intuitionist ramified analysis using set variables only of level 1 but could not in elementary number theory.

Gentzen's result has made it possible to extract further information about the power of elementary number theory. Kreisel obtained information about the relation between elementary number theory and certain quantifier-free arithmetics and also obtained a characterization of the functions which can be proved in Z to be general recursive.

A corresponding result for ramified analysis for finite levels was obtained by Lorenzen in 1951 and sharpened by Kurt Schütte. It was extended by Schütte to transfinite levels.

On the basis of these results we can say that constructive consistency proofs are available for all of predicative mathematics. In well-defined senses they are the best possible results (for instance, the above-mentioned ordinal ϵ_0 cannot be replaced by a smaller one). Nonetheless, efforts to give such a proof for impredicative classical analysis, not to speak of axiomatic set theory, have proved fruitless.

Results of quite recent research have shed considerable light on this situation. Clifford Spector (1962) proved the consistency of classical analysis relative to a quantifier-free theory (Gödel 1958) of primitive recursive functionals of arbitrary finite types, enriched by a new schema for defining functionals by "bar recursion." This amounted to generalizing Brouwer's bar theorem to arbitrary finite types. Such generalized bar recursion has not found a constructive justification, but the method has led to consistency proofs by the original bar theorem for subsystems of analysis which are, according to a reasonable criterion, impredicative.

Kreisel (1963) has shown that intuitionist analysis, with the bar theorem and a strong schema of "generalized inductive definitions" included, does not suffice to prove the consistency of classical analysis. Such a proof requires an essential extension of constructive methods beyond the established intuitionist ones.

Solomon Feferman and Schütte have given an analysis of the notion of predicativity according to which established intuitionist methods go beyond predicative ones. According to their conception, inductive definitions such as that of the class O of numbers representing the recursive ordinals are impredicative.

What has been the fate of the Hilbert program? Put most broadly, its objective was to secure the foundations of platonist mathematics by a constructive analysis of classical formal systems. The incompleteness phenomena have made it impossible, in dealing with stronger and stronger systems, to avoid the introduction of more and more abstract conceptions into the metamathematics. However interesting the information obtained about the relation between these conceptions and the platonist ones, it is not evident that these conceptions are in all respects more secure. Moreover, in the present state of research it is not certain that strong enough constructive methods can be found even to prove the consistency of classical analysis.

This state of affairs is unfavorable to those methodological views seeking to restrict mathematics to the methods which have the greatest intuitive clarity. It is evident that such methods will not suffice to resolve certain mathematical questions whose content is extremely simple, namely those concerning the truth of certain statements of the form "$(x)Fx$," where "F" stands for a primitive recursive predicate of natural numbers. Proponents of the views in question seem forced to admit that even such questions can be objectively undetermined.

4.4. FOUNDATIONS OF LOGIC. An important result concerning pure logic obtained in finitist metamathematics is a theorem, or cluster of related theorems—including Herbrand's theorem (1931) and Gentzen's theorem (1934)—to the effect that the proof of a formula of first-order logic can be put into a normal form. In such a normal-form proof the logical complexity of the formulae occurring in the proof is in certain ways limited in relation to the complexity of the conclusion; for instance, no formula can contain more nested quantifiers than the conclusion. The proof is, as it were, without detours, and *modus ponens* is eliminated. As a consequence, a quantifier-free formula deduced from quantifier-free axioms can be proved by propositional logic and substitution, which implies all the consistency results proved by the Hilbert school before the discovery of Gödel's theorem. Gentzen's theorem also applies to intuitionist logic and to other logics, such as modal logics.

These theorems, which are the fundamental theorems of the proof theory of quantification theory, are closely related to the fundamental theorem of its semantics, Gödel's completeness theorem. Every formula not formally refutable has a model—in fact, a model in which the quantifiers range over natural numbers; i.e., there are denumerably many individuals. This can be strengthened to the following: If S is any set (finite or infinite) of formulae of first-order logic, it has a denumerable model unless some finite subset of S is inconsistent—that is, unless the conjunction of the subset's members is formally refutable (Skolem-Löwenheim theorem).

This theorem has some quite startling consequences: in particular, it applies if S is the set of theorems of some system of set theory. Then if the system is consistent, S has a denumerable model even though S may contain a theorem which asserts.the existence of nondenumerable sets. That is not a contradiction: If n represents a nondenumerable set in the model, there will indeed be only countably many m's such that $m \in n$ is true in the model, but the assertion "n is nondenumerable" will be true in the model because the model will not contain an object representing the function that enumerates the objects m for which $m \in n$ is true in the model. The model is denumerable only from "outside."

This is an example of a model which is nonstandard in that it differs in some essential way from the intended one. The Skolem-Löwenheim theorem also implies the existence of nonstandard models for systems of number theory. In fact, there is a nonstandard model even for the set S of all true formulae of elementary arithmetic. The number sequence cannot be characterized up to isomorphism by any countable set of first-order formulae.

The existence of denumerable models of set theory illustrates how essential the platonist conception of set, particularly of the set of subsets of a given set, is to set theory. If there is no more to the platonist conception than is specified in any particular formal system, then apparently the cardinal number of a set cannot be objectively determined. Indeed, the cardinal number of a set depends on what mappings there are and therefore on what sets there are.

The acceptance of this relativity has been urged by many, including Skolem. A fully formalist conception would give rise even to the relativity of the natural numbers themselves.

The completeness theorem and the construction of nonstandard models are fundamental tools in a now rapidly developing branch of logic called model theory. This subject can be viewed as a development of logical semantics, but what is perhaps distinctive about the point of view underlying recent work is that it regards a model of a formal theory as a type of algebraic structure and, in general, that it integrates the semantic study of formal systems with abstract algebra. Model theory takes mathematical logic a long way from the philosophical issues with which we have been mainly concerned, in particular by taking for granted a strong form of platonism. The leaders of this development have, in fact, emphasized the application of metamathematical methods to problems in ordinary mathematics.

There are other investigations concerning the foundations of pure logic. For example, we have mentioned that there can be no decision procedure for quantification theory. Nonetheless, there is interest in the question of what subclasses of formulae are decidable. As a striking result in this direction we might mention the proof of A. S. Kahr, E. F. Moore, and Hao Wang (1962) that the existence of models of formulae of the form "$(x)(\exists y)(z)M(x,y,z)$" (or, equivalently, the provability of formulae of the form "$(\exists x)(y)(\exists z)M(x,y,z)$" where "$M(x,y,z)$" is an arbitrary quantifier-free formula, is undecidable. The development of appropriate concepts of model and completeness proofs for modal logics and intuitionist logic has come to fruition in recent years. In the case of the completeness of intuitionist logic, the situation is unclear. E. W. Beth (1956) has given a construction of models in terms of which he proves classically the completeness of intuitionist quantification theory. On the other hand, Kreisel has shown that the completeness of intuitionist logic cannot be proved by methods available in present intuitionist formal systems and, indeed, that it is incompatible with the supposition that all constructive functions of natural numbers are recursive.

4.5. AXIOMATIC SET THEORY. We shall not undertake here to survey the different axiomatic systems of set theory. We shall, however, mention some developments in the metamathematics of set theory, developments concerning the axiom of choice and Cantor's continuum problem.

The axiom of choice asserts (in one formulation) that for every set A of nonempty sets no two of which have a common element, there exists a set B which contains exactly one element from each of the sets in A. This axiom became prominent when Zermelo used it in 1904 to prove that every set can be well-ordered. Although it was much disputed, it came to be applied more and more, so that entire theories of modern abstract mathematics depend essentially on it. Naturally the question arose whether it was provable or refutable from the other axioms of various systems of set theory. A. A. Fraenkel (1922) showed that it could not be proved from Zermelo's axioms, provided that the axioms allowed individuals—that is, objects which are not sets—in the range of the quantifiers.

The continuum problem appears to be an elementary problem in the arithmetic of cardinal numbers: Is there a cardinal between $\aleph_0$, the cardinal of the integers, and $2^{\aleph_0}$, that of the continuum; stated otherwise, does the continuum contain subsets of cardinal number different from that of the continuum and that of the integers? If the answer is negative, then $2^{\aleph_0} = \aleph_1$, the first cardinal larger than $\aleph_0$, and the cardinal of the first noncountable well-ordering. Cantor's conjecture that $2^{\aleph_0} = \aleph_1$ is called the continuum hypothesis.

Gödel, in 1938, proved that the axiom of choice and a generalization of the continuum hypothesis are consistent with the other axioms. The argument applies to a number of different systems, including the Zermelo-Fraenkel system (ZF). What is proved (finitistically) is that if, say, ZF is consistent, it is likewise consistent with a new axiom, the axiom of constructibility, which implies the axiom of choice and the generalized continuum hypothesis. For the constructible sets, which are the sets obtained by extending the ramified hierarchy of types through all the ordinals, can be proved in the system to satisfy all the axioms plus the axiom of constructibility, which says that every set is constructible. In terms of models, any model of ZF contains a subclass that is a model in which all sets are constructible. The constructible sets are of interest on their own account; Gödel has remarked that the idea behind them is to reduce all impredicativities to one special kind, the existence of large ordinals. However, he does not consider the axiom of constructibility plausible.

Thus, it has been known for some time that the axiom of choice and the continuum hypothesis are not refutable from the other axioms. More recently, Paul J. Cohen proved that they are not provable either. That is, if, say, ZF is consistent, it remains so by adding the negation of the axiom of choice or by adding the axiom of choice and the negation of the continuum hypothesis. Starting from Gödel's ideas, Cohen developed a quite new method for constructing models, which has led very quickly to a large number of further independence results.

The situation with respect to the axiom of choice and the continuum problem raises anew the question of how definite our idea of a set is, whether or not such a question as the continuum problem has an objectively determinate answer. Most mathematicians today find the axiom of choice sufficiently evident. But the continuum hypothesis—perhaps because of its more special character and because of the fact that the analogy of the infinite to the finite on which the conception of the set of all subsets of a given set is based does not suggest a justification of it—is left much more uncertain by considerations of intuitive evidence or plausibility. The role of the Skolem-Löwenheim theorem in Gödel's and Cohen's constructions might encourage the idea that the continuum hypothesis is in fact undetermined. Gödel himself believes that it is false and hopes that an axiom will be found which is as evident as the axiom of choice and which suffices to refute the continuum hypothesis. At present no one seems to have a good idea of what such an axiom would be like. It would have to be of a different character from the usual strong axioms of infinity, to which the method of Gödel's consistency proof applies.

The question of the continuum hypothesis is thus very close to the general epistemological question concerning platonism. If the general conceptions of set and function are given in some direct way to the mind, if, to echo René Descartes, the idea of the infinite is in one's mind before that of the finite, there is no reason to expect a comparatively simple question like the continuum problem to be unanswerable. If, on the other hand, the platonist conceptions are developed by analogies from the area where we have intuitive evidence, if they are "ideas of reason" which, without having an intuition corresponding to them, are developed to give a "higher unity" which our knowledge cannot obtain otherwise, then it would not be particularly surprising if the nature of sets were left indeterminate in some important respect and, indeed, could be further determined in different, incompatible ways.

SUPPLEMENT (2005)

The period since 1967 has seen considerable work in all areas of the foundations of mathematics. This is most notable on the mathematical side. These developments will be discussed before turning to philosophical work.

§5. MATHEMATICAL LOGIC

Of the extensive work since the 1960s, that dealing with formalized axiomatic theories is most central to the foundations of mathematics, although there might now be more debate than earlier about the centrality of the axiomatic method. For some time mathematical logic has been divided into Proof theory, Model theory, Computability (recursion) theory, and Set theory (see the entries on those subjects), although of course there are important interconnections. Model theory and computability theory are more purely mathematical, although their methods are important for the other two areas, and some applications (such as nonstandard analysis) are of foundational interest.

One upshot of work in Proof theory is that strong subsystems of classical analysis (second-order arithmetic) have been analyzed by means that are in some sense constructive but much more powerful and abstract than was envisaged in the early history of the subject. A possibly clearer foundational gain was achieved by another proof-theoretic program, which can trace its roots to Hermann Weyl's (1918) attempt to reconstruct classical analysis predicatively. The work of Harvey Friedman, Stephen Simpson, and others, surveyed in Simpson (1998), showed that many standard theorems of analysis (and of other branches of mathematics) can, if suitably formulated, be proved in weak systems. The method of Reverse mathematics (q.v.) made it possible to calibrate exactly what axiomatic power was needed to prove a particular theorem.

The most striking developments have been in set theory, where Paul Cohen's proof in 1963 of the independence of the axiom of choice and the continuum hypothesis touched off an explosion of research. Cohen's method of forcing proved of wide applicability. In the following years, many more independence results were found in all areas of set theory and its applications. In particular, many classical conjectures were shown both consistent with and independent of the standard axiom system ZFC (or ZF in cases where the axiom of choice sufficed to prove a statement).

This body of work might suggest to a philosopher a vast indeterminacy in the concept of set or of the universe of sets, a random-seeming collection of logical relations among statements independent of ZF or ZFC. However, there is more order than this picture would suggest. The existence of important independent statements would suggest seeking new axioms, and in fact progress has been made by developing the consequences of two kinds of new axioms: strong axioms of infinity (axioms asserting the existence of certain large cardinals) and special cases of the axiom of determinacy.

The large cardinal axioms that have been studied have turned out to be linearly ordered by consistency strength (see §6 of the entry on Set theory), and this has made it possible to determine the consistency strength of other independent statements. In particular this is true of the game-theoretic axiom of determinacy. The assumption PD that the latter holds for projective sets of real numbers (roughly those definable by quantification over reals) implied solutions to the classical problems of descriptive set theory, the study of these sets. PD (and more) was shown to follow from strong large cardinal axioms.

Although this result left the continuum problem untouched, it did show that a program of investigating new axioms along lines proposed by Kurt Gödel in the 1940s could settle an important class of open problems. The large cardinal axioms implying PD have the desirable feature that their consequences in second-order arithmetic cannot be altered by forcing. W. Hugh Woodin's (2001) approach to the continuum problem (see §6 of the entry on Set theory) aims to extend this result to a higher level. But it is not regarded even by Woodin himself as a definitive solution, and even the question whether the continuum hypothesis has a determinate truth-value remains open.

§6. APPROACHES TO PHILOSOPHY OF MATHEMATICS

In 1967 philosophy of mathematics was largely ancillary to logic, and discussion centered either on logical results or on the earlier foundational programs that had contributed to the development of mathematical logic. Since then it has become more a subject in its own right. It has been influenced by the general tendencies moving the philosophy of science away from logic. In particular, historical studies have assumed a larger role, and many such studies have been of developments not close to logic.

In the earlier entry, the philosophical problems discussed concern the analysis of basic mathematical concepts (such as natural number) and the identification and justification of mathematical principles. The term *foundations* naturally suggests that focus. But the philosophy of mathematics can and does contain inquiries of other kinds. It has been charged with concerning itself only with elementary mathematics. This charge is not correct; for example, identifying the axioms required for conclusions in set theory is a matter of high-level mathematical research, and in general the justification of axioms is not independent of knowledge of the theories developed from them.

But it is true that an inquiry into basic concepts and principles will be selective in its attention to the elaboration of mathematics in current and earlier research. And one may well seek philosophical understanding of aspects of mathematical practice of a different kind. One influential strand of work of this kind is that inaugurated by Imre Lakatos, particularly in his book *Proofs and Refutations* (1976). Lakatos studied a classic theorem of Leon-hard Euler (1707–1783) relating the number of vertices, faces, and edges of a polyhedron and brought to light difficulties that had been found with proofs of it over a period of time and the refinements of the statement of the theorem that had resulted. An underlying idea was that mathematical knowledge is more fallible than a certain traditional picture has it, for a different reason from those that might be suggested by difficulties with basic principles. For reasons of space, this sort of inquiry will not be pursued here, but it should be recognized that this strand of philosophy of mathematics has grown relative to the whole since 1967.

§7. LOGICISM AND THE NEO-FREGAN PROGRAM

In §2, much attention is paid to the project of reducing arithmetic to logic and the analysis of number. Logicism in its earlier forms has not been revived, but a kind of neologicism has become an active program. It was observed that the axioms of arithmetic could be derived in second-order logic from the criterion (7) in §2.6, with numerical equivalence defined as in (8). (This is briefly sketched after (12), but the most difficult case, the proof that every natural number has a successor, is omitted.) (7) thus formulated has come (misleadingly) to be called Hume's principle (HP). The second-order theory with the number operator N_xFx and HP as a nonlogical axiom is called Frege arithmetic (FA). In 1983 Crispin Wright gave the proof that the Dedekind-Peano axioms of second-order arithmetic are provable in FA using Frege's definitions, but this was in essentials proved by Gottlob Frege and has come to be called Frege's theorem. Intuitively, Frege uses the definition of N_xFx in terms of extensions only to derive HP, and then the work is done by that principle. Richard G. Heck Jr. showed in 1993 that this was essentially true of Frege's proofs in *Grundgesetze*. Several logicians showed that FA is consistent if second-order arithmetic is.

Wright's neo-Fregean proposal is to take FA as basic arithmetic. It is a logical construction of arithmetic only if the notion of cardinal number is a logical notion and HP is a principle of logic. As a proof that arithmetic is a part of logic the construction seems to be question-begging. Still, it generated a lot of discussion by Wright and others of the status of abstraction principles like HP, which take an equivalence relation of entities of one kind as a criterion of identity for entities of another kind. Wright's initial idea seems to have been that HP is something close to a definition, although it is not an explicit definition and does not meet the usual standard for a contextual definition, that it should enable the term introduced to be eliminated by paraphrase of contexts in which it occurs. A fatal difficulty for this idea is that HP can be true relative to a domain of individuals only if the domain is infinite. Wright and his collaborators continued to argue that HP is analytic. Others have doubted that a principle that implies the existence of an infinite sequence of objects could be analytic. Another difficulty is that Frege's inconsistent axiom V is an abstraction principle, and other abstraction principles that seem plausible are either inconsistent or can be satisfied only in a finite domain.

The program of axiomatizing parts of mathematics by abstraction principles is of independent logical interest, and work has been done on analysis, and preliminary work on set theory. Kit Fine (2002) carried out an extensive analysis of abstraction principles, to distinguish those that introduce inconsistency from those that do not.

§8. PLATONISM

Since World War II, the view that classical mathematics is seriously threatened by the known paradoxes or by other unknown ones has virtually disappeared. Platonism as described in §3 has been widely accepted as a mathematical method. Taking the language of classical mathematics at face value, as implying the existence of abstract mathematical objects, even forming uncountable and still larger totalities, and allowing reasoning using both the law of excluded middle and impredicative definitions, is probably a default position among philosophers and logicians. This can be called default platonism. It is in relation to such a view, whether accepting it or rejecting it, that much of the work in the philosophy of mathematics since 1967 has concentrated on ontological problems. How might this position be rejected?

§9. CONSTRUCTIVISM

In §3.2, platonism is contrasted principally with constructivism. Intuitionism and other forms of constructivism did not accept the reasoning characteristic of classical mathematics, in the case of intuitionism the law of excluded middle.

A significant development in this area is the argument in favor of intuitionist logic based on considerations of the philosophy of language presented by Michael Dummett (1973). This has, however, had more influence on discussions of realism as a general philosophy than on the foundations of mathematics specifically. Important metamathematical work on intuitionistic theories was done especially in the 1960s and 1970s. An important development is the development of intuitionistic-type theories that are of much greater expressive power than traditional intuitionistic theories. That of Per Martin-Löf (1984) is the most developed. But although intuitionistic logic has proved to have wide application, intuitionism has declined significantly as a general approach to mathematics, competing with classical mathematics. Another constructive approach to mathematics, pioneered by Errett Bishop (1967), has been developed by several mathematicians. Although it has been more active in the last generation than intuitionism, philosophers have been more interested in the latter, perhaps justifiably because what is philosophically interesting about the Bishop approach is shared with intuitionism, and L. E. J. Brouwer and other intuitionists did more to develop philosophical arguments for their position.

§10. NOMINALISM

The term *platonism* is also used so that the view contrasts with nominalism. Since 1980 or so that opposition has been more prominent among philosophers, especially in North America. This is perhaps fundamentally due to the great influence of scientific naturalism on all theoretical parts of philosophy.

The traditional way in which nominalism rejects default platonism is by not taking the language of mathematics at face value and seeking to paraphrase it in such a way that commitment to abstract mathematical objects is avoided. Programs of this kind have been pursued especially since the 1980s, but it has proved essential to enlarge traditional nominalist resources in at least one of two ways: allowing points and possibly regions of space-time as physical or allowing modality. It is then possible to reconstruct a considerable amount of classical mathematics, at least if one accepts a controversial thesis of George Boolos (1998) that his reading of the language of monadic second-order logic by means of the English plural does not involve commitment to such entities as sets, classes, concepts, or pluralities. What has been achieved in this sort of reconstruction is surveyed in John P. Burgess and Gideon Rosen, *A Subject with No Object* (1997).

A bolder proposal was made by Hartry H. Field (1980, 1989): Where he parted from default platonism was in rejecting the view that statements of classical mathematics, taken at face value with regard to meaning, are true and even that mathematics aims at truth. He sought to account for the apparent objectivity of mathematics by viewing it instrumentally, as a device for making inferences within scientific theories. The role of truth is taken over by conservativeness: Given a nominalistic scientific theory T, a mathematical theory M is conservative if adding its resources to those of T does not enable the derivation of conclusions in the language of T that were not already derivable. This committed him to giving nominalistic versions of scientific theories, and (with the previously mentioned assumption about points and regions of space-time) he was able to give such a version of the Newtonian theory of gravitation. Difficulties stand in the way of carrying out this program for modern physical theories.

§11. STRUCTURALISM

Two related intuitions about modern mathematics are widely expressed: that it is the study of (abstract) structures and that mathematical objects have no more of a nature than is expressed by the basic relations of a structure to which they belong. The structuralist view of mathematical objects is a development of the second intuition. Its relation to default platonism is ambiguous. Some versions, which can be called eliminative structuralism, reject one part of that view, taking the language of mathematics at face value, by proposing paraphrases that eliminate reference to mathematical objects or at least to the most typical mathematical objects. Others take the structuralist idea as an explication of what the reference to objects in standard mathematical language amounts to. This noneliminative type of structuralism offers an ontological gloss on default platonism rather than a modification or rejection of it.

A simple case of an eliminative structuralist analysis is a translation of the language of second-order arithmetic into that of pure second-order logic. Suppose A is a sentence of second-order arithmetic. Since arithmetical operations such as addition and multiplication are second-order definable, it can be assumed that A contains as only primitives N (natural number), S (successor), and 0. The structure of the natural numbers is characterized by a second-order sentence with these primitives, the conjunction P of these axioms. If A is provable, the sentence $P \to A$ is provable by pure logic. If A is true, it is valid in the standard semantical sense. One can regard $P \to A$ (or the result of replacing N, S, 0 by variables) as a translation of A that eliminates reference to numbers. The translation has the difficulty that if there is no structure satisfying the axioms, then $P \to A$ and $P \to \neg A$ are both vacuously true. The translation seems to presuppose that P is satisfiable.

One version of structuralism would allow sets as basic objects. This would be a natural way of developing the first intuition, understanding structures as set-theoretic constructs. But a general structuralist view of mathematical objects would naturally aim not to exempt sets from structuralist treatment. At this point modality has been introduced. In the previous example, the assumption that it is possible that there are N, S, and 0 satisfying P is sufficient, since $P \rightarrow A$ can be strengthened to $\Box(P \rightarrow A)$. The modal structuralism of Geoffrey Hellman (1989) is a version of eliminative structuralism relying on this idea. It includes a detailed treatment of set theory. (An approach had been sketched earlier by Hilary Putnam [1967].)

What these constructions accomplish depends on the status of second-order logic, a question that arises also for the neo-Fregean program and for nominalism. Concerning this there has been much debate. Regarding set theory, there is the additional problem that the presupposition of the possibility of the structure is of a structure of such large cardinality that it could not be witnessed by objects that are in any sense concrete or physical, so that the claim of the construction to eliminate reference to mathematical objects can be questioned.

Other versions of structuralism are suggested by remarks of Willard Van Orman Quine (1969) and of some earlier writers. Noneliminative structuralisms have been worked out in some detail by Michael D. Resnik (1997), Stewart Shapiro (1997), and Charles Parsons (1990). Concerning these views, there is debate about the status of structures, as well as about questions about identity.

§12. ROBUST PLATONISM?

A more robust type of platonism is expressed in Gödel's remark that "the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false" (1964, p. 260). Such a view would be supported by whatever general considerations support philosophical realism. But something more is demanded, a certain clarity and unambiguity of set-theoretical concepts and quantification over sets. Gödel wished to argue that the continuum hypothesis (CH) must be either true or false, even though he was unable to determine which. What might reinforce his claims would be a development (such as the work of Woodin [2001]) that determines the truth-value of CH. However, the assumptions of such a result might then be incorporated into a less robust platonist view. Perhaps the greater value of Gödelian realism is as a regulative principle: one is more likely to find answers to mathematical questions if one assumes at the outset that there are answers to be found.

That decisive philosophical arguments can be given for such a realistic stance is unlikely. An alternative is to say that default platonism applied to mathematics as it develops represents the limit of what one should claim about the determinateness of the reality described by mathematical theories. This would be the application to mathematics of the naturalistic stance recommended by Quine in many writings, but without his privileging of empirical science. Such a view was advanced by Hao Wang (1974) and more recently by Penelope Maddy (1997).

Gödel's confidence in set-theoretic concepts has not been universally shared; in particular Solomon Feferman (1998, 1999) has defended a skeptical view, influenced by the earlier predicativist tradition.

§13. EPISTEMOLOGICAL PROBLEMS

In the 1967 entry, the epistemological discussion centered on the question whether mathematics can be shown to be *a priori*. It seems that there has been no decisive advance on this question, so others will be concentrated on here.

Paul Benacerraf (1973) raised in rather abstract terms a problem about mathematical knowledge: If default platonism is true, how can one have mathematical knowledge? One response would be to start from the fact that one evidently does have mathematical knowledge and then question the assumptions that generate the problem. One assumption made in Benacerraf's original formulation, the causal theory of knowledge, is relatively easy to reject. To demand a causal relation between objects referred to in a proposition for knowledge of that proposition seems to stack the deck in advance against abstract objects, and the causal theories that were current when he wrote have not stood up well in general epistemology. But one can see the problem in more general terms: Can one give an epistemology for mathematics that is naturalistic? The most fruitful approach might then be to examine actual mathematical knowledge and to consider what sort of explanation of it makes sense and whether it then meets some standard of naturalism.

No explicit program of this kind has been carried far. One place where one might naturally look for naturalistic explanation is psychology, and there has been a considerable amount of research on the development of concepts of number in young children. Although the questions are often framed in terms of the concept of set, it is not clear that that is essential or that ontology is at all central to the formulation of the problems. It can be argued that mathematical ontology only arises at a more advanced state of the development of mathematical competence than the children investigated have reached.

When one does consider even the mathematics taught in elementary college courses, then what one has to go on is history and the reflection of mathematicians (and sometimes philosophers) on the justification of their claims. That some basic statements and inferences are rationally evident seems an inescapable assumption. Examples would be simple logical inferences and the most elementary axioms of set theory, such as the pairing axiom. It does not mean that this evidence does not get crucial reinforcement from the development of theories based on these evident starting points or that the latter can never be revised in the light of the further development of knowledge. Other assumptions might become evident when an edifice of knowledge has been built up; that might be true of higher-level set-theoretic axioms such as power set and choice. What possible explanations of rational evidence would count as naturalistic is a question that has not been much explored. But now any grounds for holding that no acceptable explanation is possible would have to rely on *a priori* presuppositions.

A less abstract and perhaps more interesting epistemological question arises particularly for higher set theory. It is suggested by the indispensability argument mentioned earlier. Whatever one thinks of rational evidence in general, it is already diminished when one reaches the usual axioms for the mathematics applied in science, as is indicated by the issues about the law of excluded middle raised by Brouwer, and those about impredicativity raised by Poincaré (1908) and Weyl (1918, 1919). However, a long history of successful application convinces one, for example, that the classical mathematics of the continuum is necessary for science and at least as well established as basic physics itself. This is the claim made by the indispensability argument, and it had been suggested earlier by Bertrand Russell and then Gödel that axioms could derive their evident character from the theory they give rise to. Among the applications of mathematics, however, are those within mathematics. Gödel's view apparently was that much of mathematics (including some higher set theory) could be seen to be evident in an *a priori* way, not contaminated by evidence derived from application in empirical science. However, particularly in higher set theory axioms could obtain additional justification through the theories constructed on their basis, and such justification would be possible for stronger axioms, such as the stronger large cardinal axioms that have been proposed, where a convincing intrinsic justification is not available.

Gödel's view and the indispensability argument have in common that the justification of mathematical axioms can rest at least to a certain degree on their consequences. However, for Gödel this is compatible with the status of mathematics as rational knowledge independent of experience, whereas for the main proponents of the indispensability argument, Quine and Putnam (1971), it is not. The indispensability argument clearly runs out before higher set theory. Empirical science makes no use of it, and indeed it has been argued that from the proof theorist's point of view the mathematical theories that are applied in science are weak.

Since few are satisfied with intrinsic justifications for the strongest axioms of infinity, and little such justification is claimed for determinacy axioms, the accepted solution to the classical problems of descriptive set theory rests on assumptions whose justification depends on the theory they give rise to (see Martin 1998). The same would have to be admitted for any solution to the continuum problem that can be expected in the forseeable future.

§14. HISTORICAL STUDIES

Practically every aspect of the history of the foundations of mathematics has seen some intensive scholarly study in the period since 1967. With respect to Immanuel Kant, a decisive development was Michael Friedman's *Kant and the Exact Sciences* (1992), which integrated Kant's philosophy of mathematics with his philosophy of physics and gave the strongest version of the logical view of the role of intuition in mathematics pioneered by Evert Willem Beth (1959) and Jaakko Hintikka (1974). Younger scholars have followed up Friedman's work, often criticizing aspects of it. In particular they have explored the relation of Kant's thought about mathematics to the mathematics of his own time and earlier and to the philosophy of his immediate predecessors.

One strand of work on Frege, of which Boolos and Heck (see Demopoulos 1995) have been the leaders, has worked out perspicuously the mathematical content of Frege's work, particularly in *Grundgesetze*. Another strand has emphasized his conception of logic and how it differs from our own conception of logic. A third has drawn connections of Frege to nineteenth-century developments in mathematics, particularly geometry.

The foundations of mathematics as an object of special study arose from the revolution in mathematics in the nineteenth century, particularly developments in its second half: the rigorization of the methods of analysis, the beginning of set theory and of abstract methods, the rise of modern logic, and the role assumed early in the twentieth century by the paradoxes. Every aspect of this development has been the subject of scholarly study. The same holds of later developments such as Russell's logic, Brouwer's intuitionism, the Hilbert program, and the work of the Vienna Circle. Space does not permit describing this work, but in the bibliography selective references have been given.

See also Aristotle; Brouwer, Luitzen Egbertus Jan; Cantor, Georg; Carnap, Rudolf; Church, Alonzo; Constructivism and Conventionalism; Descartes, René; First-Order Logic; Frege, Gottlob; Geometry; Gödel, Kurt; Gödel's Theorem; Hilbert, David; Infinity in Mathematics and Logic; Intuitionism and Intuitionistic Logic; Kant, Immanuel; Knowledge, A Priori; Logic, History of; Logical Paradoxes; Mill, John Stuart; Modal Logic; Neo-Kantianism; Neumann, John von; Nominalism, Modern; Peano, Giuseppe; Poincaré, Jules Henri; Proof Theory; Quantifiers in Formal Logic; Quine, Willard Van Orman; Realism and Naturalism, Mathematical; Russell, Bertrand Arthur William; Second-Order Logic; Set Theory; Structuralism, Mathematical; Tarski, Alfred; Turing, Alan M.; Types, Theory of; Weyl, (Claus Hugo) Hermann; Whitehead, Alfred North; Wittgenstein, Ludwig Josef Johann.

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Charles Parsons (1967, 2005)

MATHER, COTTON

(1663–1728)

Cotton Mather, scholar, clergyman, and author, was the oldest son of Increase Mather, one of the leading figures in the Puritan theocracy in Massachusetts. The younger Mather was so precocious that he entered Harvard College at the age of twelve and was graduated at fifteen. Because he stammered, he felt unqualified to preach and therefore began to study medicine. After a few years, however, he overcame his speech handicap and became the assistant to his father at the Second Church, Boston. Ordained in 1685, he remained in the service of the Second Church for the rest of his life.

Mather was disappointed in many of the major quests of his life. Partly because he associated himself politically with the unpopular royal governor, Sir William Phips, partly because of the diminished prestige of the Puritan clergy, and partly because of his own often unpleasant personal qualities he lost the power to wield significant influence in public affairs. When he greatly desired to succeed his father, who retired in 1701 as president of Harvard College, he was not selected. Convinced that Harvard no longer represented the true Calvinist faith, he threw himself energetically into the foundation of Yale College, but its presidency was not offered to him until 1721, when he declined the position because of his age.

Mather's intellectual attitudes during his earlier years were extremely narrow, for he moved within the confines of a strict Puritan worldview; later, however, he became more tolerant of the differing beliefs of others. Finally, especially in his *Christian Philosopher* (1721), he moved close to the natural religion characteristic of the Age of Reason. He interpreted the theological doctrine of divine Providence in philosophical terms by asserting that the order of the universe was planned for man's good by an all-wise, all-good God. Man's appreciation of natural Beauty and his application of reason to observations drawn from nature are sufficient to prove the existence and beneficence of God. His scientific communications to the Royal Society of London led to his election as a fellow in 1713, one of the first Americans to be so honored. He was one of the earliest in the colonies to advocate inoculation against smallpox, and he ably defended his position in several pamphlets. The change in his mental attitude thus epitomizes the alteration in the intellectual life that pervaded his milieu.

Nowhere is this duality more apparent than in Mather's involvement in the witchcraft epidemic in Salem. He attempted to make a "scientific" study of the cases, but he came to the conclusion that they could be treated by prayer and fasting. He warned the judges in witchcraft trials to proceed very cautiously against the suspects and to be particularly careful in admitting "spectral evidence," yet in his *Wonders of the Invisible World* (1693) he argued that the verdicts in the Salem trials were justified. By 1700, however, he changed his mind about the fairness of the trials. In regard to the suspicion of witchcraft, as in other respects, Mather stood uneasily between traditional faith and the new scientific outlook.

See also Philosophy of Religion, History of; Scientific Method.

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