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SYLLOGISM AND QUANTIFICATION

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Anyone who reads Aristotle, knowing something about modern logic and nothing about its history, must ask himself why the syllogistic cannot be translated as it stands into the logic of quantification. It is now more than twenty years since the invention of the requisite framework, the logic of many-sorted quantification.

1. Many-sorted logic. In the familiar first-order predicate logic generality is expressed by means of variables and quantifiers, and each interpretation of the system is based upon the choice of some class over which the variables may range, the only restriction placed on this 'domain of individuals' being that it should not be empty. The only grammatical difference between this 'single-sorted' logic and the corresponding many-sorted logic is that in the latter the variables are split up into a number of different categories or sorts.[1] And the only difference in interpretation is that an interpretation of a many-sorted logic requires the choice not of one domain of individuals but of as many domains as there are sorts of variables, each domain serving as range for the variables of one sort. (The various domains must each be non-empty, as in the single-sorted logic, but it is immaterial to what extent they overlap or include one another.)

This close parallelism with the ordinary logic makes it very easy to axiomatise the logical truths of many-sorted logic. Consider the following axiom schemes and rules, in which ϕ, ψ and $\phi(a)$ are any wff., a and b any variables, and $\phi(b)$ has free occurrences of b wherever $\phi(a)$ has free occurrences of a:

A1. Axioms for the propositional calculus.
A2. $(a)(\phi \supset \psi) \supset. \phi \supset (a)\psi$, if a is not free in ϕ.
A3. $(a)\phi(a) \supset \phi(b)$, if b is of the same sort as a.
Rule of generalisation: from ϕ infer $(a)\phi$.
Rule of detachment: from ϕ and $\phi \supset \psi$ infer ψ.

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[1] Arnold Schmidt, *Uber deduktive Theorien mit mehreren Sorten von Grunddingen*, ***Mathematische Annalen***, vol. 115 (1938), pp. 485–605, and *Die Zulässigkeit der Behandlung mehrsortiger Theorien mittels der üblichen einsortigen Prädikatenlogik*, Ibid., vol. 123 (1951), pp. 187–200; Hao Wang, *Logic of many-sorted theories*, ***Journal of Symbolic Logic***, vol. 17 (1952), pp. 105–116; Alonzo Church, ***Introduction to Mathematical Logic***, Exercise 55.24. I should add that all these authors impose the restriction that each argument-place of a predicate may only be filled by variables of one particular sort. No such restriction must be made if the system is to be put to the use envisaged here.

The resulting theorems are exactly those wff. that are logically true (i.e. true under every interpretation). For our axiom system is nothing but a reduplication for each sort of variable of a standard axiomatisation of the ordinary predicate calculus,[2] and any of the standard completeness proofs is easily adapted to demonstrate its completeness[3].

Suppose now that we introduce into the many-sorted logic so far described a number of singulary predicates, one to each sort of variable, with the intention that each new predicate shall be true of exactly those individuals which constitute the range of variables of the corresponding sort. Since the intended interpretation of these 'sortal' predicates is thus tied to the assignment of ranges to the variables, an interpretation of the logic is determined equally by assigning ranges to the variables or by assigning meanings to the sortal predicates: if the sortal predicate A is interpreted as 'man' or 'is a man' then any variable a of the corresponding sort ranges over the class of men, and vice-versa. In consequence generality-statements like $(a)\phi(a)$ and $(\mathrm{E}a)\phi(a)$ can be rendered 'all As ϕ', 'some As ϕ', etc.

The immediate problem is to axiomatise the logical truths of many-sorted logic when the sortal predicates are taken into account. I shall show that the desired system is got by adding to A1–3 the following pair of axiom schemes, in which A is to be the sortal predicate corresponding to the variable a:

A4. $(a)\phi(a) \supset . A(b) \supset \phi(b)$. A5. $A(a)$.

It is easy to see that given the intended interpretation of the sortal predicates all the resulting theorems are logical truths. To show that all the logical truths are theorems (i.e. that the axioms are complete) is a little more difficult. It is easiest to prove it in a contraposed form — to prove, that is to say, that no non-theorem is a logical truth, or that given any non-theorem there is to be found an interpretation under which it is false.

Let ψ, then, be any non-theorem, and let T be the set of theorems. It cannot be the case that $\mathrm{T} \vdash \psi$, otherwise ψ would belong to T[4]. But if not $\mathrm{T} \vdash \psi$ in the system constituted by A1–5 then *a fortiori* not $\mathrm{T} \vdash \psi$ in the system constituted by A1–3 alone. In view of the completeness of A1–3 this means that there is an interpretation of this system which satisfies T and $\sim\psi$.[5] In defining this interpretation, however, no account is taken of the intended interpretation of the sortal predicates. Certainly each sortal predicate will be true of all the individuals in the corresponding domain, since $(a)A(a)$ (which follows by generalisation from A5 and therefore belongs to T) is satisfied; but the predicate may also be true of some individuals outside

[2] Cf. Church, p. 172. Following Church I shall not take the existential quantifier as primitive, but will make use of the definition $(\mathrm{E}a)\phi =_{\mathrm{df}} \sim(a)\sim\phi$.

[3] E.g. Church's **440 and **453. Cf. Wang, theorems 2.5 and 2.7.

[4] For the meaning of '$\vdash$' here see Church, p. 197.

[5] Cf. Church, **453.

the domain. Suppose then that we define a new interpretation by enlarging each domain to take in every individual of which the corresponding sortal predicate is true. If we can show that the change leaves the truth-value of wff. unaffected we shall have found an interpretation in which the sortal predicates do receive their intended interpretation and in which the non-theorem ψ is false. Since the change only concerns the range of the variables we need only consider its effect on wff. of the form $(a)\phi(a)$. If such a wff. was originally false it cannot become true by any extension of the domain. On the other hand if $(a)\phi(a)$ was originally true it could become false if there were some individual outside the original domain for which A was true but ϕ was not. Such an individual must lie in some other domain: let b be some variable of the corresponding sort which does not itself occur in $\phi(a)$. Then in the original interpretation $(\mathrm{E}b)(A(b)\ \&\ {\sim}\phi(b))$ must have been true alongside $(a)\phi(a)$. But this is impossible since $(a)\phi(a) \supset. (b)(A(b) \supset \phi(b))$, which follows from A4 by generalisation and the use of A2 and which therefore belongs to T, must also have been true.

It may be remarked that the addition of A4–5 makes A3 redundant. For if b is of the same sort as a then the same sortal predicate corresponds to both, so that $A(b)$ is as much an instance of A5 as is $A(a)$; and from this together with A4 there follows at once A3. Hence any reference in the sequel to the system constituted by A1–5 can be replaced by a reference to A1–2, 4–5 only.

2. The traditional theory. A particular case of the reading of the wff. $(a)\phi(a)$ as 'all As ϕ' is the reading of $(a)B(a)$ as 'all As are B' (or, with Aristotle, 'B belongs to all A' or 'B is predicated of all A'). In a similar way $(\mathrm{E}a)B(a)$ can be read as 'some As are B' (or 'B belongs to some A' or 'B belongs to some of the As'); $(a){\sim}B(a)$ can be read as 'no As are B' (or 'B belongs to no A' or 'B belongs to none of the As'); and $(\mathrm{E}a){\sim}B(a)$ can be read as 'not all As are B'. It is therefore natural to seek to introduce the traditional **A**, **E**, **I**, **O** forms by the following definitions[6]

$$\mathbf{A}AB =_{\mathrm{df}} (a)B(a) \qquad\qquad \mathbf{E}AB =_{\mathrm{df}} {\sim}\mathbf{I}AB$$

$$\mathbf{I}AB =_{\mathrm{df}} (\mathrm{E}a)B(a) \qquad\qquad \mathbf{O}AB =_{\mathrm{df}} {\sim}\mathbf{A}AB.$$

By a "traditional" wff. I mean either one of the **A**, **E**, **I**, **O** forms as defined above or else a wff. built up from these by means of connectives drawn from the propositional calculus. And by a 'traditional theorem' I mean a traditional wff. that can be derived by the operations of the

[6] Strictly in these definitions a ought to be specified to be some particular variable of the A-sort, say the first in an alphabetical ordering, though of course the wff. produced by different choices of variable are synonymous.

propositional calculus from the following axiom schemes[7]

$$\mathbf{A}AA \qquad\qquad \mathbf{A}BC \,\&\, \mathbf{A}AB \supset \mathbf{A}AC$$
$$\mathbf{I}AA \qquad\qquad \mathbf{A}BC \,\&\, \mathbf{I}BA \supset \mathbf{I}AC.$$

It is known that this system is complete with respect its interpretation in terms of arbitrary non-empty classes, with **A** standing for class-inclusion and **I** for class-overlap[8]. But this is precisely the interpretation which traditional wff. receive in our many-sorted logic. In view of the completeness of our axioms A1–5 this means that *a traditional wff. is provable from* A1–5 *if and only if it is a traditional theorem.*

There no more exists a decision procedure for the many-sorted predicate calculus than for the single-sorted one. But for wff. which contain only sortal predicates (and thus in particular for all 'traditional' wff.) there is a simply described decision procedure: *a wff. containing letters (predicates or variables) of not more than n different sorts is a theorem if and only if it is true under every interpretation in terms of the non-empty subclasses of a domain of* 2^n-1 *individuals.* The justification of this runs parallel to the justification of the corresponding decision procedure for the ordinary singulary predicate calculus[9]: in any interpretation we can class together those individuals which belong to the same selection of the n domains involved, and since every individual must belong to at least one domain there are at most 2^n-1 classes to be formed in this way. Then we show that the individuals in each class can be lumped together without affecting the truth-values which wff. receive under the interpretation.

The traditional 'immediate inferences' fall under the case $n=2$ and are therefore decidable in terms of the non-empty subclasses of a domain of 3

[7] Jan Łukasiewicz, ***Aristotle's Syllogistic,*** §§ 25–6. Łukasiewicz' lower-case variables correspond to my capitals, except that his variables are *replaceable by* particular terms whereas I have found it convenient to use 'syntactical' variables which *stand for* predicates. Either system could be re-written to follow either usage. In one respect Łukasiewicz' system is not 'traditional', in that he uses the full propositional calculus as an auxiliary. But the reader should be warned that in many other respects the book is firmly in the tradition of Prantl and Maier. Its most persistent failing is the author's ignorance of the idea of a principle of inference (expressed by the sign '$\vdash$') as opposed to inference ('$\therefore$') and implication ('$\supset$'). It is this which among other things vitiates his criticisms of Aristotle's proofs by *reductio ad impossibile* and his discussion of syllogistic necessity, and which leads him to confuse his own 'rejection' (a brilliant way of formulating elegant decision procedures, based on the fact that an effectively enumerable set with an effectively enumerable complement is effectively decidable) with Aristotle's straightforward *a fortiori* proofs of non-deducibility. For convenience' sake I have accepted in the text the formulation of the traditional theorems as implications.

[8] J. C. Shepherdson, *On the interpretation of Aristotelian syllogistic,* ***Journal of Symbolic Logic,*** vol. 21 (1956), pp. 137–147, with references to an earlier proof by Słupecki (not available to me).

[9] Cf. Church, **466.

individuals. (In fact the same process continued reduces the interpretations involved to one or other of the five types depicted in the familiar Eulerian diagrams.) Wff. of the traditional syllogistic fall under the case n=3 and so are decidable in terms of the non-empty subclasses of a domain of 7 individuals. But the decision procedure is also applicable to certain wff. which are not strictly 'traditional', for example to wff. of the form $(a)(B(a) \supset C(a))$, which can be shown in this way to occupy a position intermediate between **A**BC and **A**$AB \supset$ **A**AC.

The existence of a decision procedure means that there is in principle no need to establish the validity of the traditional theorems by deducing them from our axioms — we could instead validate them by testing them in accordance with the procedure described. But it seems to me that the best method in practice is the one adopted by Aristotle himself: to alternate between formal deduction (for validating theorems) and interpretation (for rejecting non-theorems)[10]. We are indeed in a position to provide a theoretical justification for this two-sided method of Aristotle's. That every wff. which is not a logical truth is false under some interpretation is true by definition: what our decision procedure adds is (a) a method for actually finding a counter-example whenever one exists in theory, and (b) an assurance that when found the counter-example will be of the very simple kind considered by Aristotle. On the other hand the completeness of our axioms ensures that every logical truth has a proof, and if it should be difficult to find one the systematic enumeration of all possible proofs is available as a last resort. But it is perhaps worth indicating convenient lines of proof for some of the more important of the traditional theorems:

The axiom A5 itself answers to the *law of identity*, in the form '(any) A is A'. In the form in which Łukasiewicz takes it as an axiom, **A**AA, i.e. $(a)A(a)$, the law follows at once from A5 by generalisation.

To derive the laws embodied in the traditional *square of opposition* we need the theorem scheme $(a)\phi(a) \supset (\mathrm{E}a)\phi(a)$, which is proved exactly as in the single-sorted predicate calculus[11]. Taking B and $\sim B$ in turn for ϕ establishes the relations of subalternation, **A**$AB \supset$ **I**AB and **E**$AB \supset$ **O**AB. All the other laws of the square follow from our definitions of the **A, E, I, O**

[10] It might be thought anachronistic to invest Aristotle's method of proceeding with the status of a conscious solution of the decision problem. Łukasiewicz indeed says positively (p. 75) that Aristotle was unaware of the existence of the decision problem, but it is possible that he is prevented from doing Aristotle justice here by his belief that the whole method of providing a concrete counter-example by interpretation is a "flaw of exposition" which brings into logic things "not germane to it". To me it seems that in a work whose professed purpose was to "state by what means, when, and how every syllogism is produced" (*An. Pr.* 25ᵇ26) the machine-like alternation of deduction and interpretation stands out as something too important to be under-estimated, at whatever risk of anachronism.

[11] Cf. Church, *331.

forms either immediately or with the help of the above theorem. Also, from $(a)A(a)$ and $(a)A(a) \supset (\mathrm{E}a)A(a)$ there follows by detachment $(\mathrm{E}a)A(a)$, which is Łukasiewicz' axiom **I**AA.

Conversion. The conversion of **E**-propositions is expressed in the implication **E**$AB \supset$ **E**BA, so what we must try to prove is $(a){\sim}B(a) \supset (b){\sim}A(b)$. By substitution in A4, $(a){\sim}B(a) \supset. A(b) \supset {\sim}B(b)$. But by A5, $B(b)$. Hence $(a){\sim}B(a) \supset {\sim}A(b)$, whence the result by generalisation and the use of A2. An alternative proof could have been based on Aristotle's own proof by 'exposition'. He argued that if no B is A, neither can any A be B, "For if some A (say C) were B, it would not be true that no B is A, for C is a B."[12] I should analyse the quoted sentence as embodying three steps: (1) from $(\mathrm{E}a)B(a)$ to $B(c)$; (2) from $B(c)$ to $A(c) \& B(c)$; (3) from $A(c) \& B(c)$ to $(\mathrm{E}b)A(b)$. The first of these steps, corresponding to Aristotle's "say C", needs careful handling in a purely formal treatment — see, for example, J. B. Rosser, ***Logic for Mathematicians***, Ch. VI, § 7, "The formal analogue of an act of choice". But the second step is straightforwardly justified by A5 (for if c is a variable of the A-sort then $A(c)$ is an instance of this axiom), and the third step is made by simply contraposing A4.[13]

The syllogistic. From the axiom A4 by generalisation and the use of A2 we derive (provided b does not occur free in $\phi(a)$) $(a)\phi(a) \supset (b)(A(b) \supset \phi(b))$. And just as in the single-sorted predicate calculus we can prove the implications $(b)(A(b) \supset \phi(b)) \supset. (b)A(b) \supset (b)\phi(b)$ and $(b)(A(b) \supset \phi(b)) \supset. (\mathrm{E}b)A(b) \supset (\mathrm{E}b)\phi(b)$. Combining these with the first result yields the two schemes:

(1) $(a)\phi(a) \supset. (b)A(b) \supset (b)\phi(b)$

(2) $(a)\phi(a) \supset. (\mathrm{E}b)A(b) \supset (\mathrm{E}b)\phi(b)$.

Taking C and ${\sim}C$ for ϕ in (1) produces the syllogisms *Barbara* and *Celarent* respectively, and doing the same thing in (2) produces the syllogisms *Darii* and *Ferio* respectively. Having proved these four syllogisms, we can conveniently derive the remainder by the traditional methods of reduction.

Note that the axiom A4, $(a)\phi(a) \supset. A(b) \supset \phi(b)$, which plays a crucial part in almost all the proofs, itself answers to the *Dictum de Omni et Nullo* of the traditional theory.

3. Singular and negative terms. It is no more difficult to introduce singular terms (names and definite descriptions) into the basic many-

[12] *An. Pr.* 25^{a} 15–17.

[13] This last step recurs in the corresponding analysis of Aristotle's other proof by exposition (*An. Pr.* 28^{a} 22–6), where the argument — "If both P and R belong to all S, should one of the Ss, e.g. N, be taken, both P and R will belong to this, and thus P will belong to some R" — would be analysed into the steps (1) from $(s)P(s) \& (s)R(s)$ to $P(n) \& R(n)$, and (2) from $P(n) \& R(n)$ to $(\mathrm{E}r)P(r)$.

sorted logic (i.e. before the introduction of the sortal predicates) than it is to introduce them into single-sorted logic. Just as the axioms A1–3 simply reproduce for each sort axioms for the predicate calculus without singular terms, so it is simply a matter of making additions to them which reproduce for each sort any one of the standard treatments of singular terms. Of course what this will result in is a theory of singular terms each of which is assigned to one particular sort, just as the variables are. So far as descriptions are concerned this is just what one would expect, since the variable used to form the description '$\imath a\phi(a)$' is itself necessarily restricted to some one sort. In this way we can directly represent descriptions like 'the man who ...' or 'the proposition that ...' which when they stand for anything stand for a man or a proposition. No doubt too the idea of 'sorted' names whose grammar gives partial information about the bearer is in accord with the ordered, classified world which one associates with the philosophy of Aristotle and which is reflected in a language in which e.g. 'Earl Russell' necessarily denotes an Earl, 'Fido' a dog, and in which it is still pertinent to ask "Ah! but what was his name before it was Robinson?". But once the sortal predicates are introduced there is no need to confine ourselves to 'sorted' names: if in the axiom A4 b is allowed to be a name as well as a variable the result is a theory of names which are not assigned to any particular sort (the sorted names are distinguishable by the fact that for them the appropriate sentence $A(b)$ is a logical truth). The traditional theory of singular terms has hardly been developed in a way that would permit the establishment of any exact statement of equivalence, but the reader may verify that such familiar theorems as 'If all men are mortal then if Socrates is a man Socrates is mortal' are provable — in this particular case by direct substitution in the axiom A4.

To reconstruct the traditional theory of negative terms in many-sorted logic we must suppose that every sort has a complement. If a is any variable and a' a variable of the complementary sort, the intention is that in any interpretation the ranges of a and a' shall be exclusive and together exhaustive — every individual shall belong to one or other but not to both. If a'' is a variable of the sort complementary to that of a' it follows that the range of a'' will always coincide with that of a. Thus although each fundamental sort gives rise to infinitely many derived sorts (its complement, the complement of the complement, etc.) they are not independent: in fact each interpretation of the system is uniquely determined by the choice of a domain of individuals and of non-empty non-universal subclasses of it to serve as ranges for the variables of the various fundamental sorts, with complementary subclasses being assigned to complementary sorts.

To axiomatise the resulting logical truths observe that if A and A' are sortal predicates of complementary sorts then the requirement that the ranges corresponding to the two sorts should be exclusive and exhaustive is equivalent to the requirement that A' should be true of exactly those individuals of which A is not true. And this latter condition is easily secured by adding to A1–5 the following axiom scheme:

A6 $A'(b) \equiv \sim A(b)$.

This axiom is tautologically equivalent to $(A(b) \vee A'(b))$ & $\sim(A(b)$ & $A'(b))$, and if it were desired the two halves of this conjunction could be posited separately as axioms in place of A6. The point is that the first half of the conjunction, $A(b) \vee A'(b)$, answers to the traditional law of excluded middle, in the form 'anything is either A or non-A'; while the second half, $\sim(A(b)$ & $A'(b))$, answers in a similar way to the traditional law of non-contradiction.

By the traditional theory of negative terms I mean the system got by adding the following axioms to the framework of the propositional calculus:

$$\mathbf{A}AA'' \qquad \mathbf{A}A''A \qquad \mathbf{A}AB \supset \mathbf{A}B'A'$$

$$\mathbf{A}AB \;\&\; \mathbf{A}BC \supset \mathbf{A}AC \qquad \mathbf{A}AB \supset \sim\mathbf{A}AB'.$$

It is known that this set of axioms is complete with respect to its interpretation in terms of the non-empty, non-universal subclasses of an arbitrary class, with **A** standing for class-inclusion and the dash indicating class-complementation[14]. As before, the equivalence of the two theories is a consequence of the completeness of Wedberg's axioms on the one hand and A1–6 on the other; but as before it may be of interest to furnish proofs of some traditional theorems, say those which Wedberg takes as axioms:

$\mathbf{A}AA''$, i.e. $(a)A''(a)$, can be derived as follows: both $A'(a) \equiv \sim A(a)$ and $A''(a) \equiv \sim A'(a)$ are instances of A6, and together they yield $A(a) \equiv A''(a)$. Since $A(a)$ is an instance of A5 the result follows by detachment and generalisation. $\mathbf{A}A''A$ is proved in a similar way, and the third axiom, *Barbara*, was proved in § 2.

The axiom $\mathbf{A}AB \supset \mathbf{A}B'A'$ is one of the laws of contraposition, and our translation of it, $(a)B(a) \supset (b')A'(b')$, is derived as follows: from A5, $B'(b')$, and A6, $B'(b') \equiv \sim B(b')$, there follows $\sim B(b')$. Moreover $A'(b') \equiv \sim A(b')$ is an instance of A6. Hence from $(a)B(a) \supset . A(b') \supset B(b')$, which is an instance of A4, there follows $(a)B(a) \supset A'(b')$, and from this the desired result follows by generalisation and the use of A2.

The remaining axiom, $\mathbf{A}AB \supset \sim\mathbf{A}AB'$ or $(a)B(a) \supset \sim(a)B'(a)$, follows from the fact that both $(a)B(a) \supset B(a)$ and $(a)B'(a) \supset B'(a)$ are instances of A3. Since their consequents are incompatible, by A6, their antecedents are incompatible also.

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[14] A. Wedberg, *The Aristotelian theory of classes*, **Ajatus**, vol. 15 (1948), pp. 299–314, and Shepherdson, *op. cit.*, Theorem 6. Since in our system each predicate A'' is equivalent to the original A, I suppose that we need not have posited a whole infinity of derived sorts but could have stopped short at complementary pairs of sorts, *defining* A'' to be A itself. For an axiomatisation of the traditional theory incorporating this idea, see Ivo Thomas, *CS(n): an Extension of CS*, **Dominican Studies**, vol. 2 (1949), pp. 145–160.

In Wedberg's system **A** is the only primitive operator, and **I** is defined in terms of it in such a way as to make the relevant law of obversion follow from the definition ($\mathbf{I}AB =_{\mathrm{df}} \sim\mathbf{A}AB'$). If we retain the independent definition of **I** in terms of quantifiers we must check that obversion is still possible. In fact the law in question, $\mathbf{I}AB \equiv \sim\mathbf{A}AB'$ or $(\mathrm{E}a)B(a) \equiv \sim(a)B'(a)$, follows almost at once from A6.

The decision procedure of § 2 is easily extendible to take in wff. with singular or negative terms or both: a wff. which contains, besides any number of unsorted singular terms, letters of not more than n different sorts (and for this purpose sorts related by complementation count together as one), is a theorem if and only if it is true under every interpretation in terms of the non-empty, non-universal subclasses of a domain of 2^n individuals.

4. Existential Import. Since the interpretation of our many-sorted logic demands that all the relevant domains of individuals shall be non-empty, there is a sense in which all the wff., whether cast in affirmative or negative form, have an existential import. But this is something implicit rather than explicit — the existence of the various As is a pre-condition of the successful application of the system rather than an assumption formulated or even formulable within the system. Nor is the position altered by the introduction of the sortal predicates. For in many-sorted logic, however much may be comprehended by the various domains of individuals taken all together, the expression of generality is automatically confined to one domain at a time: there is not (unless the system is deliberately enlarged to provide for it) any universal sort of variable with an answering domain embracing all the others. Hence there is still no wff. on the lines of $(\mathrm{E}x)A(x)$ that could be used to express the existence of As as a piece of information within the system.

The reader will not tail to notice that the feature which under the proposed translation accounts for the existential import of the Aristotelian logic, is nothing more than the reduplication of a feature (the assumption of a non-empty domain of individuals, with the consequent appearance of existentially quantified wff. as theorems) which characterises the dominant system of modern logic. It may be noted too that the whole burden of existential import is borne by the subjects of our propositions, and that there is absolutely no existential commitment involved in the use of predicate terms. It is only the recognition of a category or sort of subject terms (free or bound variables or other sorted singular terms) which carries with it any existential commitment. Thus the reader will have noticed that in our proofs in § 2 we were able to make free use of negated predicates ($\sim A$, $\sim B$, etc.) without in any way anticipating the theory of negative terms of the present section. Again, it is well-known that the introduction of conjunctive terms into the traditional theory brings with it such widespread existential commitments (in the form of **I**-theorems) as to make the theory virtually unworkable[15]. If in many-sorted logic we allow for the conjunction of all the various sorts, with the consequent requirement that in every interpretation every domain shall overlap with every other, we shall run into exactly the same trouble. But there is nothing similar to prevent us from using predicates conjoined or compounded in any other way[16].

It is notorious that if one wants to reproduce the traditional theory in terms of the ordinary single-sorted logic of quantification the existential import of the traditional theorems can no longer be left implicit but must be expressed by adding extra existential premisses or conditions. It is possible to state exactly how these additions should be made and to prove that they do produce exactly the desired result. Consider the following translation from the many-sorted into the single-sorted predicate logic (assuming that the latter, which has of course variables only if the one universal sort, contains a sufficient number of singulary predicates to represent the sortal predicates of the many-sorted system). To translate a wff. we must (i) replace each part of the form $(a)\phi(a)$ by $(a)(A(a) \supset \phi(a))$; (ii) replace each sorted variable by a variable of the universal sort, understanding that different variables in the original get replaced by different variables in the translation. The effect of the translation can be roughly expressed by saying that $(a)\phi(a)$ goes into $(x)(A(x) \supset \phi(x))$, and hence that $(\mathrm{E}a)\phi(a)$ goes into $(\mathrm{E}x)(A(x) \,\&\, \phi(x))$.

By the 'existential import' of a many-sorted wff. in the light of its translation into a single-sorted formula, let us understand the aggregate of all wff. of the form $(\mathrm{E}x)A(x)$ whenever a bound variable of the A-sort occurs in the original, together with all wff. of the form $A(x)$ whenever a free variable of the A-sort occurs in the original (x being the variable which replaces it in the translation). Let $\phi_1, \ldots, \phi_n, \psi$, be many-sorted wff.; let $\phi_1^*, \ldots, \psi^*$, be their single-sorted translations; and let '$\mathrm{E}xx$.' denote the existential import (as defined above) of $\phi_1, \ldots, \phi_n$ and ψ taken together. Then we have the following theorem: $\phi_1, \ldots, \phi_n \vdash \psi$ in the system constituted by A1–5 if and only if $Exx., \phi_1^*, \ldots, \phi_n^* \vdash \psi^*$ in the single-sorted predicate calculus.

This theorem extends the scope of a theorem proved by Schmidt and by

[15] Cf. K. R. Popper, *The trivialisation of mathematical logic*, ***Proceedings of the Xth International Congress of Philosophy*** (Amsterdam, 1949), pp. 722–727.

[16] In this connexion I find it interesting that the philosopher Locke, having observed that "the common names of substances, as well as other general terms, stand for *sorts*", should contrast the formation of complex ideas of substances, when the mind "never puts together any that do not really, or are not supposed to, co-exist", with that of complex or mixed modes, which are "not only made by the mind but made very arbitrarily, made without patterns, or references to any real existence". ***Essay concerning Human Understanding,*** III.6.1, III.6.29, III.5.3.

Wang, to cover the case of wff. with free as well as bound variables. Its proof is an easy corollary of the respective completeness theorems for the two systems concerned[17]. When the translation used in the theorem is coupled with our previous translation of the **A**, **E**, **I**, **O** forms into many-sorted wff. it will induce a translation of the traditional forms directly into the single-sorted logic, $\mathbf{A}AB$ going into $(x)(A(x) \supset B(x))$ and $\mathbf{I}AB$ going into $(\mathrm{E}x)(A(x) \,\&\, B(x))$. The theorem just stated will therefore have an immediate corollary relating the traditional theory and the ordinary predicate calculus, without any overt mention of many-sorted logic. This corollary bears out my suggestion above that it is only the subject-terms of the traditional theory that need be thought to carry existential import, for, on the definition used in the theorem, $\mathbf{A}AB$, $\mathbf{E}AB$, $\mathbf{I}AB$, and $\mathbf{O}AB$ all have the same existential import, namely $(\mathrm{E}x)A(x)$.

It may finally be asked, what does happen if we do deliberately enlarge the scope of the many-sorted logic to include a universal sort of variable along with the other sorts? The short answer is that the resulting calculus is marked by two features. One is the provability of the equivalences $(a)\phi(a) \equiv (x)(A(x) \supset \phi(x))$ and $(\mathrm{E}a)\phi(a) \equiv (\mathrm{E}x)(A(x) \,\&\, \phi(x))$, of which the first recalls Aristotle's definition of 'to be predicated of all'[18]. But although the respective l.h.s. and r.h.s. of these equivalences are inter-deducible they are still different formulae expressing different idioms: if e.g. the wff. $(\mathrm{E}a)B(a)$ is chosen as the formal analysis of 'B belongs to some A' then the wff. corresponding to 'there are As', $(\mathrm{E}x)A(x)$, is not itself a part of the analysis, although it is entailed by it. A second feature of the enlarged calculus is that existential import can no longer be left tacit in it — each wff. of the form $(\mathrm{E}x)A(x)$ is a logical truth. It is possible that some readers' understanding of the phrase 'logical truth' may invest this with more significance than in fact it has. A logical calculus is after all only a means for codifying certain methods of argument, and the theorems of the calculus are only those formulae which it is understood need no justification *within the system* (i.e. as between one user and another). Since the existence of the various As is a pre-condition of the successful (truth-preserving) application of a calculus of the Aristotelian type, it is hardly remarkable that (in this extension of the calculus) it should appear as something that cannot be questioned within the calculus once it has been adopted.

5. Quantification of the predicate. In this section I shall return to the basic many-sorted predicate logic, without singular or negative terms. Suppose that we add the relation-sign '$=$' to the vocabulary, giving it the natural interpretation as a sign of identity between individuals. To

[17] Cf. Wang, *op. cit.*, theorem 3.2.

[18] *An. Pr.* 24b 28–30. Cf. ***Collected Papers of Charles Sanders Peirce***, § 3.396 (a reference suggested, among other helpful criticisms, by Prof. A. N. Prior).

axiomatise the resulting logical truths we must add to the original rules and axioms A1–3 of § 1 the following pair of axiom schemes:

A7. $a=a$.

A8. $a=b \supset . \phi(a) \supset \phi(b)$.

These reduplicate a set of ordinary axioms for identity, and any of the standard completeness proofs for the predicate calculus with identity is easily adapted to demonstrate their completeness. Moreover, the argument of § 1 can be repeated to show that the further addition of the axioms A4–5 yields a complete set of axioms for the manysorted logic when not only '$=$' but also the sortal predicates are taken into account.

The addition of the identity sign makes it possible to develop the traditional theory of the 'quantification of the predicate'. For we can now express the 'quantified' predicates '... is every A', '... is some A', '... is no A', '... is not every A', by the forms $(a)(a=\ldots)$, $(\mathrm{E}a)(a=\ldots)$, $(a)(a\neq\ldots)$, $(\mathrm{E}a)(a\neq\ldots)$, respectively. As far as I know the traditional theory has never been developed in a way which would permit one to formulate any exact statement of equivalence, but the reader may verify, for example, the relations between the quantified predicates analogous to those involved in the square of opposition, or the more complex relations which arise when both subject and predicate are quantified. (Here the eight doubly quantified forms $(a)(b)(a=b)$, $(\mathrm{E}a)(\mathrm{E}b)(a\neq b)$, $(a)(\mathrm{E}b)(a=b), \ldots,$ $(\mathrm{E}a)(\mathrm{E}b)(a=b)$, answer exactly to the eight forms distinguished by De Morgan[19].) I shall confine myself to providing a proof of one particularly important result, the equivalence between the simple predicate '(is) A' and the quantified predicate 'is some A', which is expressed in the following theorem scheme:

(1) $A(b) \equiv (\mathrm{E}a)(a=b)$.

(Proof: by substitution in A4, $(a)(a\neq b) \supset . A(b) \supset b\neq b$. But $b=b$ is an instance of A7, so by propositional calculus there follows $A(b) \supset (\mathrm{E}a)(a=b)$. In proving the converse implication we must distinguish two cases: (i) a and b are the same variable. Then $A(b)$ is itself an instance of the axiom A5, whence the implication follows by propositional calculus. (ii) a and b are distinct. By A8, $a=b \supset . A(a) \supset A(b)$. Hence by A5 and detachment, $a=b \supset A(b)$, whence by generalisation $(a)(a=b \supset A(b))$. But since a does not occur in $A(b)$, $(\mathrm{E}a)(a=b) \supset A(b)$ follows from this just as in the ordinary predicate calculus.)

The fact that no predicate appears on the right-hand side of (1), but only variables, quantifiers and the identity sign, suggests the possibility of using (1) to *define* the sortal predicates in the basic many-sorted logic with

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[19] ***Transactions of the Cambridge Philosophical Society,*** vol. 9 (1856), at p. 91; cited and discussed in Prior, ***Formal Logic,*** Pt. II, Ch. II, § 4.

identity. Consider therefore the many-sorted analogue of the ordinary predicate calculus with identity (the system constituted by the axioms A1–3, 7–8), and suppose that the sortal predicates are introduced by definitions on the following pattern: $A(b) =_{\mathrm{df}} (\mathrm{E}a)(a=b)$.

We want to show that by using these definitions we can reconstruct the theory of the sortal predicates exactly as it would be given by adding A4–5 as further axioms. The fact that (1) is provable in the full theory shows that the definitions are not, as it were, too strong. To show that they are strong enough it will be sufficient to show that we can prove A4–5 (or rather the corresponding wff. built up by using the definitions) from the remaining axioms. In deriving the wff. corresponding to A4, viz. $(a)\phi(a) \supset. (\mathrm{E}a)(a=b) \supset \phi(b)$, we must distinguish two cases: (i) a and b are the same variable. Then by A3, $(a)\phi(a) \supset \phi(b)$, whence the result by propositional calculus. (ii) a and b are distinct. By A8, propositional calculus and generalisation, $(a)(\phi(a) \supset. \sim\phi(b) \supset a\neq b)$, from which there follows $(a)\phi(a) \supset (a)(\sim\phi(b) \supset a\neq b)$ just as in the ordinary predicate calculus. But a does not occur free in $\sim\phi(b)$ so by A2 $(a)(\sim\phi(b) \supset a\neq b) \supset. \sim\phi(b) \supset (a)(a\neq b)$. The desired result is got by putting these two implications together. The wff. corresponding to A5, $(\mathrm{E}a)(a=a)$, follows from A7, $a=a$, just as in the ordinary predicate calculus.

It is, however, not quite correct to say that the wff. we have just proved correspond to A4–5. For in the definition $A(b) =_{\mathrm{df}} (\mathrm{E}a)(a=b)$ of each sortal predicate some particular variable a of the appropriate sort ought to be specified on the r.h.s., whereas in the wff. we have just proved a different variable is used in every different instance. We must show therefore that if a and a_1 are any variables of the same sort then $(\mathrm{E}a)(a=b)$ is synonymous with $(\mathrm{E}a_1)(a_1=b)$. Just as in the single-sorted logic, synonymity (i.e. inter-replaceability) is ensured by inter-deducibility, so it will be enough to prove $(\mathrm{E}a)(a=b) \equiv (\mathrm{E}a_1)(a_1=b)$. Once again we must distinguish two cases: (i) b is distinct both from a and a_1. Then by A3, $(a)(a\neq b) \supset a_1\neq b$, whence by generalisation and the use of A2, $(a)(a\neq b) \supset (a_1)(a_1\neq b)$, whence $(\mathrm{E}a_1)(a_1=b) \supset (\mathrm{E}a)(a=b)$ follows by contraposition. The converse implication is derived similarly. (ii) b is the same as a, say. Then $(\mathrm{E}a)(a=b)$ is the same wff. as $(\mathrm{E}a)(a=a)$ and so is itself a theorem. By A3, $(a_1)(a_1\neq a) \supset a\neq a$, whence $(\mathrm{E}a_1)(a_1=a)$ follows by A7 and propositional calculus. Since both sides of the equivalence are thus separately theorems, so, trivially, is the equivalence itself.

I should like at this point to summarise what I have tried to establish in the way of translations between the Aristotelian logic and many-sorted logic. In § 2 I showed how to reproduce the traditional theory in a system developed axiomatically by adding to the rules of generalisation and detachment the following axiom schemes (two of which themselves echo principles of the old logic, the law of identity and the *Dictum de omni et* *nullo*): A1. Axioms for the propositional calculus. A2. $(a)(\phi \supset \psi) \supset. \phi \supset (a)\psi$, if a is not free in ϕ. A4. $(a)\phi(a) \supset. A(b) \supset \phi(b)$. A5. $A(a)$.

In § 3 I showed how the traditional theory of negative terms can be reproduced if we introduce the complementation of sorts and with it an axiom (A6) echoing two more principles of the old logic, the laws of contradiction and excluded middle.

In the present section I have shown that the traditional theory together with the somewhat conjectural theory of the 'quantification of the predicate' can be reproduced in the many-sorted logic of identity. Since it was also shown that when identity is present the sortal predicates are redundant and can be introduced by a suitable definition, it follows (what could also have been proved directly) that the traditional theory can as an alternative be reproduced in the straight many-sorted analogue of the predicate calculus with identity, determined by the following axioms: A1. Axioms for the propositional calculus. A2. $(a)(\phi \supset \psi) \supset. \phi \supset (a)\psi$, if a is not free in ϕ. A3. $(a)\phi(a) \supset \phi(b)$, if b is of the same sort as a. A7. $a=a$. A8. $a=b \supset. \phi(a) \supset \phi(b)$.

(It hardly needs saying that similar axiomatisations could have been modelled on other formulations of the ordinary logic.)

If the Aristotelian logic, after a long pre-eminence and a shorter period of disrepute, is now more temperately regarded, the change is surely due to Łukasiewicz' formalisation of the traditional syllogistic in the 1930's, and his bringing modern techniques and ideas to bear on the resulting system. But the price paid for a rehabilitation of the traditional logic through an algebra of the Łukasiewicz type is a certain divorce from the main current of modern logic: Łukasiewicz was even led to conclude (*op. cit.*, p. 130) that the syllogistic of Aristotle "exists apart from other deductive systems, having its own axiomatic and its own problems." The result is a certain ambivalence in the current attitude towards the old logic — when we compile our World Team of logicians we tend to include Aristotle as (non-playing) captain. This attitude, at once admiring and dismissive, is well illustrated in Łukasiewicz' conclusion that "The syllogistic of Aristotle is a system the exactness of which surpasses even the exactness of a mathematical theory, and this is its everlasting merit. But it is a narrow system and cannot be applied to all kinds of reasoning, for instance to mathematical arguments. ... The logic of the Stoics, the inventors of the ancient form of the propositional calculus, was much more important than all the syllogisms of Aristotle. We realize today that the theory of deduction and the theory of quantifiers are the most fundamental branches of logic." (p. 131.)

It would of course be absurd and anachronistic for me to try to vindicate Aristotle's choice of subject-matter by suggesting that he was consciously guided by anything like the modern idea of quantification. But without committing this mistake there are two observations which I think may properly be made. One is that if it is anachronistic to suggest that Aristotle's logic is 'really' a theory of quantification then it is equally anachronistic to suggest that it is 'really' a theory of primitive functors **A**, **I**, etc. As Łukasiewicz himself remarks in his book, "the logic of Aristotle is formal without being formalistic"; and what I have for the sake of convenience called the 'traditional' theory in § 2 is, both in its conscious conception as an algebra of non-empty classes and in its formalistic vocabulary and axiomatisation, as distinctively 'modern' as the logic of quantification. The other remark to be made is that the logic of many-sorted quantification is in no sense something existing "apart from other deductive systems". Not only is it formally no more than a systematic reduplication of the standard single-sorted logic, but it is also the obvious framework for the formalisation of a whole range of mathematical theories: any branch of geometry will furnish one example and Russell's or von Neumann's set theories another. I should like therefore to think that the translations introduced above would help to counter the suggestion of even a residual incompatibility between the modern and the Aristotelian formal logic.

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