Mathematics and the Infinite (part 1)

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O Creator ineffable, who of the riches of Thy wisdom didst appoint three hierarchies of Angels and didst set them in wondrous order over the highest heavens, and who didst apportion the elements of the world most wisely: do Thou, who art in truth the fountain of light and wisdom, deign to shed upon the darkness of my understanding the rays of Thine infinite brightness, and remove far from me the twofold darkness in which I was born, namely, sin and ignorance. Do Thou, who givest speech to the tongues of little children, instruct my tongue and pour into my lips the grace of Thy benediction. Give me keenness of apprehension, capacity for remembering, method and ease in learning, insight in interpretation, and copious eloquence in speech. Instruct my beginning, direct my progress, and set Thy seal upon the finished work, Thou, who art true God and true Man, who livest and reignest world without end. Amen.

(St. Thomas Aquinas *Oratio ante studium*)

Number is Defined by Correspondence

- Aristotle's mathematics always applied mathematics.
  - He was a "mathematical realist"
  - In contrast to "mathematical nominalism," which claims "mathematicals" (numbers, geometric figures, etc.) are only names.
- How does the theory of number relate to motion?
  - It relates to it indirectly.
    - Philosophical physics and mathematics involve two different degrees of abstraction
      - 1st degree: physical abstraction
      - 2nd degree: mathematical abstraction

Number is Defined by Correspondence

- Science of *ens mobile* (philosophical physics) and mathematics overlap in two respects:
  - "[M]athematical theory of one age often seeps down into practice later on".
    - e.g., Einstein adopted non-Euclidean geometry for his Special Theory of Relativity.
  - Is the actually infinite mathematically possible?
- Aristotle defines number as a multitude measured by unity.
- Contemporary account of number is different.

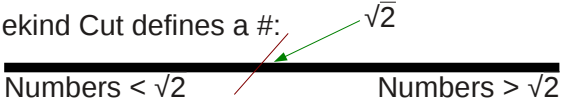
Number is Defined by Correspondence

- Contemporary theory defines number by correspondence.
  - Example 1: For every point on a line there corresponds a real number.
  - Example 2: Two barrels of white and red beans.
    - $W_n = n^{th}$ white bean, $R_n = n^{th}$ red bean
    - One-to-one correspondence:

$$\begin{array}{ccccccc} W_1 & W_2 & W_3 & W_4 & \dots & W_n \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & & \Updownarrow \\ R_1 & R_2 & R_3 & R_4 & \dots & R_n \end{array}$$

Numbers are Formed from Infinite Decimals

- Real numbers can "be reduced to countable quantities."
  - Imaginary numbers like $\sqrt{-1}$ cannot.
- Real numbers are transcendental (e.g., $e \cong 2.71828\ldots$ or $\pi \cong 3.14159265358979\ldots$), algebraic (e.g., $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$), or rational (e.g., ½, ¾, ⅓, etc.)
  - One can put these real numbers into correspondence with points on a line.
- Dedekind Cut defines a #:



Numbers are Formed from Infinite Decimals

DEFINING THE TRANSCENDENTAL NUMBER π USING THE DEDEKIND CUT:

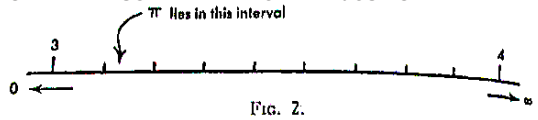


FIG. 2.

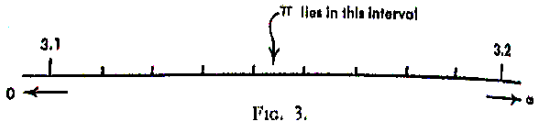


FIG. 3.

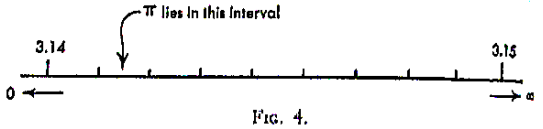


FIG. 4.

Numbers are Formed from Infinite Decimals

- Cantor definition of irrationals: “an infinite sequence of rationals”
- Another definition: All numbers made up of infinite decimals.
  - 2.9999999999… is a periodic decimal.
    - So is ⅐ = 0.142857142857142857… etc.
  - Non-periodic decimal:
    - $\sqrt{2}$ = 1.41421356237309504880168 9… etc.
    - Φ = 1.61803398874989484820458 7… etc.
      - (Φ is called the Golden Ratio.)

Cantor Defined a Transfinite Number

- Some terminology:
  - “Transfinite” a more precise term than “infinity.”
  - “cardinal number n. (Arith.): a number which answers the question ‘how many?’; one of the primitive or ‘natural’ numbers (one, two, three, etc.), as distinguished from the ordinal n.1 numbers (first, second, third, etc.).” (*OED*).
- Leibniz said there is no greatest cardinal number.
- What if there is a number “at a higher level”?

Cantor Defined a Transfinite Number

- Georg Cantor defined the first transfinite number as $\aleph_0$ (aleph-zero).
  - It is the number of all finite numbers, number of denumerable sets, a “number of a number.”
- Odd integers have $\aleph_0$ as their cardinal number.
- Prime numbers also have $\aleph_0$ as their cardinality.
  - Prime numbers are numbers which are divisible only by themselves and 1.
  - 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, … are primes.

Cantor Defined a Transfinite Number

- Cantor also uses “bi-unique correspondence” to show the set of rational numbers is denumerable.
  - Viz., all fractions can be put into a one-to-one correspondence with the integers.

0/1	1/1	1/2	2/1	1/3	3/1	1/4	2/3 …
⇕	⇕	⇕	⇕	⇕	⇕	⇕	⇕ …
1	2	3	4	5	6	7	8 …

- Cantor shows there are transfinite numbers greater than $\aleph_0$.

Cantor Proposed a Hierarchy of Transfinites

- Transfinite number $c > \aleph_0$.
  - c is the number of points on the continuum.
  - It includes algebraic and transcendental numbers.
- The continuum is non-denumerable.
  - i.e., it cannot be put into a one-to-one correspondence with the integers.
- Cantor’s proof that $c > \aleph_0$ is called “Cantor’s diagonal method.”

Cantor Proposed a Hierarchy of Transfinites

- Each line represents a number between 0 and 1, where each symbol is a number 0, 1, … 9:

0. **A1** a2 a3 a4 a5 a6 a7 a8 a9 … a_n
0. b1 **B2** b3 b4 b5 b6 b7 b8 b9 … b_n
0. c1 c2 **C3** c4 c5 c6 c7 c8 c9 … c_n
0. d1 d2 d3 **D4** d5 d6 d7 d8 d9 … d_n
0. e1 e2 e3 e4 **E5** e6 e7 e8 e9 … e_n
⋮ ⋮ ⋱

Cantor Proposed a Hierarchy of Transfinites

- The number formed by the diagonal, "0. A1 B2 C3 D4 E5…," is a number that is not in the set of the infinite numbers in the previous slide.
- ∴ the set of numbers of the form "0. A1 B2 C3 D4 E5…" form a set with a higher cardinality $c > \aleph_{\emptyset}$.
- This process can be repeated *ad infinitum* to find sets of higher and higher cardinalities.
- There is even a whole algebra of transfinite numbers!

Russell Reduces Infinity to a Logical Hierarchy

- Rudolf Carnap coined the term "logicalism."
  - It is the reduction of mathematics to logic.
- Russell wished to derive all of mathematics from logic.
  - He defined number as "the class of similar classes."
- Number is a property of a class.
  - e.g., the two classes of 10 ostriches and 10 humans both share the number-class 10.
- Isn't this definition circular? A *definitum* cannot be included in a definition.
  - e.g., the statement "All men are liars" manifests this problem.
- But the definition of infinity needn't be infinitely complex.

References

- V. E. Smith's *Philosophical Physics*
  - Please begin reading ch. 9 (Mathematics and the Infinite).
    - The required reading will be in the reginacoeli.box.com account.