late propositional logic as a deductive preliminary, but he stated a generalized form of *modus ponens,* to the effect that a true proposition could be suppressed when it occurred as an antecedent or as part of a conjunction of antecedents in a theorem.

Peano had already obtained his five axioms of arithmetic, which contain the principle of mathematical induction, by 1889, when he published *Arithmetices Principia Nova Methodo Exposita.* The year before, J. W. R. Dedekind had reached substantially the same result in *Was sind und was sollen die Zahlen?* (Brunswick, Germany, 1888) with the induction principle provable, however, owing to his having started further back in logic, with sets and projections, rather than with sets, number, and successor. Frege, as Dedekind did not know at that time, had gone still further in the same direction. The fact that Peano, even in 1908, did not refer to either Frege or Dedekind but explicitly left the possibility of defining "number" an open question may indicate that he continued to be interested in logic more as a means of attaining brevity and rigor, and an occasional new insight, than as material from which the basic arithmetical notions might be constructed.

Cantor. Peano did draw on the theory of sets of Georg Cantor (1845–1918), including Cantor's proofs that the algebraic numbers can be put in one-to-one correspondence with the positive integers and that the real numbers cannot be so made to correspond (the "diagonal" proof). Cantor's work had grown out of a reorganization of analysis parallel to that of algebra and geometry. He was influenced, of course, by the work of Cauchy, Riemann, and Hankel on functions of complex variables, but his principal predecessor was Karl Weierstrass (1815–1897), who was greatly interested in foundational matters, especially in regard to irrational numbers and points of condensation of infinite sets. Cantor became convinced that without extending the concept of number to actually infinite sets it would hardly be possible to make the least step forward without constraint. The arithmetic that he thus created was welcomed by Frege; its influence is widely apparent and was acknowledged in Russell's *Principles of Mathematics* (Cambridge, U.K., 1903), which plotted the future progress of *Principia Mathematica.*

See also Aristotle; Boole, George; Cantor, Georg; De Morgan, Augustus; Frege, Gottlob; Geometry; Helmholtz, Hermann Ludwig von; Hilbert, David; Jevons, William Stanley; Łukasiewicz, Jan; Many-Valued Logics; Peano, Giuseppe; Peirce, Charles Sanders; Proof Theory; Russell, Bertrand Arthur William; Whitehead, Alfred North.

Bibliography

R. C. Archibald, "Outline of the History of Mathematics," in *American Mathematical Monthly* 56 (1949), cites standard histories of nineteenth-century mathematics. See also the general histories of logic listed above, as well as Roberto Bonola, *Non-Euclidean Geometry* (New York: Dover, 1955); Alonzo Church, "Schroder's Anticipation of the Simple Theory of Types," in *Erkenntnis* 9 (1939): 149–152; Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers,* translated, with an introduction, by P. E. B. Jourdain (Chicago: Open Court, 1915); Ettore Carruccio, *Mathematics and Logic in History and in Contemporary Thought,* translated by Isabel Quigly (Chicago: Aldine, 1965); H. B. Curry, *Foundations of Mathematical Logic* (New York: McGraw-Hill, 1963); and Giuseppe Peano, *Formulario Mathematico* (Turin, 1908; facsimile reprint, Rome: Edizioni Cremonese, 1960).

Ivo Thomas (1967)

FREGE. Modern logic began with the publication in 1879 of the *Begriffsschrift* of Gottlob Frege (1848–1925). In the *Begriffsschrift* we find for the first time a comprehensive treatment of the ideas of generality and existence, because sentence forms which were hitherto accommodated only by complicated ad hoc theories are here provided with an adequate symbolization by the device of quantification, rules for which are adjoined to the first complete formalization of the classical propositional calculus. The result closely approximates a modern formal axiomatic theory. It meets Frege's aim of a codification of the logical principles used in mathematical reasoning, although the rules of inference (substitution and *modus ponens*) and the definition of other logical constants in terms of the primitives (negation, implication, the universal quantifier, and identity) are not explicitly formalized but are mentioned as obviously justified by reference to the intended interpretation. A proof of completeness was not to be had in Frege's day, but he demonstrated the power of his system by deriving a large number of logical principles from his basic postulates and took an important step toward the formulation of arithmetical principles by showing, with the aid of second-order quantification, how the notion of serial order may be formalized.

After the *Begriffsschrift*, Frege's next major work was *Die Grundlagen der Arithmetik* (Breslau, 1884), an analysis of the concept of cardinal number presented largely in nontechnical terms. It opens the way for Frege's theories with a devastating criticism of the views of various writers on the nature of numbers and the laws of arithmetic. Difficulties encountered in the analyses of number find explanation and resolution in the celebrated claim that a statement of number contains an assertion about a concept. To say, for instance, that there are three letters in the word *but* is not, on Frege's view, to attribute a property to the actual letters; it is to assign the number 3 to the concept "letter in the word 'but'." If we now say that two concepts *F* and *G* are numerically equivalent (*gleichzahlig*) if and only if there is a one-to-one correspondence between those things which fall under *F* and those which fall under *G*, we can define the number that belongs to a concept *F* as the extension of the concept "numerically equivalent to the concept *F*."

In terms of this definition any two numerically equivalent concepts, such as "letter in the word 'but'" and "letter in the word 'big'," can be seen to determine the same extension, and therefore the same number, and it remains only to specify concepts to which the individual numbers belong. In sketching this and subsequent developments Frege found that the notions used appear to allow of resolution into purely logical terms. He concluded that it is probable that arithmetic has an a priori, analytic status, a view that places him in opposition to Immanuel Kant, who held that propositions of arithmetic were synthetic a priori, and to J. S. Mill, who regarded them as inductive generalizations.

In papers published after the *Grundlagen*, Frege turned his attention to problems of a more general philosophical nature, and the development of his thought in this period led to a revised account of his logic, which is incorporated in his most ambitious work, *Die Grundgesetze der Arithmetik* (2 vols., Jena, Germany, 1893–1903), in which he extended and formalized the theory of number adumbrated in the *Grundlagen*. In the *Begriffsschrift* he had rejected the traditional subject-predicate distinction but had retained one predicate, "is a fact" (symbolized "$\vdash$"), which indicated that the judgment which it prefaced was being asserted. In his essay "Über Sinn und Bedeutung" this view was abandoned on the ground that the addition of such a sign, conceived as a predicate, merely results in a reformulation of the same thought, a reformulation which in turn may or may not be asserted.

The logic of the *Grundgesetze* is based on Frege's theory of sense and reference, the interpretation of the symbolism of the *Begriffsschrift* being modified accordingly. The formal system of the *Begriffsschrift* is further changed by replacing certain of the axioms with transformation rules, but a more important innovation is the extension of the earlier symbols to cover classes. Corresponding to any well-defined function $\Phi(\xi)$ is the range, or course of values (*Wertverlauf*), of that function, written $\dot{\epsilon}\Phi(\epsilon)$, which Frege introduced via an axiom stipulating that $\dot{\epsilon}\Phi(\epsilon)$ is identical with $\epsilon\psi(\epsilon)$ if and only if the two associated functions $\Phi(\xi)$ and $\psi(\xi)$ agree in the values which they take on for all possible arguments ξ. In particular, this axiom licenses the passage from a concept to its extension, the course-of-values notation providing a means of representing classes and foreshadowing Bertrand Russell's class-abstraction operator, $\hat{z}(\phi z)$. Another device that found a close analogue in Russell's logic is Frege's symbol $\backslash\xi$. If a course of values ξ has a unique member, then $\backslash\xi$ is this member; otherwise $\backslash\xi$ is the course of values ξ itself. In the first case $\backslash\xi$ provides a translation of expressions of the form "the *F*" and so corresponds to Russell's description operator, $(\imath x)(\phi x)$; the second case ensures that when ξ has no unique member, $\backslash\xi$ is nevertheless well defined.

The preliminary development of logic and the theory of classes is followed by the main subject of the *Grundgesetze*, the theory of cardinal number, developed with respect to both finite and infinite cardinals. The theory of real numbers is begun in the second volume but the treatment is incomplete, and Frege was probably loath to advance further in this direction after learning, while the second volume was in the press, that the very beginnings of his theory harbored a contradiction. This contradiction, discovered by Russell, resulted from the axiom allowing the transition from concept to class, an axiom in which Frege had not had the fullest confidence. Russell's communication is discussed in an appendix to the second volume, where an emended version of the axiom is put forward. This emendation was not, in fact, satisfactory, and although Frege apparently did not know that a contradiction could still be derived, he eventually abandoned his belief that the program of the *Grundgesetze* could be carried out successfully and claimed that geometry, not logic, must provide a basis for number theory.

See also Frege, Gottlob; Kant, Immanuel; Mill, John Stuart; Russell, Bertrand Arthur William.

Bede Rundle (1967)

PEANO. Giuseppe Peano (1858–1932), professor of infinitesimal analysis at Turin and a prolific writer on a wide range of mathematical topics, contributed to the early development of both logicism and the formalism to which it is partly opposed. His first book, published under the name of a former teacher, Angelo Genocchi, was devoted to the calculus and featured a careful, systematic treatment of the subject that contrasted favorably with customary texts in rejecting loosely phrased defini-