Dominus vobiscum, *fratres*,

I wonder, Alan, if your mathematician friend in Paris read Lebesgue's 1904 *Leçons sur l'intégration et la recherche des fonctions primitives*, *professées au Collège de France*: https://archive.org/stream/leconegrarecher00leberich#page/n5/mode/2up

It is, with his 1901 article and 1902 dissertation *Intégrale*, *longueur*, *aire* (under Émile Borel's supervision), Lebesgue's most complete resource for his extended integration theory.

Always important to remember (as Lebesgue himself insists in his 1901 article): he developed his generalized integral in order to solve the problem of primitives for unbounded functions. His definition of "summable" functions on a subspace A of R, denoted $L^1(A)$, is predicated upon a concept introduced a few years before by Borel himself: that of measure of a bounded set $E \subset R$. Lebesgue integral is a linear functional on the vector space $L^1(A)$.

Likewise his definition of "measure" relies on the generalization of Riemann by another French mathematician, Camille Jordan, who, the "first" (along with Giuseppe Peano), came up with the ideas of substituting intervals in step functions by bounded sets (called Jordan measurable sets, i.e. sets with a "Peano-Jordan" measure). Lebesgue measurability represents a direct extension of both Peano-Jordan measure and Borel measure. Thus a set $E \subset R$ is Lebesgue measurable if, for every $A \subseteq R$:

$$\mu^* = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

In other words, if $E \subset [a, b]$ is Jordan measurable, follows Lebesgue's outer measurability:

$$\mu_e(E) + \mu_e([a, b] - E) = b - a$$

His extended measure directly leads to his measurability theorem, which directly bears on his new definition of integration:

Let f_n be a sequence of measurable functions defined on $[a, b]$, such that $f(x) = \lim_{n \to +\infty} f_n(x)$ for $\forall x \in [a, b]$. Then f is measurable.

In other words, a bounded function $f : [a, b] \to R$ can be integrated over $[a, b]$ in terms of Lebesgue integration if for any $c < d$ falling in the range of f, the set $\{x \in [a, b] : c \le f(x) < d\}$ is measurable.

Also, notice (often overlooked) the simple connection obtaining as an equation of Lebesgue and Riemann respective integrals. Let f be a bounded function defined on a bounded interval $[a, b]$, with $b > a$. If f can be integrated in terms of Riemann integration (i.e. provided f is continuous almost everywhere), then f can be summed and its Lebesgue integral is equal to its Riemann integral:

$$\int_{[a,b]}^{\square} f(x)\, d\mu(x) = \int_a^b f(x)\, d(x)$$

Laus ad pulchritudinem aeternam.

Sébastien

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