Mathematics and the Infinite (part 2)

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O Creator ineffable, who of the riches of Thy wisdom didst appoint three hierarchies of Angels and didst set them in wondrous order over the highest heavens, and who didst apportion the elements of the world most wisely: do Thou, who art in truth the fountain of light and wisdom, deign to shed upon the darkness of my understanding the rays of Thine infinite brightness, and remove far from me the twofold darkness in which I was born, namely, sin and ignorance. Do Thou, who givest speech to the tongues of little children, instruct my tongue and pour into my lips the grace of Thy benediction. Give me keenness of apprehension, capacity for remembering, method and ease in learning, insight in interpretation, and copious eloquence in speech. Instruct my beginning, direct my progress, and set Thy seal upon the finished work, Thou, who art true God and true Man, who livest and reignest world without end. Amen.

(St. Thomas Aquinas *Oratio ante studium*)

Russell Reduces Infinity into a Logical Hierarchy

- We will discuss three philosophies of mathematics:
  - Logicalism
    - of Peano and Frege, culminating in Russell
    - Reduction of mathematics to logic
      - We briefly discussed this last time.
        - Number is "the class of similar classes."
  - Formalism
    - David Hilbert and his school
  - Intuitionism
    - Brouwer, Weyl, and Poincaré

Russell Reduces Infinity into a Logical Hierarchy

- Russell's goal is "to make the modern method more discursive and, like matter, more controllable."
- If number is the "class of similar classes," then infinity would have to include itself as a member!
  - This is Russell's "paradox of the infinite."
    - But technically infinity is not a number.
- Example paradox: "A Cretan says: 'All Cretans are liars.'"
  - Russell resolved this with "logical types."
    - In this example, "logical types" put the Cretan outside.
- Infinity need not be infinitely complex.
  - Thus, e.g., $\aleph_\odot$ is finite "in the structure of its definition," although it represents the infinite series of algebraic numbers.

Formalism Makes Infinity into a Rule

- Russell believed that his logicalism would have to be supplemented by empirical data.
- Hilbert's formalism takes logicalism to the extreme.
  - Hilbert dismisses the need for the empirical.
    - "Content is sacrificed to form, inner meaning to mere rule."
- Kurt Gödel: "the foundations of mathematics must always contain at least a small number of undefined terms ['the undecidables'] which cannot be treated in the system which they organize."
- Gödel thus proved both logicalism and formalism false : these undecidables will preclude them "from ever saying the final word about reality or mathematics."

Formalism Makes Infinity into a Rule

- Hilbert responded to Gödel by saying that mathematics need not conform to reality.
  - Mathematics for Hilbert is just a manipulation of symbols.
- Hilbert: "A number is not an object in the proper sense but a property."
- Infinity is an idea.
- Inductive side of mathematics: "One formula [of mathematics] is conceived and derived from another."

Formalism Makes Infinity into a Rule

- "When we generalize this situation, then mathematics arrives at a number of formulas, first those which correspond to the matter expressed in finite propositions, in essentially numbered equalities and inequalities, and second at broader formulæ that in themselves have no meaning and form the ideal representations of our theory." (Hilbert's *Grundlagen der Geometrie*)
- Infinity is in thought alone.
- Hilbert: "The infinite divisibility of the continuum is an operation which exists in thought only, is just an idea, an idea which is refuted by our observations of nature, as well as by physical and chemical experiments."
- "Infinity here exists only logically."

Intuitionism Denies Hierarchy of Infinites

- Intuitionists: L. E. J. Brouwer, Herman Weyl, Leopold Kronecker, Henri Poincaré
- Intuitionism results from Kant's synthetic *a priori*
  - Kant thought Euclidean geometry was *a priori*
- Mathematics must begin with the intuitive.
- "The primary intuition, the bedrock of all mathematics, is the 'intuition of the bare two-oneness'." (Brouwer, "Intuitionism and Formalism," *Bul. AMS*, XX (1913) 85-86)

Intuitionism Denies Hierarchy of Infinites

- Brouwer: "intuitions of the bare two-oneness"
  - "Objectively and subjectively, a thing is prior to its relations. *One* is what it is, first, and is then related to *two*, and the order of thought follows along with this arrangement of reality itself."
  - Brouwer believes, however, one and two are "drawn much more tightly together."
  - "[I]n a single flash of insight, there arise the concepts of one, two, and their relation."
- Intuitionists recognize only denumerable sets.

Intuitionism Denies Hierarchy of Infinites

- Brouwer can indefinitely "split" the "two-oneness" to form the concept of "between."
- "The notion of the infinite is precipitated from considering this idea of 'between' which can be strewn indefinitely with repetitions of the 'two-oneness.'"
- "No infinity can be finited to a point of being compared with other infinities" because Brouwer considers this opposed to his "two-oneness" intuition.
  - Thus, Brouwer only admits the $\aleph_0$ infinity.

Intuitionism Denies Hierarchy of Infinites

- Mathematics has a priority over logic in intuitionism.
- Intuitionists emphasize mathematical *content*; formalists "emphasize the *regulative* and *logical*."
- Intuitionists reject Cantorian infinities ∵ they have "content" only "among the rules of logic."
- Intuitionists reject the "domination by the logical order."

Intuitionism Denies Hierarchy of Infinites

- Kronecker's finitism: "God made the integers and man everything else."
- "the notion of all the real numbers between 1 and 2":
  - Russel et al. formalists: the "middle area is mapped out in terms of a series of infinite decimals, requiring an infinite number of operations."
  - Intuitionists: "the stretch from 1 to 2 is spanned by a law to construct the series of intervening decimals, a law that is involved by the analysis of 'between'; there is thus a finite number of operations."
    - "no such thing as a more 'between' or a less 'between.'"

Intuitionism Denies Hierarchy of Infinites

- Intuitionism denies the principle of the excluded middle.
  - "A thing is either *A* or *not-A*, without the possibility of a middle ground."
- It holds instead a "three-valued logic" or "three-valued mathematics":
  - "either *A* or *not-A* or *indeterminate*"
    - Is this indeterminate Aristotle's potency, the intermediary between being and non-being?

Intuitionism Denies Hierarchy of Infinites

- Hermann Weyl succinctly outlines the intuitionists' understanding of the infinite in four steps:
  1) Concrete individual judgment (e.g., 2 + 5 = 7), from which intuitions are obtained
  2) Symbolic stage (e.g., $m + n = n + m$)
  3) Concept of sum in general obtained
  4) From this, any number can be formed *ad infinitum*.

References

- V. E. Smith's *Philosophical Physics*
  - Please continue reading ch. 9 (Mathematics and the Infinite).
    - The required reading will be in the reginacoeli.box.com account.
- Have a blessed summer break!