primitive, ancient, feudal, capitalist, and socialist. The tension between classes builds up (quantitative changes) to a ''revolutionary situation'': revolution is a leap in which a new qualitative situation comes to be wherein one class negates the other. The ideologies are reflections of the base, i.e., of the class-conditions of their exponents. Thus, bourgeois ideology (philosophy, art, etc.) serves the interests of the exploiters, while proletarian ideology (Marxism-Leninism) serves the interests of the proletarian masses; the latter are the best interests of all of humanity. The Communist party, which is the ''vanguard,'' ''conscience,'' and ''honor'' of the proletariat, therefore represents the best interests of all of humanity and is entitled to govern all domains. Mankind, finally, is moving toward a final state of paradise on earth, communism, and it is led to this goal by the proletariat, i.e., by the Communist party.

Origins and Development. Elements of historical materialism are to be found in the writings of Jean Jacques ROUSSEAU, SAINT-SIMON, Fourrier, et al., but its true founder was Karl Marx. Engels contributed little to its formulation. Lenin attenuated Marx's economic determinism with his revolutionary voluntarism; he affirmed that revolution did not have to wait for changes in the economic base, but could be effected by a disciplined party of professional revolutionaries. Stalin's only contribution of note was the interpretation of language mentioned above.

Influence and Critique. It is difficult to overestimate the influence that this doctrine had, especially on the nonphilosophical intelligentsia. For those who put a price on certainty in the explanation of social and historical events, historical materialism purported to offer a doctrine that is simple (two basic elements), clear (everything follows from the conflict of these two), inspiring (history has a glorious goal), and directive (one knows what to do to aid history). Add to this the fact that the whole doctrine is couched in seemingly simple terms, and it becomes easy to see why it was most popular among the substitutes for religion in the twentieth century.

Yet historical materialism was also an a priori schematism with a conceptual coherence almost completely lacking in the events it was supposed to explain. Secondly, its conceptual apparatus was much too rough: e.g., the notion of class, which is fundamental, was always left vague. Finally, it was based on several gratuitous assumptions, e.g., on the nature of man, the origin of society, and ''communism of the future,'' which were all unacceptable, the last shown to be false with the fall of the USSR in 1989.

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[T. J. BLAKELEY]

MATHEMATICS, PHILOSOPHY OF

Philosophy of mathematics is a broad term including any theory on the nature of mathematics as a whole or on the nature of any part or aspect of mathematics. A specialized branch of learning, it deals with the following and related topics: the origin of mathematical knowledge and its relation to the real world; the nature and type of existence peculiar to mathematical entities; mathematics as viewed by mathematicians and by other scientists—and this either absolutely, or in terms of its relationship to other human and cultural values (hence the ''logic of'' or the ''psychology of'' mathematics). A thorough presentation of the philosophy of mathematics would, then, include a discussion of each of these aspects. The analysis here is restricted to a consideration of various theories on the nature of mathematics and a critical evaluation of these theories.

Various Theories

Since a philosophy of mathematics is based on a given mathematical content, this exposition is divided into three parts, roughly corresponding to the three major developments within mathematics itself: the classical, the transitional, and the contemporary.

Classical Period: c. 3000 B.C. to A.D. 1600. From the time of its origin among the Egyptians and of its development as a science by the Greeks (*c.* 600 B.C.) until the 16th century, the content of pure mathematics embraced arithmetic and basic number theory, along with geometry, the conic sections, and basic trigonometry. Music, astronomy, geography, mechanics, and hydrostatics constituted applied mathematics.

Greek Theories. It is currently held that the philosophy of mathematics originated with the same Greeks who have also been credited with being the founders of mathematics as a science. Among the Pre-Socratics, the PYTHAGOREANS proposed the first complete theory: the principles of mathematics, number and form, are the principles and the basic reality of all things. Each thing has its own mathematical number and form: justice is the number four (and therefore a square); time is seven; perfection is ten, etc. The Pythagoreans were largely responsible also for the division of mathematics into arithmetic (and music, as applied arithmetic) and geometry (and astronomy, as applied geometry).

PLATO modified this view, placing mathematics within the framework of knowledge in general: ordinary knowledge of material things (''opinion'') is unstable because the objects known are constantly changing; mathematics achieves more stable knowledge because its objects, numbers and forms, do not change; while the most perfect kind of knowledge (''dialectic'') is the contemplation of ideas or forms of which the sense, and even the mathematical words, are but a participation and imitation. Mathematical entities, then, differ from sense objects in being eternal and unchangeable; they differ from the forms in that there are many alike (many circles), while the form is in each case unique. The Pythagorean-Platonic view is adopted totally or with modifications by PLOTINUS, PROCLUS, NICHOLAS OF CUSA, COPERNICUS, KEPLER, GALILEO, and many others.

The Aristotelian view classifies mathematics as one of three speculative sciences (*see* SCIENCES, CLASSIFICATION OF). Mathematics arises by means of ABSTRACTION from sense data and deals with ''quantified substance'' as discrete (number), whence arise the arithmetical branches, or as continuous (form), whence arise the geometrical branches. Mathematics is but one way of fulfilling man's ability to know reality; it has a pedagogical and useful value, but it is not a knowledge necessary for, or leading to, an other-worldly contemplation. BOETHIUS, AVICENNA, AVERROËS, THOMAS AQUINAS, and many medieval scholars support this view with modifications, and the writings of Francis BACON, G. W. LEIBNIZ, Auguste COMTE, and many others, show its influence.

The skeptical view of mathematics is ably expressed by Sextus Empiricus (*c.* 250 A.D.) whose analysis of mathematical notions denies that they are clear, universal, and necessary, and proposes that they are inexact, empirical, and conventional (*see* SKEPTICISM). The British empiricists of the 16th and 17th centuries and some contemporaries adopt a similar interpretation.

Medieval Theory. The view of St. Thomas Aquinas is an instance of the medieval theory of mathematics. Essentially that of Aristotle, whose ideas he enriches with insights and refinements of earlier commentators and by his own metaphysical and terminological precision, Aquinas's theory considers mathematics, physics, and metaphysics as the three speculative (pure) sciences. In mathematics, one considers those aspects of things that depend not on their qualitative modifications but only on the fact that they are ''quantified,'' that is, subject to quantity—either the numerableness of things, or the shapes and forms in which quantity is, or might be, arranged.

In common with the other speculative sciences, mathematics is based upon what might be called preliminary knowledge and experience (now referred to as ''prereflective'' knowledge) of the quantitative aspects of things. This preliminary knowledge must include at least a vague understanding of the terms with which one expresses quantity (one, two; circle, square). After adding to this prereflective data an explicit understanding of logical procedure, one establishes axioms and postulates, defines mathematical objects (e.g., a square), and then proceeds to deduce the characteristics of those objects that are necessarily implied by the given definition (e.g., that a square is equiangular). The conclusions that are thereby reached are restricted only by the scope of man's imagination. Mathematics is the most exact and certain of the three speculative sciences and exercises a certain hegemony over the sciences of nature.

Greek and medieval thought was familiar with several sciences in which the propositions of pure mathematics are applied to other sciences. Aquinas names astronomy and optics as instances of applied geometry, and music, of applied arithmetic. They are called ''intermediate'' (*scientiae mediae*) because they employ the mathematical method of DEMONSTRATION (formally mathematical) on subject matter from sensible nature (materially physical).

The liberal arts for Thomas Aquinas are ordered to knowledge and involve some sort of making, working, or producing, as making a syllogism or a speech, composing melodies, and reckoning the course of the stars. Mathematics (arithmetic and music; geometry and astronomy) form the quadrivium (four-ways), while logic, grammar, and rhetoric form the trivium (three-ways) of the seven liberal arts—all of which formed part of the medieval scheme of education, especially aimed at learning philosophy. As a liberal art, mathematics designates the mind's capacity to make concepts symbolic of quantity as discrete (number-symbols) or as continuous (form-symbols).

Transitional Period: 1600 to 1850. During this period, geometry lost its dominance over arithmetic through the invention of analytic geometry. A second significant discovery, the infinitesimal calculus, opened the way for the mathematical study of change and motion. Arithmetic and algebra, trigonometry, and various other branches and aspects of both pure and applied mathematics were either developed or founded anew.

Unfortunately, the philosophy of mathematics was given less attention, and no new and original theories arose to match the new mathematics. The philosopher George BERKELEY, offered some sound criticism of the loose reasoning that accompanied the first invention of the calculus. But the main growth in the philosophy of mathematics consisted in the dream of a universal science and method that haunted such thinkers as F. BACON, R. DESCARTES, B. PASCAL, and especially G. W. Leibniz. At the close of this period, this dream was fulfilled through the formulation of symbolic logic.

Immanuel KANT is well known for his analysis of mathematics and of science, but apparently he was unaware of the basic changes that mathematics had undergone. He considered mathematics as a system of absolute conceptions (synthetic a priori judgments) constructed within and unified by the central and regulative intuitions of space and time. Other theories during this period generally followed the philosophical outlook of their proponents, as in RATIONALISM, EMPIRICISM, IDEALISM, and POSITIVISM.

Contemporary Period: 1850 to the Present. In contrast to its earlier status, mathematics in the contemporary period is characterized by two significant changes. First, the invention of symbolic logic and the refinement of the axiomatic method have endowed it with a richer symbolism and led to its presentation as a system of purely deductive structures based on primitive axioms and propositions. Secondly, the change of outlook on such subjects as algebra and geometry, the development of analysis, and the introduction of the infinite into mathematics through Cantor's theory of sets have greatly enlarged the content of contemporary mathematics. Of special importance today are three theories included under the title of ''foundations of mathematics.''

Logicism, originating principally with G. Boole, G. Frege, and G. Peano, holds mathematics to be a branch of logic; the identity of mathematics with logic was formally proposed in B. Russell and A. N. Whitehead's *Principia Mathematica.*

Formalism, deriving mainly from David Hilbert (who claims Euclid as forerunner), originally began as a refinement of axiomatic method (as in Euclid's *Elements*). It was further developed by Hilbert and his followers to ease the crises caused by the paradoxes of *Principia Mathematica,* and to demonstrate the consistency of classical mathematics challenged by Brouwer and his school. Hilbert formulated classical mathematics by means of ''meaningless'' symbols as formal axiomatic theory (especially in his and P. Bernays' *Grundlagen der Mathematik*); this ''meaningless'' formalization, in turn, acquires meaning by becoming an object of a mathematical study called ''proof-theory'' or metamathematics.

Intuitionism, advocated by L. E. J. Brouwer, as a general philosophy holds that man's primordial experience is mathematical, that mathematics is identical with the exact part of human thought, and that no science (not even philosophy or logic) exercises priority or hegemony over mathematics. As a philosophy of mathematics, it holds to the dependence of mathematics on intuition alone and requires constructibility in terms of the natural numbers as the sole method of mathematical proof. Certain parts of classical mathematics and the logical law of excluded middle are rejected. Language and symbols are instruments for communicating mathematical ideas, but are not to be considered mathematics.

Critical Evaluation

It has been traditional in philosophy to view pure mathematics as a science dealing with quantity as discrete (arithmetic) or as continuous (geometry), and to consider astronomy and music as branches of applied mathematics. But this view needs now to be greatly enlarged by a deeper appreciation of tradition, an understanding of the development of mathematics itself, and an integration of current views on the foundations of mathematics.

Foundations. Mathematics can still be considered a highly abstract science having both a pure, or speculative, and a practical, or applied, aspect. As the intuitionists have partly shown, however, there are two ways in which pure mathematics originates from sense data. (1) Before mind actually ''abstracts'' the notion of circle or number, it must have previously generalized from sense experience such basic notions as structure (quantitative form), correspondence, the notions of singularity (unity) and group (multitude), and of sequence and order—a group of notions that form the basic architecture of the mind's mathematical universe. (2) It is only into this potentially prepared mathematical world that mind can ''abstract'' and then ''localize'' such mathematical entities as group, aggregate, circle, or number.

Once the mind has been equipped with these basic laws and foundational entities, it is to the credit of logicism and formalism to have shown that one can ''mathematicize'' in three ways. (1) Aided by creative imagination and by renewed recourse to sense imagery, the mind can tend to purify and perfect the entities (e.g., figures and numbers), or to embellish and create new instances of them (as inventing new numbers, or constructing topology). (2) The mind can neglect the basic mathematical paradigms themselves—even to the extent of considering them formless and meaningless inhabitants of this universe—and concentrate on their arrangement, i.e., their relations of priority and posteriority or of simplicity and complexity. Once this has been done, one can reintroduce mathematical entities to see if they can be made conformable to this new structural arrangement. At this point, the formalist school is content if no contradiction can be shown, while the intuitionists require that each new mathematical entity must be ''shown'' or constructed. (3) Another way of manipulating mathematical entities, pointed up by the invention of the infinitesimal calculus, is to consider them as relatively moveable and changeable, and to determine their laws of generation and their mutual reducibility within the mathematical universe. It is here that the import of such mathematical ''actions'' as squaring and differentiating is made clear, and the possibility of the infinitely small and the infinitely large is seen to be consistent.

Further Integration. A yet further advance is made when mind compares the mathematical with, say, the logical or the metaphysical universes or with the universe of language; or, again, when mind considers the methodology it employs in mathematics as similar to, or as contrasted with, that of other sciences. These considerations form a large area of interest in contemporary philosophy of mathematics.

A final way of considering mathematical entities is that proper to applied mathematics. This is the ''projective'' technique of matching such entities with, or imposing mathematical structures upon, the world of experience. Tradition has not always been clear on the distinction between pure and applied mathematics, and was given to emphasizing the static and immobile aspect of mathematics and its closeness to sense experience. The fact of various geometries and algebras shows that in pure mathematics, at least, there is no unique geometry or algebra of the real world, and that in applied mathematics the operational or pragmatic definition of truth applies: whatever mathematical system works best for the problem at hand is true.

A clear notion of mathematical abstraction, however, enables one to hold that mathematics is still a science of reality. To say that mathematics studies ''quantity'' is a traditional but inexact shorthand for stating that it studies any ''ordering or structuring of the parts of quantified substance'' (since, even in tradition, no ''accident'' as ''quantity'' can be the subject of science). The speculative mathematical universe, then, is the world of the traditional ''intelligible'' (imaginable) matter within which the mind engages in any of the various types of mathematical activity outlined above.

See Also: QUANTITY; CONTINUUM; EXTENSION.

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[E. A. MAZIARZ]

MATHER, INCREASE AND COTTON

Father and son, Puritan clergymen.

Increase, b. Dorchester, Mass., June 21, 1639; d. Boston, Mass., Aug. 23, 1723. He was the youngest son of Richard Mather, a prominent Puritan clergyman and Katherine (Holt) Mather. He attended Harvard (B.A. 1656), but spent most of his time in Ipswich and Boston studying under Rev. John Norton. Later he entered Trinity College, Dublin, Ireland (M.A. 1658). In 1661, after serving as chaplain to the soldiers on Guernsey, he returned to Boston and married Maria Cotton, daughter of John. Mather was a leader in Puritan circles, becoming pastor (1664) of the Second Church in Boston, a post he retained throughout his life. From 1685 to 1701 he was president of Harvard, but spent little time in Cambridge, preferring to devote his time to church affairs. In 1688 the colony sent him to England, where, after three years, he finally obtained a new charter. After 1692 his influence declined. His numerous writings include theological, historical, and biographical works. *Cases of Conscience Concerning Evil Spirits* (1693) appeared during the witchcraft hysteria and cautioned against the abuses of the witch trials.

Cotton, b. Boston, Feb. 12, 1663; d. there, Feb. 13, 1728. He graduated from Harvard (B.A. 1678, M.A. 1681) and was ordained (1685), serving at the Second Church during his father's absences and after his father's death. In 1718, with his father, he assisted at the ordination of a Baptist minister, and three years later he championed the unpopular cause of innoculation against smallpox. He was one of the founders of Yale and was the first native American to be a fellow of the Royal Society. His publications include *Magnalia Christi Americana* (1702), a collection of materials on the ecclesiastical history of New England; *Wonders of The Invisible World* (1693); and *Essays to Do Good* (1710).

[E. DELANEY]

MATHEW, THEOBALD

Known as the apostle of temperance; b. Thomaston, County Tipperary, Ireland, Oct. 10, 1790; d. Cobh, County Cork, Dec. 8, 1856. He was the fourth of the 12 children of James Mathew, of a distinguished Catholic